Cours : Dynamique Non-Linéaire

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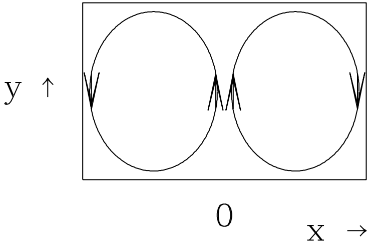
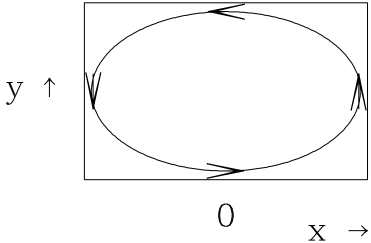
Symmetry

Reflection Symmetry





Symmetric 2D vector field

Antisymmetric 2D vector field

$$\kappa \begin{pmatrix} u \\ v \end{pmatrix} (x, y) \equiv \begin{pmatrix} -u \\ v \end{pmatrix} (-x, y)$$

Group table for $\{I, \kappa\}$:

	I	κ
I	I	κ
κ	κ	I

Other reflection operators

$$(\kappa f)(x) \equiv f(-x)$$

$$\kappa a \equiv -a$$

Evolution equation: $\dot{a} = g(a)$

System has reflection symmetry $\Longleftrightarrow$ g **is** *equivariant* $\Longleftrightarrow$ $g\kappa = \kappa g$

$$(g\kappa)(a) = (\kappa g)(a) \Longleftrightarrow g(-a) = -g(a) \Longleftrightarrow g \text{ odd}$$

$$\Longrightarrow \dot{a} = g_1 a + g_3 a^3 = (g_1 + g_3 a^2) a$$

to cubic order $\Longrightarrow$ **pitchfork bif**

$$\kappa \begin{pmatrix} a \\ s \end{pmatrix} \equiv \begin{pmatrix} -a \\ s \end{pmatrix} \quad \Longrightarrow \quad \kappa \begin{pmatrix} 0 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix} \quad \textbf{and} \quad \kappa \begin{pmatrix} a \\ 0 \end{pmatrix} = - \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} a \\ s \end{pmatrix} = G(a, s) \equiv \begin{pmatrix} g \\ h \end{pmatrix} (a, s)$$

$$\kappa \begin{pmatrix} g \\ h \end{pmatrix} (a, s) = \begin{pmatrix} g \\ h \end{pmatrix} \kappa(a, s)$$

$$\begin{pmatrix} -g \\ h \end{pmatrix} (a, s) = \begin{pmatrix} g \\ h \end{pmatrix} (-a, s)$$

$$\begin{aligned} \begin{pmatrix} g \\ h \end{pmatrix} (a, s) &= \begin{pmatrix} g_{10}a + g_{11}as + g_{30}a^3 + g_{12}as^2 \\ h_{00} + h_{01}s + h_{20}a^2 + h_{02}s^2 + h_{21}a^2s + h_{03}s^3 \end{pmatrix} \\ &= \begin{pmatrix} (g_{10} + g_{11}s + g_{12}s^2 + g_{30}a^2)a \\ h_{00} + h_{01}s + h_{02}s^2 + h_{03}s^3 + (h_{20} + h_{21}s)a^2 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{g}(a^2, s)a \\ \tilde{h}(a^2, s) \end{pmatrix} = \tilde{g}(a^2, s) \begin{pmatrix} a \\ 0 \end{pmatrix} + \tilde{h}(a^2, s) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

invariants: $\tilde{g} = \tilde{g}\kappa, \quad \tilde{h} = \tilde{h}\kappa$ **equivariant:** $(g, h)\kappa = \kappa(g, h)$

Solutions to κ-equivariant systems are not all symmetric. (That is why bifurcation theory is interesting.)

If (a, s) is a solution to a κ-equivariant system then $\kappa(a, s) \equiv (-a, s)$ is also a solution.

