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FORMATION OF TAYLOR VORTICES IN SPHERICAL COUETTE FLOW

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We have conducted a numerical study of spherical Couette Flow -- the flow between differentially rotating concentric spheres. When the gap ratio $\sigma \equiv (R_2 - R_1)/R_1$ (where R_1 and R_2 are the radii of the inner and outer spheres) is small, the flow near the equator resembles that between cylinders, the classic Taylor-Couette problem. As in Taylor-Couette flow, when the angular momentum gradient, measured by the Reynolds number $Re \equiv R_1{}^2\Omega_1/\nu$ (where Ω_1 is the angular velocity of the inner sphere and ν the kinetic viscosity) exceeds a critical value, Taylor vortices form to redistribute angular momentum between radial shells.

For $\sigma = 0.18$, the gap ratio studied experimentally by Sawatzki and Zierep (1970) and Wimmer (1976), there exist three steady axisymmetric steady states, each with a different number of Taylor vortices (zero, one, or two) per hemisphere. The equilibrium attained by the flow depends on the history of its acceleration. Previous initial value codes (Bonnet & Alziary de Roquefort 1976, Yavorskaya et al. 1978, and Bartels 1982) have reproduced the steady states and some of the transitions, but have been unable to generate the one-vortex state as a transition from the basic zero-vortex flow.

We answer two questions arising from these previous studies, namely : 1) Why has generation of the one-vortex state eluded previous initial value studies ?

2) What is the mechanism by which the history of the flow determines the final steady state ?

Methods

We have written an initial value code to solve the axisymmetric incompressible time-dependent Navier-Stokes equations in a spherical geometry. We use a pseudospectral method (Gottlieb & Orszag 1977). Functions are represented as sums of basis functions -- in this case, Chebyshev polynomials in radius multiplied by sines in theta. Derivatives are taken in the spectral representation, multiplications are performed in physical space, and Fast Fourier Transforms are used to transform between the two representations. Evolution in time is accomplished by an algorithm of global accuracy $O(\Delta t)^2$: on the nonlinear terms, the

Adams-Bashforth approximation is used, while on the linear terms we employ the Crank-Nicolson approximation. We used a resolution of 16 polynomials in radius and 128 sine functions in angle θ, and 70 time steps per inner sphere revolution.

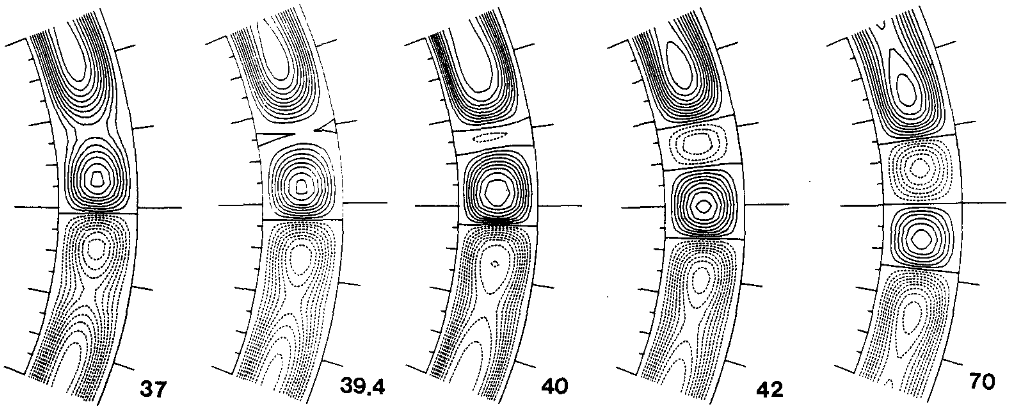
The elliptic operator resulting from the Crank-Nicolson approximation is block-upper triangular in the sine series basis. It can therefore be inverted using a sub-matrix back-solve, analogous to an ordinary back-solve for upper triangular matrices. The sub-matrix equations, one for each sine basis function, are solved using an eigenvector-eigenvalue decomposition. The sub-matrices differ only by a multiple of the identity, so the eigenvector-eigenvalue factorization need be done only once. Dirichlet and Neumann boundary conditions on the meridional stream function, which obeys a fourth-order equation, are imposed by a Greens function technique.

