An informal derivation of the Stuart-Landau equation

Laurette S. Tuckerman

http://www.pmmh.espci.fr/ laurette

We consider a bifurcation in a system which is homogeneous in x from an x-independent state to a periodic state with x-wavenumber k. The governing equation is

$$\partial_t u(x,t) = \mathcal{L}u + \mathcal{N}(u,u) \tag{1}$$

where $\mathcal{L}$ is linear and $\mathcal{N}$ is bilinear. We decompose u into a Fourier series in x:

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t)e^{ikx} \quad \text{where} \quad u_{-k} = u_k^* \tag{2}$$

Because $\mathcal{L}$ is homogeneous in x, it does not couple the different Fourier components; it acts on $u(x,t)$ by multiplying each Fourier component u_k by coefficient λ_k. For example, if $\mathcal{L}$ is $\mu - \partial_x^2$, then $\lambda_k = \mu - k^2$.

$$\mathcal{L}u(x,t) = \sum_{k=-\infty}^{\infty} \lambda_k u_k(t)e^{ikx} \tag{3}$$

Similarly, the bilinear operator $\mathcal{N}$ also acts on $u(x,t)$ via coefficients $\tilde{\nu}_{l,m}$:

$$\begin{aligned}
\mathcal{N}\left(\sum_{l=-\infty}^{\infty} u_l(t)e^{ilx}, \sum_{m=-\infty}^{\infty} u_m(t)e^{imx}\right) &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{ilx}e^{imx}u_l(t)u_m(t)\,\tilde{\nu}_{l,m} \\
&= \sum_{k=-\infty}^{\infty} e^{ikx}\left(\sum_{m=-\infty}^{\infty} u_m(t)u_{k-m}(t)\,\tilde{\nu}_{m,k-m}\right) \\
&= \sum_{k=-\infty}^{\infty} e^{ikx}\left(\sum_{m\geq k/2} u_m(t)u_{k-m}(t)\,\nu_{m,k-m}\right) \qquad (4)
\end{aligned}$$

where

$$\nu_{m,k-m} = \begin{cases} \tilde{\nu}_{m,k-m} + \tilde{\nu}_{k-m,m} & \text{if} \quad m > k/2 \\ \tilde{\nu}_{k/2,k/2} & \text{if} \quad m = k/2, (k \text{ even}) \end{cases} \tag{5}$$

We separate the terms in e^{ikx}

$$\partial_t u_k(t) = u_k(t)\lambda_k + \sum_{m\geq k/2} u_m(t)u_{k-m}(t)\nu_{m,k-m} \tag{6}$$

and write out the first few equations explicitly

$$\partial_t u_0 = \lambda_0 u_0 + \nu_{1,-1}\, u_1 u_{-1} + \nu_{2,-2}\, u_2 u_{-2} \ldots \tag{7a}$$

$$\partial_t u_1 = \lambda_1 u_1 + \nu_{1,0}\, u_1 u_0 + \nu_{2,-1}\, u_2 u_{-1} \ldots \tag{7b}$$

$$\partial_t u_2 = \lambda_2 u_2 + \nu_{1,1}\, u_1 u_1 + \nu_{2,0}\, u_2 u_0 \ldots \tag{7c}$$

We assume

$$\lambda_1 > 0 > \lambda_0, \lambda_2 > \lambda_3 \cdots \qquad \text{and} \qquad |u_{\pm 1}| > u_0, |u_{\pm 2}| > |u_{\pm 3}| \cdots \tag{8}$$

u_0, $u_{\pm 2}$ are *slaved* to u_1; they are damped, but fed by u_1. Their eigenvalues are sufficiently negative that the amplitudes of these eigenmodes quickly equilibrate and obey algebraic equations, which describe the center/slow manifold on whch u_1 evolves:

$$0 = \lambda_0 u_0 + \nu_{1,-1}|u_1|^2 + \nu_{2,-2}|u_2|^2 \ldots \tag{9a}$$

$$0 = \lambda_2 u_2 + \nu_{1,1} u_1^2 + \nu_{2,0} u_2 u_0 \ldots \tag{9b}$$

We retain only the largest nonlinear terms. Since $u_0, |u_{\pm 2}| \ll |u_{\pm 1}|$, we will write

$$u_0 = \frac{\nu_{1,-1}|u_1|^2}{-\lambda_0} \qquad u_2 = \frac{\nu_{1,1} u_1^2}{-\lambda_2} \tag{10}$$

We then substitute these expressions in the evolution equation (7b) for u_1

$$\begin{aligned} \partial_t u_1 &= \lambda_1 u_1 + \nu_{1,0}\, u_1 u_0 + \nu_{2,-1}\, u_2 u_{-1} \\ &= \lambda_1 u_1 + \nu_{1,0}\, u_1 \left(\frac{\nu_{1,-1}|u_1|^2}{-\lambda_0} \right) + \nu_{2,-1}\, u_1^* \left(\frac{\nu_{1,1} u_1^2}{-\lambda_2} \right) \end{aligned} \tag{11}$$

to obtain the Stuart-Landau equation:

$$\boxed{\partial_t u_1 = \lambda_1 u_1 + |u_1|^2 u_1 \left[\left(\frac{\nu_{1,0}\nu_{1,-1}}{-\lambda_0} \right) + \left(\frac{\nu_{2,-1}\nu_{1,1}}{-\lambda_2} \right) \right]} \tag{12}$$