Asymmetric solutions (a, s), $a \neq 0$ come in pairs $(\pm a, s)$.

If a κ-equivariant system has a unique solution, then that solution is symmetric.

e.g. linear system, or Navier-Stokes equations at low Re

Linear system G which commutes with κ:

$$\begin{aligned} Gu &= \lambda u \\ \kappa G u &= \kappa \lambda u \\ G \kappa u &= \lambda \kappa u \end{aligned}$$

$(\lambda, \kappa u)$ is also an eigenpair of G. κu could be a multiple of u:

$$\begin{aligned} \kappa u &= c u \\ \kappa^2 u &= \kappa c u \\ u &= c^2 u \\ c &= \begin{cases} 1 & \Longrightarrow \ u \text{ is symmetric} \\ -1 & \Longrightarrow \ u \text{ is antisymmetric} \end{cases} \end{aligned}$$

Or else if u and κu are linearly independent, then:

$$\begin{aligned} u + \kappa u & \quad \text{is symmetric eigenvector} \\ u - \kappa u & \quad \text{is antisymmetric eigenvector} \end{aligned}$$

Verify equivariance of Navier-Stokes equations

$$(\kappa f)(x) \equiv f(-x) \Longrightarrow \begin{cases} (\kappa f)'(x) = -f'(-x) \\ (\kappa f)''(x) = f''(-x) \end{cases}$$

Demonstration:

Let $\tilde{f}(x) \equiv (\kappa f)(x) \equiv f(-x)$

$$\begin{aligned} \tilde{f}'(x) &\equiv \lim_{\Delta x \to 0} \frac{\tilde{f}(x+\Delta x) - \tilde{f}(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(-x-\Delta x) - f(-x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(-x+\Delta x) - f(-x)}{-\Delta x} = -f'(-x) \end{aligned}$$

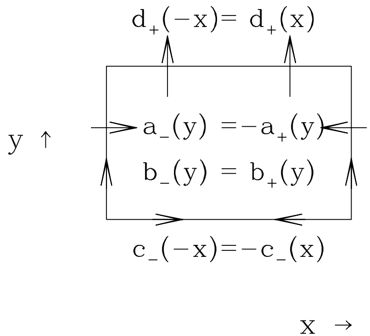
$$\left[(NS)\begin{pmatrix} u \\ v \end{pmatrix}\right](x,y) \equiv \begin{pmatrix} -(u\partial_x + v\partial_y)u - \partial_x p + \nu(\partial_x^2 + \partial_y^2)u \\ -(u\partial_x + v\partial_y)v - \partial_y p + \nu(\partial_x^2 + \partial_y^2)v \end{pmatrix}(x,y)$$

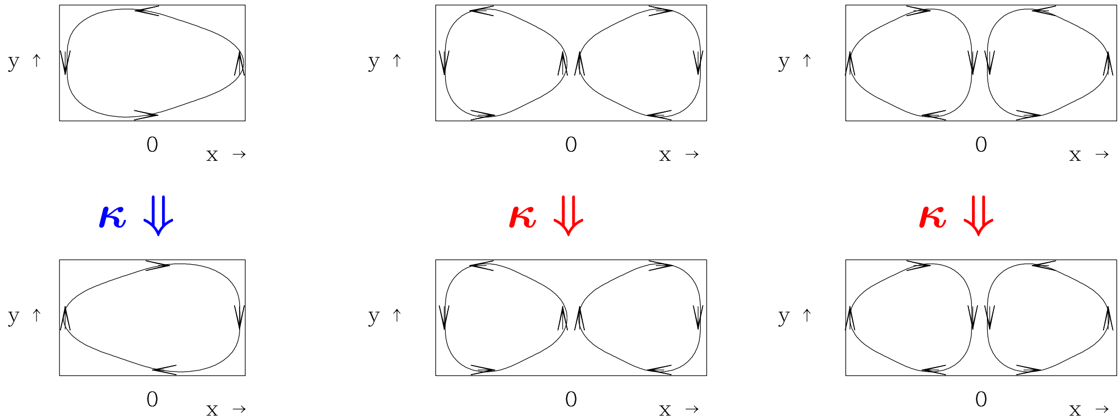
$$\left[\kappa\begin{pmatrix} u \\ v \end{pmatrix}\right](x,y) \equiv \begin{pmatrix} -u \\ v \end{pmatrix}(-x,y)$$