The algorithm requires little modification to perform linear stability analysis. To calculate the eigenvalues and eigenvectors of the Navier-Stokes equations linearized about a steady state $\vec{U}$, it suffices to replace the full nonlinear interaction $(\vec{u}.\nabla)\ \vec{u}$ by the linearized term $(\vec{u}.\nabla)\ \vec{U} + (\vec{U}.\nabla)\ \vec{u}$, and to impose homogeneous boundary conditions. Iteration in time is then equivalent to the power method, and causes an initial guess to converge to the eigenvector with the largest eigenvalue. The Rayleigh quotient is used to estimate the eigenvalue from two successive approximations to the eigenvector.

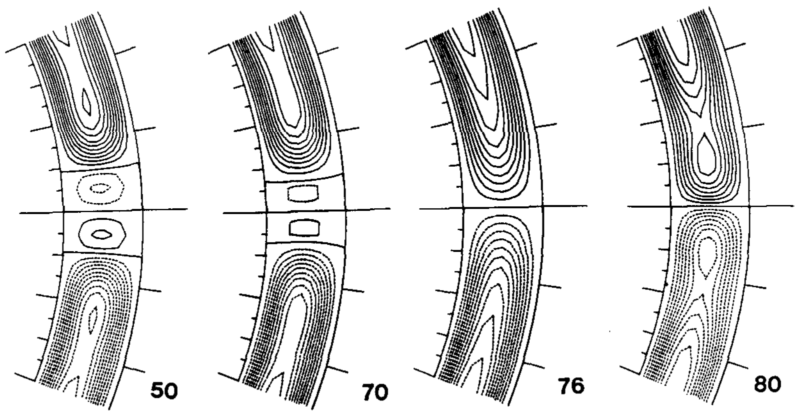
Results

We have numerically reproduced the experimentally observed states and the transitions between them for Re <1000. In particular, we find that the transition from the zero- to the one- vortex state takes place asymmetrically with respect to the equator, despite the equatorial symmetry of the initial and final states. This is why the transition was not seen in previous inital value simulations, which imposed equatorial symmetry in addition to axisymmetry. The Reynolds number Re = 652 at which the transition occurs agrees exactly with that found experimentally by Wimmer.

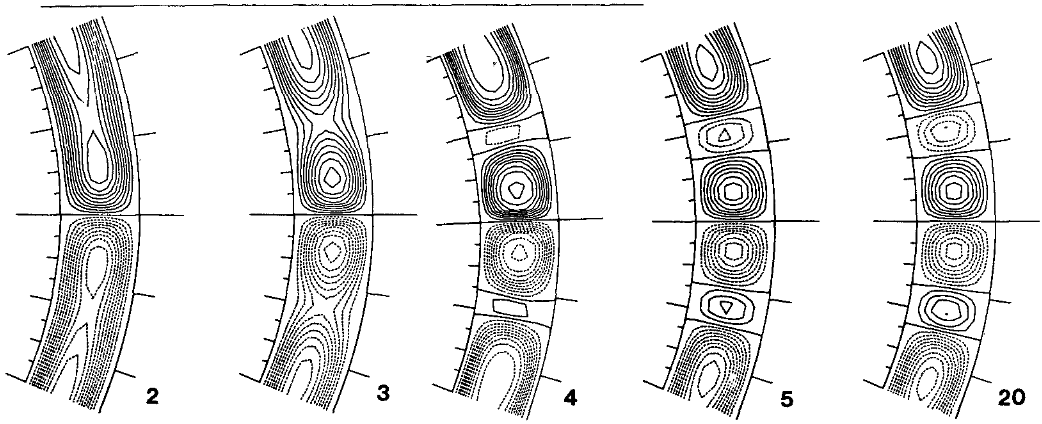
The reverse transition, from the one- to the zero- vortex state, when Re is decreased, takes place symmetrically with respect to the equator, as does the transition from the zero- to the two- vortex state. We observe another asymmetric transition, not in the previous published literature, from the two- to the one- vortex state. Bühler (private communication) has confirmed experimentally the qualitative form of all of these transitions.



