Radiative heat transfer

22 mars 2017

Energy can be transported by the electromagnetic field radiated by an object at finite temperature. A very important example is the infrared radiation emitted towards space by the surface of the earth. The partial absorption of this radiation by various gases in the atmosphere (water vapor, carbon dioxide, methane, ...) is the origin of the greenhouse effect (fig. 1) which determines the average temperature of the atmosphere.

1 Radiation from black, grey and real bodies

1.1 Black body radiation

1.1.1 Photons inside a cavity

To characterize the radiation emitted by a body of matter at finite temperature, it is first useful to define the emission characteristics of an idealized *black body*. A black body is a system which absorbs completely any incoming radiation and is in equilibrium with a reservoir at constant temperature T. The system which would represent closely a black body is a cavity with a very small opening. Radiation entering the cavity through the small aperture would be trapped inside by multiple reflections. Using the laws of statistical physics, it is possible to derive the distribution of states of an assembly of photons within a closed cavity in thermal equilibrium. The number of photons in state s having an energy ϵ_s is the Planck distribution [1] :

$$n_s = \frac{1}{e^{\beta\epsilon_s} - 1}$$

where $\beta = 1/k_B T$. Each photon is characterized by its wavevector $\mathbf{k}$ and its state of polarization (there are two states of polarization possible) ; its energy ϵ is given as a function of wavevector $k = |\mathbf{k}|$ or angular frequency ω by $\epsilon = \hbar ck = \hbar\omega$. If we let $f(\mathbf{k})d^3\mathbf{k}$ be the number of photons per unit volume, with one specified direction of polarization, having a wavector between $\mathbf{k}$ and $\mathbf{k}+d\mathbf{k}$. The number of such photon states per unit volume is $(2\pi)^{-3}d^3\mathbf{k}$. Taking into account Planck's distribution, we have :

$$f(\mathbf{k})d^3\mathbf{k} = \frac{1}{e^{\beta\hbar\omega} - 1}\frac{d^3\mathbf{k}}{(2\pi)^3}$$

Now, we compute the number of photons per unit volume with both directions of polarizations, with angular frequency in the range ω - $\omega + d\omega$, $n(\omega)d\omega$. This is given by summing $f(\mathbf{k})d^3\mathbf{k}$ over the domain of Fourier space contained between radii $k = \omega/c$ and $k + dk = (\omega + d\omega)/c$, i.e. $4\pi k^2 dk$, and multiplying by two to account for both polarizations :

$$n(\omega)d\omega = 2f(\mathbf{k})4\pi k^2 dk = \frac{8\pi}{(2\pi c)^3}\frac{\omega^2 d\omega}{e^{\beta\hbar\omega} - 1}$$

Finally, since each photon has an energy $\hbar\omega$, the mean energy per unit volume of the photon assembly in the range ω - $\omega + d\omega$ is :

1. For the derivation of this result, see a statistical physics textbook

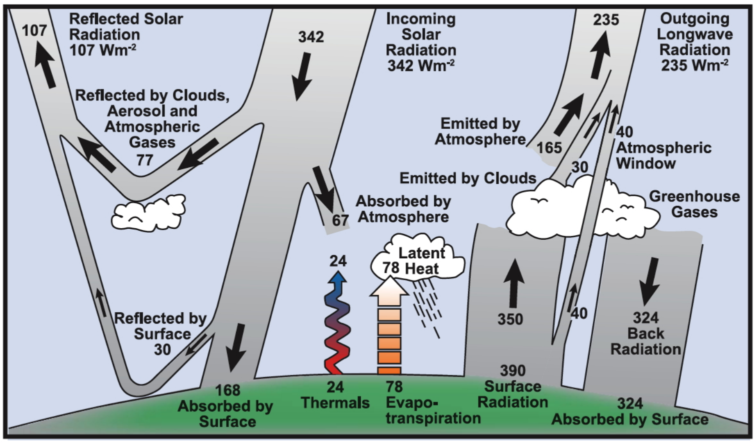

FIGURE 1 – Heat fluxes through the earth atmosphere. Figure from the 4th report of the IPCC (2007)

$$\bar{u}(\omega, T)d\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1}. \tag{1}$$

To insure thermal equilibrium within the cavity, the associated radiation field has the following properties : it is homogeneous (independent of position), isotropic (independent of the direction of propagation), unpolarized (all polarizations are equally probable) and independent of the shape of the cavity.