$$\left[(NS)\kappa\begin{pmatrix} u \\ v \end{pmatrix}\right](x,y) \stackrel{?}{=} \left[\kappa(NS)\begin{pmatrix} u \\ v \end{pmatrix}\right](x,y)$$

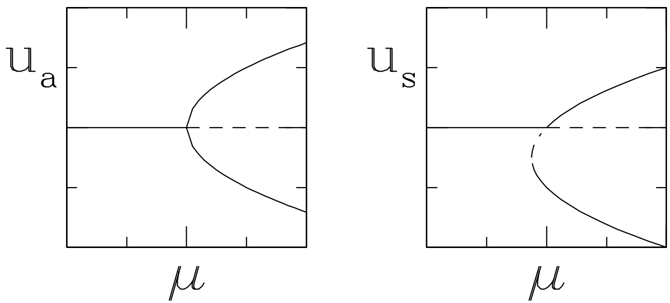
$$\begin{pmatrix} -[-(u\partial_x + v\partial_y)u - \partial_x p + \nu(\partial_x^2 + \partial_y^2)u] \\ -(u\partial_x + v\partial_y)v - \partial_y p + \nu(\partial_x^2 + \partial_y^2)v \end{pmatrix}(-x,y)$$

Boundary conditions & external forces determine if problem is κ-equivariant:

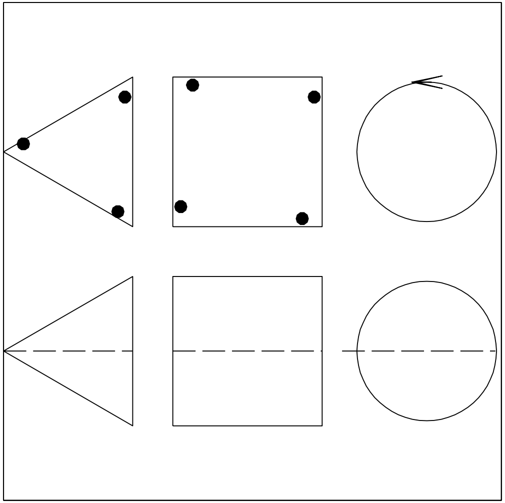




Not equivalent flows



Rotations and reflections of the plane



Z_2	Z_3	Z_4	$\ldots$	$SO(2)$
D_2	D_3	D_4	$\ldots$	$O(2)$

Natural representation of $O(2)$ on (x, y) uses $z = x + iy$

$$\begin{aligned} S_\theta z &\equiv e^{i\theta} z \\ \kappa z &\equiv \bar{z} \end{aligned}$$

$$\begin{aligned} \kappa S_\theta z &= \kappa(e^{i\theta} z) = e^{-i\theta} \bar{z} \\ S_\theta \kappa z &= S_\theta \bar{z} = e^{i\theta} \bar{z} \\ \kappa S_\theta z &= S_{-\theta} \kappa z \end{aligned}$$

$$\begin{aligned} (S_{\theta_0} w)(\rho, \theta) &\equiv w(\rho, \theta + \theta_0) \\ (\kappa w)(\rho, \theta) &\equiv w(\rho, -\theta) \end{aligned}$$

