Microchip problem

January 30, 2018

Assuming that the transport boundary layer has a thickness $\delta \ll L$, the transport equation reduces to :

$$y\frac{U}{H}\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial y^2} \tag{1}$$

A dimensional analysis of the equation provides a scaling law for $\delta(x)$:

$$\delta(x) = x^{1/3} H^{2/3} Pe_H^{-1/3} \tag{2}$$

where Pe_H is the Peclet number based on the channel thickness H.

We seek solutions for the concentration field as self-similar profiles : $C = C_0 f(\zeta)$ where $\zeta = y/\delta(x)$ is the rescaled vertical coordinate.

We have the following relations for the spatial derivatives of C :

$$\frac{\partial^2 C}{\partial y^2} = \frac{C_0}{\delta^2}\frac{d^2 f}{d\zeta^2} \tag{3}$$

$$\frac{\partial C}{\partial x} = C_0 \frac{df}{d\zeta}\frac{\partial \zeta}{\partial x} = -C_0 \frac{df}{d\zeta}\frac{y}{\delta^2}\frac{d\delta}{dx} = -\frac{C_0}{3}\frac{df}{d\zeta}\frac{y}{\delta x} \tag{4}$$

Substituting in (1), we get :

$$-\frac{1}{3}\frac{df}{d\zeta}\frac{y^2}{\delta x}\frac{U}{H} = D\frac{1}{\delta^2}\frac{d^2 f}{d\zeta^2} \tag{5}$$

or :

$$-\frac{1}{3}\frac{df}{d\zeta}\frac{\zeta^2 \delta^3}{x H^2} Pe_H = \frac{d^2 f}{d\zeta^2} \tag{6}$$

Since $\delta^3 = xH^2/Pe_H$, we get finally an ordinary differential equation for f :

$$-\frac{\zeta^2}{3}\frac{df}{d\zeta} = \frac{d^2 f}{d\zeta^2} \tag{7}$$

or :

$$\frac{\partial}{\partial \zeta}\left(\ln \frac{df}{d\zeta}\right) = -\frac{\zeta^2}{3} \tag{8}$$

Integrating once, we have :

$$\frac{df}{d\zeta} = A \exp(-\zeta^3/9) \tag{9}$$

The boundary conditions are $C = 0$ at the wall ($f = 0$ at $\zeta = 0$) and $C \to C_0$ when $y \gg \delta$, or $f \to 1$ when $\zeta \to \infty$.

The solution is :

$$f = \frac{\int_0^\zeta \exp(-\xi^3/9)d\xi}{\int_0^\infty \exp(-\xi^3/9)d\xi} \approx 0.54 \int_0^\zeta \exp(-\xi^3/9)d\xi \tag{10}$$

The local mass flux at the wall is :

$$J_D(x) = -D\frac{\partial C}{\partial y} = -\frac{DC_0}{\delta(x)}f'(0) \approx -0.54\frac{DC_0}{\delta(x)} \tag{11}$$