1.1.2 Emission and absorption of radiation

The electromagnetic emission of a body can be characterized by the quantity $P_e(\mathbf{k}, \alpha)d\omega d\Omega$ which is the power emitted per unit area, with polarization α, in the frequency range ω - $\omega + d\omega$, within a small solid angle $d\Omega$ around the direction given by the wavevector $\mathbf{k}$. This body can also receive radiation from a direction $\mathbf{k}'$ defined as an incident power $P_i(\mathbf{k}', \alpha)d\omega d\Omega$. The body will in general adsorb a fraction $a(\mathbf{k}', \alpha)$ of this incident power. $a(\mathbf{k}', \alpha)$ is called the *absorptivity* of the body. In equilibrium, conservation of energy requires that the fraction of radiation which is not adsorbed should be reflected. The consequence of this equilibrium is the following relation between the emitted and incident powers :

$$P_e(-\mathbf{k}, \alpha) = a(\mathbf{k}, \alpha)P_i(\mathbf{k}, \alpha) \tag{2}$$

A black body is a perfect absorber, regardless of wavelength and polarization, i.e. $a(\mathbf{k}, \alpha) \equiv 1$ and we have $P_e(-\mathbf{k}, \alpha) = P_i(\mathbf{k}, \alpha)$. We can compute P_i for a black body, using the mean number of photons $f(\mathbf{k})d^3\mathbf{k}$ determined above and noting that, if the angle of incidence with respect to the normal is θ, the number of photons crossing the surface per unit time is : $c\ \cos\theta\ f(\mathbf{k})d^3\mathbf{k}$. Each photon carrying an energy $\hbar\omega$, we have :

$$P_i(\mathbf{k}, \alpha)d\omega d\Omega = \hbar\omega\ c\ \cos\theta\ f(\mathbf{k})d^3\mathbf{k}.$$

In spherical coordinates, we have $d^3\mathbf{k} = k^2 dk d\Omega = (\omega^2/c^3)d\omega d\Omega$ and :

$$P_i(\mathbf{k}, \alpha) = \frac{\hbar\omega^3}{c^2} f(k)\cos\theta \tag{3}$$

From eqn. 2 and eqn. 3, we can see that the emitted power $P_e(k, \alpha) = P_i(\mathbf{k}, \alpha)$ varies with the cosine of the angle with respect to the normal. This result, known as Lambert's law, is simply a geometrical consequence of the projection of a unit area of the body onto the direction of propagation.

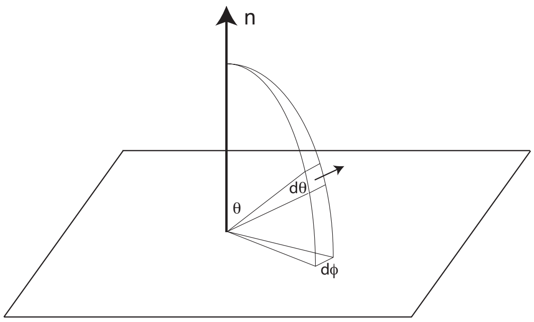

FIGURE 2 – Radiation within a solid angle $d\Omega = \sin\theta\ d\theta\ d\phi$ in a direction making an angle θ with the normal $\mathbf{n}$.