Circle Pitchfork

$$f(z,\bar z) = \sum_{m,n} f_{mn} z^m \bar z^n$$

$$\kappa f(z,\bar z) = \overline{f(z,\bar z)} = \bar f_{mn} \bar z^m z^n \qquad f(\kappa(z,\bar z)) = f_{mn} \bar z^m z^n$$

f **real**

$$S_\theta f(z,\bar z) = e^{i\theta} f(z,\bar z) \qquad f(S_\theta(z,\bar z))$$

$$= f_{mn}(e^{i\theta} z)^m \overline{(e^{i\theta} z)}^n$$

$$= e^{i\theta} f_{mn} z^m \bar z^n \qquad = f_{mn} e^{im\theta} z^m e^{-in\theta} \bar z^n$$

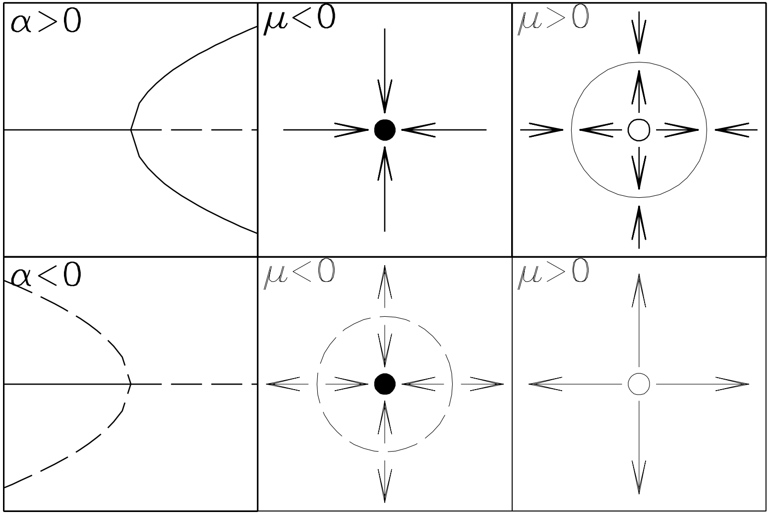
$$m - n = 1 \;\; \textbf{or} \;\; f_{mn} = 0$$

$$\begin{aligned} f(z,\bar z) &= f_{10} z + f_{21} z^2 \bar z + f_{32} z^3 \bar z^2 + \cdots \\ &= (f_{10} + f_{21}|z|^2 + f_{32}|z|^4 + \cdots) z \\ &= \tilde f(|z|^2) z \end{aligned}$$

$|z|^2$ **invariant** $\qquad$ z **equivariant**

Circle Pitchfork

$$\dot{z} = (\mu - \alpha|z|^2)z$$



Circle Pitchfork

$$\dot{z} = (\mu - |z|^2)z$$

Cartesian Form:

$$\begin{aligned}
\dot{x} + i\dot{y} &= \mu(x + iy) - (x^2 + y^2)(x + iy) \\
\dot{x} &= \mu x - (x^2 + y^2)x \\
\dot{y} &= \mu y - (x^2 + y^2)y
\end{aligned}$$

Polar Form:

$$\begin{aligned}
(\dot{r} + ir\dot{\theta})e^{i\theta} &= (\mu - r^2)re^{i\theta} \\
\dot{r} &= \mu r - r^3 \\
\dot{\theta} &= 0
\end{aligned}$$

Subcritical Form:

$$\dot{z} = (\mu + |z|^2)z$$

Stability of origin (use Cartesian coordinates):

$$J(x=0,y=0) = \begin{pmatrix} \mu - (3x^2+y^2) & -2xy \\ -2xy & \mu - (x^2+3y^2) \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

double eigenvalue μ.