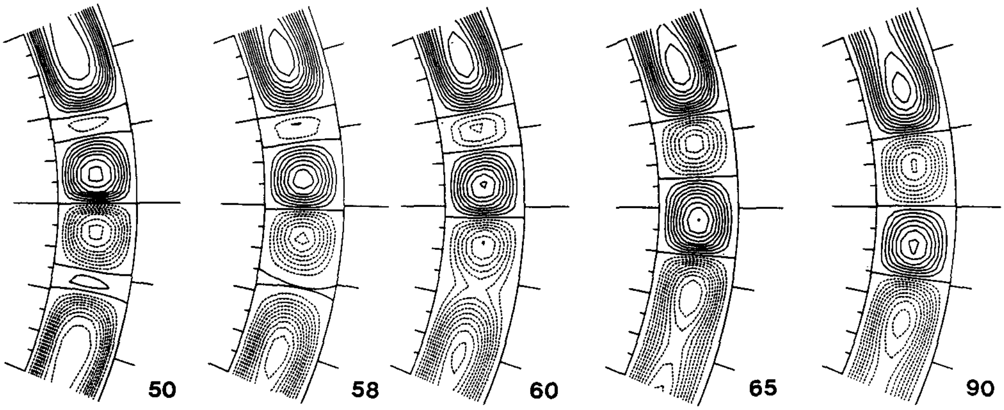
Zero- to one- vortex transition. Pictured are meridional streamlines in the equatorial region. Solid and dashed streamlines represent counter-clockwise and clockwise circulation, respectively. Time is shown in inner sphere revolutions. Note the breaking of equatorial symmetry. Re = 700.



One- to zero- vortex transition at Re = 645.



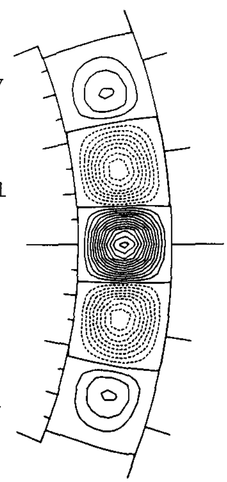
Zero- to two-vortex transition at Re = 800.



Two- to one- vortex transition at Re = 750.

Schrauf (1983)'s steady state calculation has revealed that the one-vortex states lie on a separate solution branch which never intersects the branch containing zero-vortex states. Schrauf's study and ours discovered that the zero- and the two- vortex states lie on the same solution branch, called the primary branch. That is, the zero-vortex states evolve continuously into the two-vortex states as the Reynolds number is increased, the demarkation between the two occurring at Re = 740. This branch structure is similar to that predicted by Benjamin (1978) for Taylor-Couette flow between cylinders of finite length.

By calculating eigenvectors and eigenvalues, we find that an interval of the primary branch is linearly unstable to an equatorially antisymmetric eigenvector. This instability initiates the transition to the one-vortex state ; therefore, we call the unstable interval 651 < Re < 775 a "window" from the primary branch to the one-vortex branch. The window contains both zero- and two- vortex states, and the two-vortex states at Re $\geq$ 775 are stable. This explains the non-uniqueness seen experimentally by Wimmer : in accelerating the system to its final angular velocity, a one-vortex state will be generated if the time spent in the window is sufficient for the antisymmetric instability to attain the threshold level necessary for transition. Otherwise a two-vortex state will be generated.



Antisymmetric eigenvector at Re = 700.

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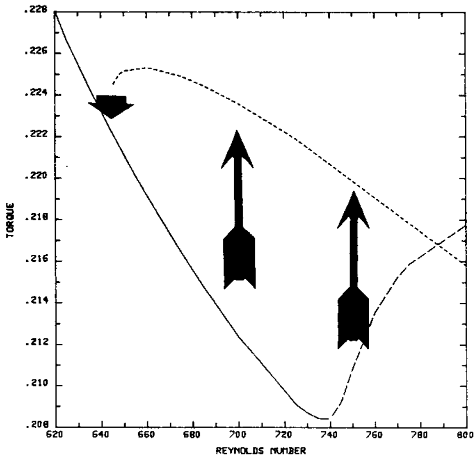

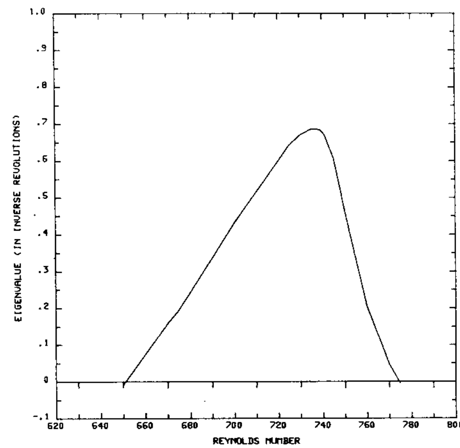

Left : Torque vs.Re of steady states in region of window. Solid, short-dashed, and long-dashed curves represent zero-, one-, and two- vortex states, respectively. Arrows show schematically transitions between states. Note that the curves representing zero- and two- vortex states join continuously, but that the one-vortex states are on an unconnected curve. ("Intersection" at Re ≃ 790 is a projection effect).

Right : Growth rate vs.Re of antisymmetric eigenvector to which the primary branch is unstable.

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