To get the total power emitted in all directions of space $P_e(\omega)d\omega$, we need to integrate $P_e(k,\alpha)d\omega d\Omega$ over all solid angles defined by the polar angle $0 < \theta < \pi/2$ and the azimuthal angle $0 < \phi < 2\pi$ and multiply by 2 to account for the two possible polarizations :

$$P_e(\omega)d\omega = \frac{2\hbar\omega^3}{c^2} f(k)d\omega \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta = \frac{2\pi\hbar\omega^3}{c^2} f(k)d\omega \tag{4}$$

Using the value found above for $f(k)$, we have the power emitted per unit area by a black body at temperature T in the angular frequency range ω - $\omega + d\omega$:

$$P_e(\omega, T)d\omega = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3 d\omega}{e^{\beta\hbar\omega} - 1} \tag{5}$$

If we need this spectral distribution in terms of the wavelength λ, we use $\omega = 2\pi c/\lambda$ and $|d\omega| = 2\pi c/\lambda^2\ d\lambda$ to get

$$P_e(\lambda, T)d\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{d\lambda}{\exp(hc/\lambda k_B T) - 1}. \tag{6}$$

This spectral distribution is shown on fig. 3 for four different values of the temperature T. This distribution has a maximum at a wavelength inversely proportional to temperature :

$$\lambda_M = \frac{2,88 \times 10^{-3}}{T}. \tag{7}$$

This proportionality to the inverse of the absolute temperature is called Wien's displacement law. It comes from the fact that the average energy of the radiated photons is equal to the thermal energy :

$$k_B T \propto \frac{hc}{\lambda_M}.$$

At $T = 300$K, the emission is in the infrared range, with $\lambda_M = 9.6\mu$m and at $T = 5800$K, the surface temperature of the sun the maximum of emission is in the visible range at $\lambda_M = 0.5\mu$m. There is a diffuse electromagnetic radiation in the universe with a wavelength in the mm range, equivalent to a temperature of 3K and which is interpreted as a remnant of the primordial explosion.

When we integrate the spectral distribution over the whole wavelength range, we get the total power emitted by unit surface :

$$P_{BB}(T) = \int_0^\infty P_e(\lambda, T)d\lambda = \frac{2\pi^5 k_B^4}{15\ h^3 c^2} T^4 = \sigma T^4. \tag{8}$$

This is Stefan-Boltzmann's law, the value of the constant σ is $5,67 \times 10^{-8}$ W.m^{-2}.K^{-4}. For example, we can compute from the Stefan-Boltzmann's law that a black body at 300 K emits

460 W.m^{-2}. The dependance of the emitted power with the fourth power of temperature can be understood by a scaling argument : each photon has an energy which is on average proportional to $k_B T$ and the average wavelength is inversely proportional to T. Consequently the average wave vector k of these photons is proportional to T. The number of possible states n_S is proportional to the volume of the sphere of radius k in the Fourier space, that is $n_S \propto k^3 \propto T^3$. As a result, the total average energy of the assembly of photons is proportional to $n_S k_B T \propto T^4$.

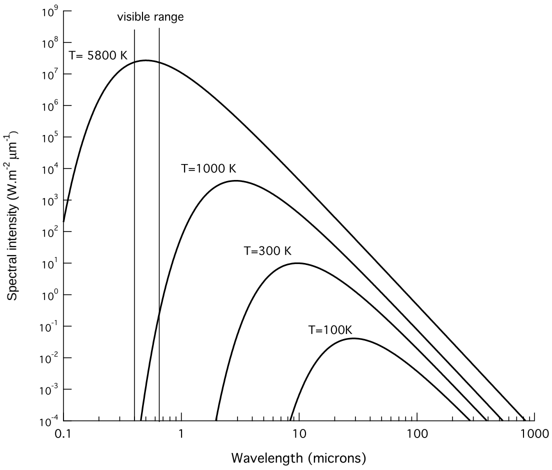

FIGURE 3 – Spectral distribution of the black body radiation at four different temperatures.