Stability of states on circle (use polar coordinates):

$$J(r=\sqrt{\mu},\theta) = \begin{pmatrix} \mu - 3r^2 & 0 \\ 0 & 0 \end{pmatrix}\Bigg|_{\sqrt{\mu},\theta} = \begin{pmatrix} -2\mu & 0 \\ 0 & 0 \end{pmatrix}$$

eigenvalue -2μ along r and marginal eigenvalue 0 along θ.

$$\frac{dw}{dt}(\theta) = \mathcal{F}(w)(\theta)$$

$\mathcal{F}$ indep of θ

$$0 = \mathcal{F}(W)$$

W a steady solution on circle

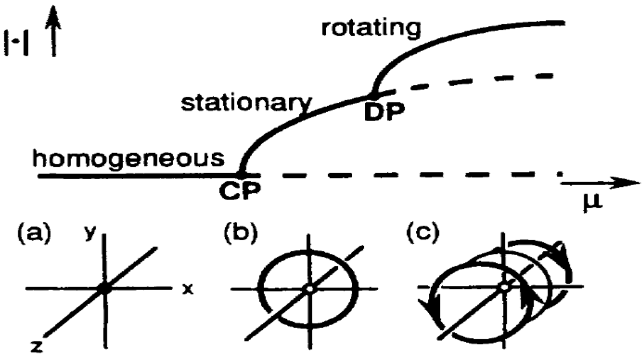
$$0 = \frac{d\mathcal{F}(W)}{d\theta} = \frac{\delta\mathcal{F}}{\delta W}\frac{\partial W}{\partial\theta}$$

Jacobian or Frechet derivative

marginal eigenvector

Drift Pitchfork

$$\dot{r} = (\mu - r^2)r$$
$$\dot{\theta} = \zeta$$
$$\dot{\zeta} = (r^2 - 1 - \zeta^2)\zeta$$



$\mu < 0$ $\quad 0 \leq \mu \leq 1 \quad 1 < \mu$

$r = 0$	$r = \sqrt{\mu}$	$r = \sqrt{\mu}$
	$\theta = \theta_0$	$\theta = \theta_0 + \zeta t$
$\zeta = 0$	$\zeta = 0$	$\zeta = \pm\sqrt{\mu - 1}$

Drift Pitchfork

Speed at onset is slow: $\zeta = \sqrt{\mu - \mu_{DP}} = \sqrt{\mu - 1}$

Motion along the circle: group orbit

Symmetry-breaking variable: ζ

Function $W(\theta)$ is even about some θ_0.

Marginal and bifurcating eigenvectors are odd about θ_0

Drifting W is asymmetric in θ.

Jacobian:

$$\begin{bmatrix} \frac{\partial \dot{r}}{\partial r} & \frac{\partial \dot{r}}{\partial \zeta} & \frac{\partial \dot{r}}{\partial \theta} \\ \frac{\partial \dot{\zeta}}{\partial r} & \frac{\partial \dot{\zeta}}{\partial \zeta} & \frac{\partial \dot{\zeta}}{\partial \theta} \\ \frac{\partial \dot{\theta}}{\partial r} & \frac{\partial \dot{\theta}}{\partial \zeta} & \frac{\partial \dot{\theta}}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \mu - 3r^2 & 0 & 0 \\ 2r\zeta & r^2 - 1 - 3\zeta^2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Eigenvalues of diagonal matrix are along the diagonal.

Also true for upper or lower diagonal matrices! So eigenvalues are:

State	λ_1 (along r)	λ_2 (along ζ)	λ_3 (along θ)
General case	$\mu - 3r^2$	$r^2 - 1 - 3\zeta^2$	0
$r = \zeta = 0$	μ	-1	0
$r = \sqrt{\mu}, \zeta = 0$	-2μ	$\mu - 1$	0
$r = \sqrt{\mu}, \zeta = \pm\sqrt{\mu - 1}$	-2μ	$-2(\mu - 1)$	0