Multidimensional system

What if our problem is multidimensional, e.g. u depends on a non-homogeneous direction y? We decompose u into a Fourier series in x:

$$u(x,y,t) = \sum_{k=-\infty}^{\infty} \tilde{u}_k(y,t) e^{ikx} \quad \text{where} \quad \tilde{u}_{-k} = \tilde{u}_k^* \tag{13}$$

Since the linear operator $\mathcal{L}$ is homogeneous in x, it acts on $u(x,y,t)$ by acting on each $\tilde{u}_k(y,t)$ with an operator $\mathcal{L}_k$. For example, if $\mathcal{L}$ is $\mu - \nabla^2$, then $\mathcal{L}_k = \mu - k^2 + \partial_y^2$.

$$\mathcal{L}u(x,y,t) = \sum_{k=-\infty}^{\infty} \mathcal{L}_k \tilde{u}_k(y,t) e^{ikx} \tag{14}$$

We diagonalize $\mathcal{L}_k$ and we assume that all eigenvalues are real. Let λ_k and $\psi_k(y)$ be the leading eigenvalue and eigenvector of operator $\mathcal{L}_k$. We include in $u(x,y,t)$ only these leading modes, assuming that the other (non-leading) eigenvectors of $\mathcal{L}_k$ are rapidly damped. An example of a rapidly damped eigenmode in y would be in Rayleigh-Bénard convection, in which non-leading eigenmodes would contain two or more vertically superposed rolls rather than a single roll spanning the gap. Thus we assume that $\tilde{u}_k(y,t) = u_k(t)\psi_k(y)$ with $\mathcal{L}_k \psi_k = \lambda_k \psi_k$ and so

$$\mathcal{L}u(x,y,t) = \sum_{k=-\infty}^{\infty} u_k(t)\, \mathcal{L}_k\, \psi_k(y)\, e^{ikx} = \sum_{k=-\infty}^{\infty} u_k(t) \lambda_k\, \psi_k(y)\, e^{ikx} \tag{15}$$

Because the bilinear operator $\mathcal{N}$ is homogeneous in x, it acts on

$$u(x,y,t) = \sum_{k=-\infty}^{\infty} u_k(t)\,\psi_k(y)\,e^{ikx} \tag{16}$$

via bilinear operators $\mathcal{N}_{l,m}$ in y:

$$\begin{aligned}\mathcal{N}\left(\sum_{l=-\infty}^{\infty} u_l(t)\psi_l(y)e^{ilx}, \sum_{m=-\infty}^{\infty} u_m(t)\psi_m(y)e^{imx}\right) &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{ilx}e^{imx}u_l u_m\,\mathcal{N}_{l,m}(\psi_l(y),\psi_m(y)) \\ &= \sum_{k=-\infty}^{\infty} e^{ikx}\left(\sum_{m=-\infty}^{\infty} u_m u_{k-m}\,\mathcal{N}_{m,k-m}(\psi_m,\psi_{k-m})\right) \end{aligned} \tag{17}$$

We separate the terms in e^{ikx}:

$$\partial_t u_k \psi_k(y) = \lambda_k\, u_k\, \psi_k(y) + \sum_{m=-\infty}^{\infty} u_m u_{k-m}\,\mathcal{N}_{m,k-m}(\psi_m,\psi_{k-m}) \tag{18}$$

Let ϕ_k be the leading left eigenvector of the operator $\mathcal{L}_k$ such that $\langle\phi_k,\psi_k\rangle = 1$. We take the inner product of (18) with ϕ_k:

$$\partial_t u_k = \lambda_k u_k + \sum_{m=-\infty}^{\infty} u_m u_{k-m}\,\langle\phi_k, \mathcal{N}_{m,k-m}(\psi_m,\psi_{k-m})\rangle \tag{19}$$

We define

$$\nu_{m,k-m} = \begin{cases} \langle\phi_k, \mathcal{N}_{m,k-m}(\psi_m,\psi_{k-m})\rangle + \langle\phi_k, \mathcal{N}_{k-m,m}(\psi_{k-m},\psi_m)\rangle & \text{if} \quad m > k/2 \\ \langle\phi_{k/2}, \mathcal{N}_{k/2,k/2}(\psi_{k/2},\psi_{k/2})\rangle & \text{if} \quad m = k/2, (k \text{ even}) \end{cases} \tag{20}$$

This yields the same equations as (6)

$$\partial_t u_k = u_k\lambda_k + \sum_{m\geq k/2} u_m u_{k-m}\nu_{m,k-m} \tag{21}$$

and so, using (7)-(11), we again obtain the Stuart-Landau equation:

$$\boxed{\partial_t u_1 = \lambda_1 u_1 + |u_1|^2 u_1\left[\left(\frac{\nu_{1,0}\nu_{1,-1}}{-\lambda_0}\right) + \left(\frac{\nu_{2,-1}\nu_{1,1}}{-\lambda_2}\right)\right]} \tag{22}$$