1.2 Real bodies. Grey bodies.

1.2.1 Emissivity and Kirchoff's law

Materials in general have an absorptivity a and emissivity $e(\lambda, T, \theta)$ smaller than one and depending on the wavelength, the temperature and the angle of incidence. The emissivity e is defined as the ratio between the power emitted P and the power that would be emitted by a black body at the same temperature, wavelength and orientation, i.e. $P = e(\lambda, T, \theta) P_{BB}(T, \lambda, \theta)$. There is a general relation between absorptivity and emissivity which can be derived by the following thought experiment : imagine that we place a black body (a body absorbing all radiation) inside a cavity at temperature T. The radiation inside the cavity P_{cav} is determined by Planck's distribution. If the black body is also at T, there should not be a net exchange of energy between the cavity and the blackbody. The power emitted by the blackbody should be equal to the incident radiation : $P_{BB} = P_{cav}$. Now we place in the cavity a body which is not black and characterized by its absorptivity a. The power absorbed is aP_{cav} and for the body to be in equilibrium it should be equal to the emitted power P. So we get : $P = aP_{cav} = aP_{BB}$. From the definition of emissivity $P = eP_{BB}$, we get Kirchoff's law :

$$a = e \tag{9}$$

the emissivity of a body is equal to its absorptivity.

1.2.2 Grey bodies

For some materials in a range of wavelengths, the emissivity e is constant and the power emitted has a spectral distribution which is identical to a black body. Those materials are referred to as *grey bodies*. Their total emissive power is simply given by :

$$P_{GB}(T) = eP_{BB}(T) = e\sigma T^4 \tag{10}$$

For example, polished metals have an emissivity which depends weakly on wavelength in the infrared range (fig. 4). For copper surfaces, either polished or covered with an oxide layer, the hemispherical emissivity (i.e. the emissivity integrated over all possible directions and wavelengths) depends weakly on temperature in the range 200-1000 K.

Polished metal surfaces are highly reflective and their absorptivity a is very small. As a consequence of Kirchoff's law, their emissivity $e = a$ is also very small. Polished copper is a very good reflector of radiation and a very poor emitter of radiation. Thin metal layers deposited on the glass surface of Dewar flasks prevent the exchange of heat by radiation. Likewise, blankets made of a thin polymer sheet bonded to a reflective metal layer prevent the radiative loss of heat from an injured person.

When a metal surface is covered by an oxide layer, the emissivity increases strongly (fig. 4)