$$\mu = \mu_{\mathrm{CP}} = 0$$

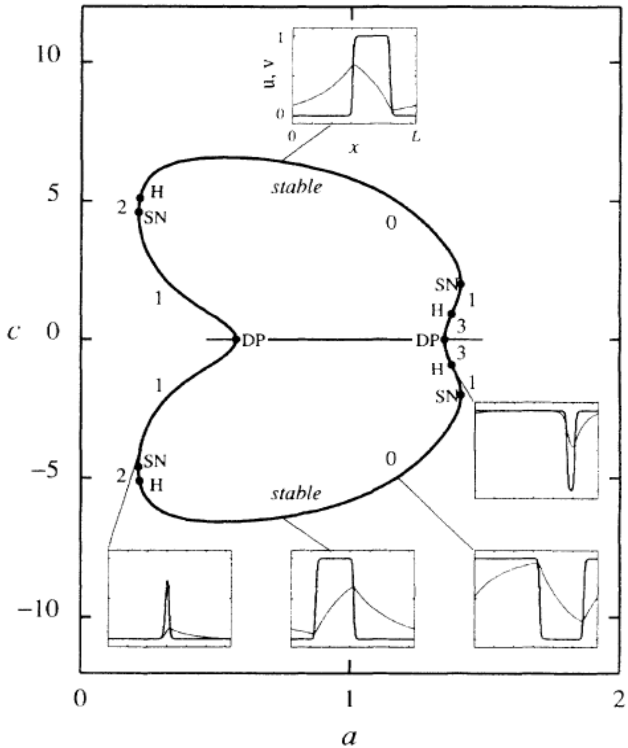
$$\mathcal{J} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mu = \mu_{\mathrm{DP}} = 1$$

$$\mathcal{J} = \left[\begin{array}{c|cc} -2 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

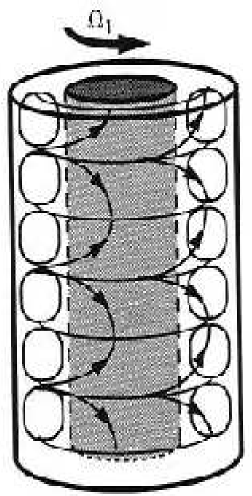
Jordan block

Drift pitchfork in reaction-diffusion equations

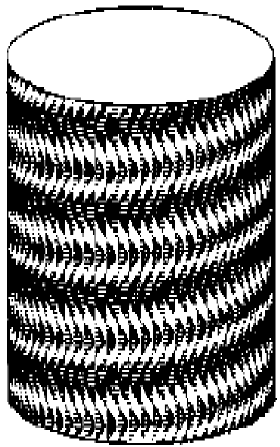


From Kness, Tuckerman & Barkley, Phys. Rev. A 46, 5054 (1992).

$O(2)$ and $SO(2)$



Axisymmetric Taylor vortices
Tagg, Nonlinear Science Today 4, 1 (1994).



Spiral Taylor vortices
Antonijoan et al., Phys. Fluids 10, 829 (1995).

No requirement of $\kappa f = f\kappa \Longrightarrow f$ complex

$$\dot{z} = (\mu + i\omega - (\alpha + i\beta)|z|^2)z$$

$$(\dot{r} + ri\dot{\theta})e^{i\theta} = ((\mu - \alpha r^2) + i(\omega + \beta r^2))\; r\; e^{i\theta}$$

$$\begin{aligned}\dot{r} &= (\mu - \alpha r^2)\; r \\ \dot{\theta} &= \omega + \beta r^2\end{aligned}$$

Breaking of $SO(2) \Longrightarrow$ motion along θ direction

Hopf bifurcation and $O(2)$ symmetry

Need four-dimensional eigenspace:

$$u(\theta, t) = (z_+(t) + z_-(t))e^{i\theta} + (\bar{z}_+(t) + \bar{z}_-(t))e^{-i\theta}$$

where $z_\pm$ are complex amplitudes (i.e. amplitude and phase) of left-going and right-going traveling waves.

At linear order:

$$\frac{d}{dt}\begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} i\omega z_+ \\ -i\omega z_- \end{pmatrix}$$

$$u(\theta, t) = z_+(0)e^{i(\theta+\omega t)} + z_-(0)e^{i(\theta-\omega t)} + \bar{z}_+(0)e^{-i(\theta+\omega t)} + \bar{z}_-(0)e^{-i(\theta-\omega t)}$$

$z_\pm(0)$ arbitrary initial amplitudes.

Addition of nonlinear terms compatible with $O(2)$ symmetry greatly restricts possible equilibria.

Appropriate representation of $O(2)$ on (z_+, z_-):

$$\begin{aligned} S_{\theta_0}(z_+, z_-) &= (e^{i\theta_0} z_+, e^{i\theta_0} z_-) \\ \kappa(z_+, z_-) &= (\bar{z}_-, \bar{z}_+) \end{aligned}$$

Simplest cubic order equivariant evolution equations:

$$\frac{d}{dt}\begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} \left[\mu + i\omega + a|z_-|^2 + b(|z_+|^2 + |z_-|^2)\right] z_+ \\ \left[\mu - i\omega + \bar{a}|z_+|^2 + \bar{b}(|z_+|^2 + |z_-|^2)\right] z_- \end{pmatrix}$$

Substitute $z_\pm = r_\pm e^{i\phi_\pm}$:

$$\begin{aligned} \dot{r}_\pm &= (\mu + a_r r_\mp^2 + b_r(r_+^2 + r_-^2)) r_\pm \\ \dot{\phi}_\pm &= \pm(\omega + a_i r_\mp^2 + b_i(r_+^2 + r_-^2)) \end{aligned}$$

Equations do not involve phases $\phi_\pm$ (all phases equivalent).

Solutions with $\dot{r}_\pm = 0$:

$$\text{origin}: \quad r_+ = 0,\ r_- = 0$$

$$\text{left traveling waves}: \quad r_+ = \sqrt{\frac{-\mu}{b_r}},\ r_- = 0,$$

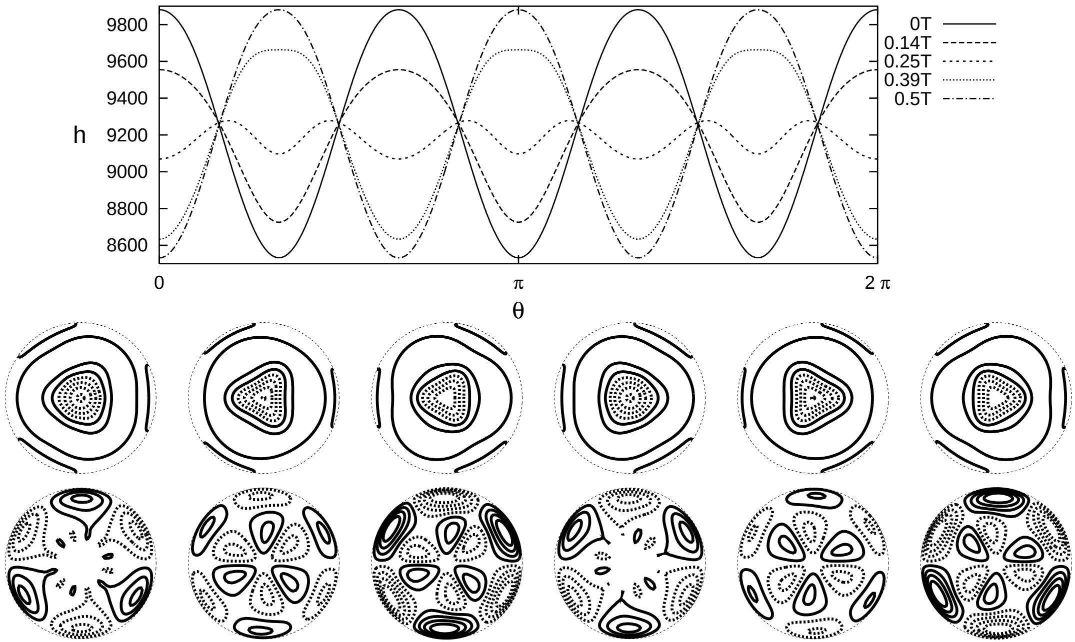
$$\dot{\phi}_+ = \omega - \mu\frac{b_i}{b_r}$$