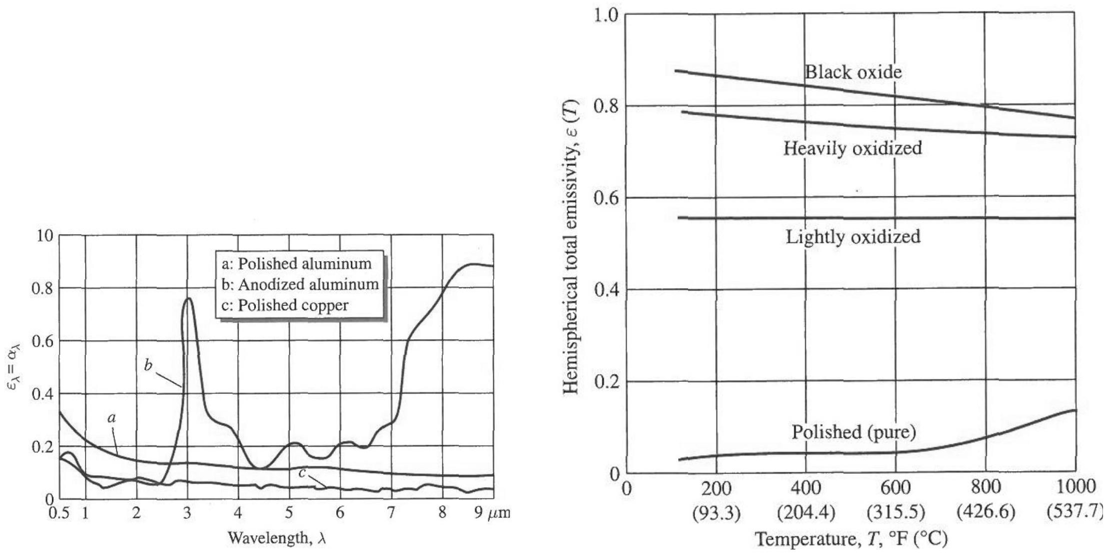

FIGURE 4 – Left : emissivity as a function of wavelength for three metal surfaces : polished aluminum and copper, anodized aluminum. Right : hemispherical emissivity (emissivity integrated over wavelengths and directions) for copper with and whitout oxide layers. Figures reproduced from Kreith et al., Principles of heat transfer.

1.2.3 Infrared thermography

The power emitted by a body varies with the fourth power of the absolute temperature according to Stefan's law. As a result, even over a limited range of temperatures, say from 10°C to 110°C (283 K to 383 K) the power emitted increases by a factor of three (fig.5) . From a measurement of the intensity radiated and a knowledge of the emissivity of the surface, it is possible to determine the temperature of a surface. This is the principle of infrared thermography. Infrared cameras operating at wavelengths between 7 and 14 μm are typically able to "image" temperatures in the range -40°C - 2000°C with a precision in the measurement of $\pm$ 1°C in the range 5°C-150°C. Passive thermography is used routinely to evaluate the thermal insulation properties of buildings or to detect problems in electrical circuits leading to enhanced heat dissipation. There is another mode of operation : active of flash thermography, in which a heat pulse is locally applied to a material by a focalized strong source of light. The camera records the evolution of the temperature field after the application of the pulse. If there is a defect inside the material, the diffusion of heat will be altered and the temperature field will be different from what we expect from diffusion in a homogeneous material.

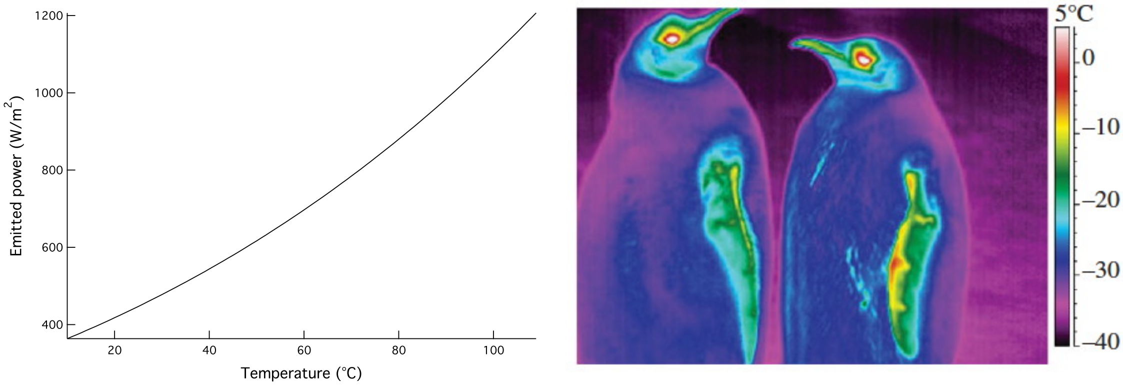

FIGURE 5 – Left : power emitted by a black body as a function of temperature. Right : thermal imaging of emperor penguins. Image reproduced from McCafferty et al., Biol. Lett. 2013

2 Radiative exchange between two bodies. Shape factors.

To compute the radiative transfer of energy between two bodies, it is necessary to take into account the angular distribution of radiation with respect to the surface normal. Even if the emissivity is isotropic, the intensity of radiation depends on the orientation with regards to normal to the surface. The emission in a direction making an angle θ with the normal must be multiplied by a factor $\cos\theta$ corresponding to the projection onto the normal. Likewise, for the radiation arriving on a surface, one has to take into account the inclination with respect to the normal.

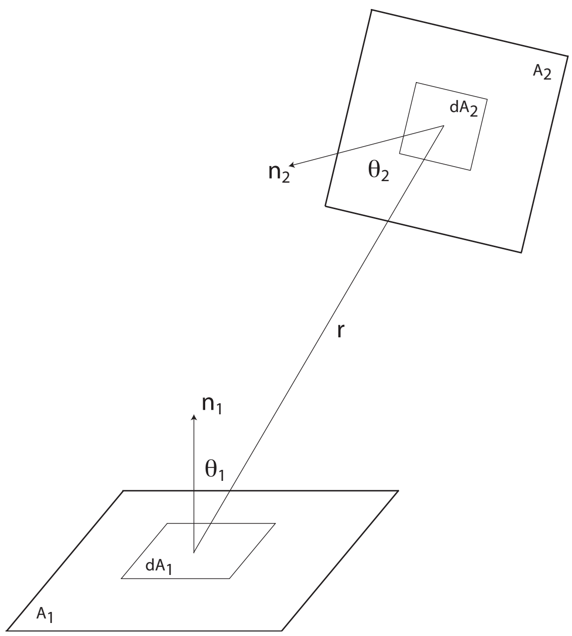

FIGURE 6 – Determination of the shape factor for the radiative transfer between two bodies.

For each particular geometry, it is necessary to compute a shape factor (or view factor) F_{1-2} giving the fraction of energy radiated by body 1 which reaches body 2. Let us consider two black bodies with surfaces A_1 et A_2 (fig. 6). The energy flux from 1 to 2 is : $q_{1\to 2} = P_{b1}A_1F_{1-2}$. Conversely, the flux exchanged between 2 and 1 is : $q_{2\to 1} = P_{b2}A_2F_{2-1}$. If the two bodies are blackbodies, the energy received is entirely adsorbed and the net balance of exchanged energy is : $\Delta q_{1-2} = P_{b1}A_1F_{1-2} - P_{b2}A_2F_{2-1}$. If the two bodies are at the same temperature, the net energy balance is zero and the power densities radiated by each body are the same : $(P_{b1} = P_{b2})$. We then get the reciprocity relation for the shape factors :

$$A_1F_{1-2} = A_2F_{2-1} \tag{11}$$

and the energy flux exchanged between two black bodies at different temperatures can be written : $\Delta q_{1-2} = A_1F_{1-2}(P_{b1} - P_{b2}) = A_2F_{2-1}(P_{b1} - P_{b2})$.

The energy radiated by an elementary surface dA_1 reaching the elementary surface dA_2 is :

$$dq_{1\to 2} = I_1 \cos\theta_1 dA_1 d\Omega_{1-2} \tag{12}$$

where I_1 is the *radiation intensity* $I_1 = P_{b1}/\pi$ [2] and where $d\Omega_{1-2}$ is the solid angle defined by dA_2 seen from dA_1. This solid angle is given by : $d\Omega_{1-2} = \cos\theta_2 dA_2/r^2$. Then :

$$dq_{1\to 2} = P_{b1}\frac{\cos\theta_1\cos\theta_2 dA_1 dA_2}{\pi r^2} \tag{13}$$

with a symmetrical expression for the flux exchanged between 2 and 1. The net flux exchanged between the two elementary surfaces is :

$$dq_{1-2} = (P_{b1} - P_{b2})\frac{\cos\theta_1\cos\theta_2 dA_1 dA_2}{\pi r^2}. \tag{14}$$

To compute the total flux exchanged between bodies 1 and 2, we integrate the elementary fluxes over the entire surfaces A_1 and A_2 :

$$q_{1-2} = (P_{b1} - P_{b2})\int_{A_1}\int_{A_2}\frac{\cos\theta_1\cos\theta_2 dA_1 dA_2}{\pi r^2}. \tag{15}$$

The calculations of shape factors are in general rather complicated but can be done analytically for a few standardized geometries, for example two rectangles facing each other or at right angle, two disks facing each other.

When the bodies involved in the radiation exchange are not black, it is of course necessary to account for their emissivity.

2.1 Energy exchange between two disks facing each other

A simple example of shape factor calculation is the case of a small disk or area A_1 and a large disk of area $A_2 = \pi a^2$, facing each other and separated by a distance D (fig. 7).

The shape factor F_{1-2} is such that :

$$A_1F_{1-2} = \int_{A_1}\int_{A_2}\frac{\cos\theta_1\cos\theta_2\ dA_1\ dA_2}{\pi r^2}. \tag{16}$$

Area A_1 being very small compared to A_2, the angles θ_1 and θ_2 vary only very little when we move the point of interest over the surface A_1 and, instead of integrating over A_1, we can just multiply by A_1 :

$$A_1F_{1-2} = \frac{A_1}{\pi}\int_{A_2}\frac{\cos\theta_1\cos\theta_2\ dA_2}{r^2}. \tag{17}$$

Using the following relations : $\cos\theta_1 = \cos\theta_2 = D/r$, $r = \sqrt{D^2+\rho^2}$ et $dA_2 = 2\pi\rho\ d\rho$, we get :

$$A_1F_{1-2} = 2A_1\int_0^a\frac{D^2\rho\ d\rho}{(D^2+\rho^2)^2}. \tag{18}$$

and after integration :

$$F_{1-2} = \frac{a^2}{a^2+D^2} \tag{19}$$

2. The π factor comes from the integration over all possible angles on an hemisphere.

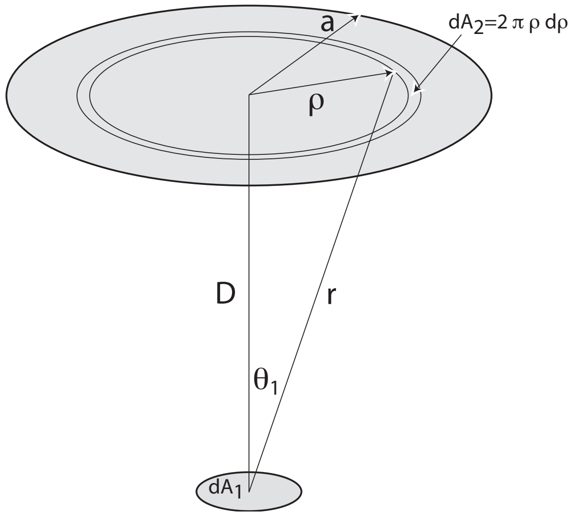

FIGURE 7 – Radiative transfer between coaxial disks.

3 Summary

— bodies at a finite temperature T radiate electromagnetic waves with an intensity depending on temperature, wavelength and direction with respect to the normal $I(T, \lambda, \theta)$ defined by the emissivity $e(T, \lambda, \theta)$ such as $I(T, \lambda, \theta) = e(T, \lambda, \theta) I_{BB}(T, \lambda, \theta)$ where $I_{BB}(T, \lambda, \theta)$ is the intensity radiated by an ideal black body and defined as :

$$I_{BB} = \frac{2hc^2}{\lambda^5} \frac{\cos\theta}{\exp(hc/\lambda k_B T) - 1}$$

— the emissivity e is equal to the absorptivity a (Kirchoff's law)
— the radiated intensity is maximum at a wavelength λ_M inversely proportional to T (Wien's law)
— the total power radiated by a black body is proportional to T^4 (Stefan's law)
— to compute the net exchange of energy by radiation between two bodies 1 and 2 of area A_1 and A_2, we need to evaluate shape factors F_{1-2} or F_{2-1} possessing the reciprocity relation : $A_1 F_{1-2} = A_2 F_{2-1}$.