$$\text{right traveling waves}: \quad r_+ = 0,\ r_- = \sqrt{\frac{-\mu}{b_r}},$$

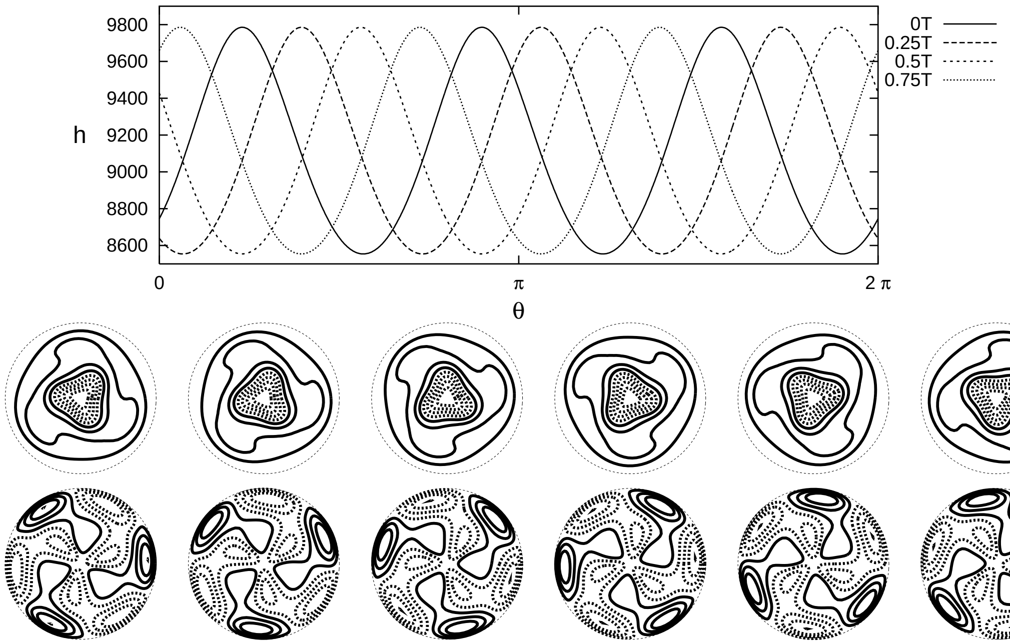
$$\dot{\phi}_- = -\left(\omega - \mu\frac{b_i}{b_r}\right)$$

$$\text{standing waves}: \quad r_+ = r_- = \sqrt{\frac{-2\mu}{a_r + 2b_r}},$$

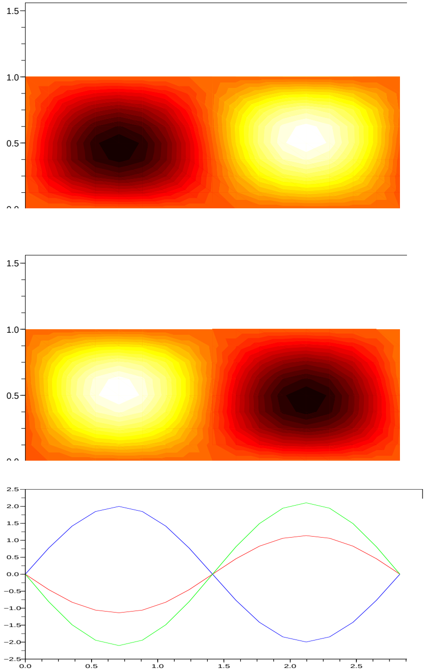
$$\dot{\phi}_\pm = \pm\left(\omega - 2\mu\frac{a_i + 2b_i}{a_r + 2b_r}\right)$$



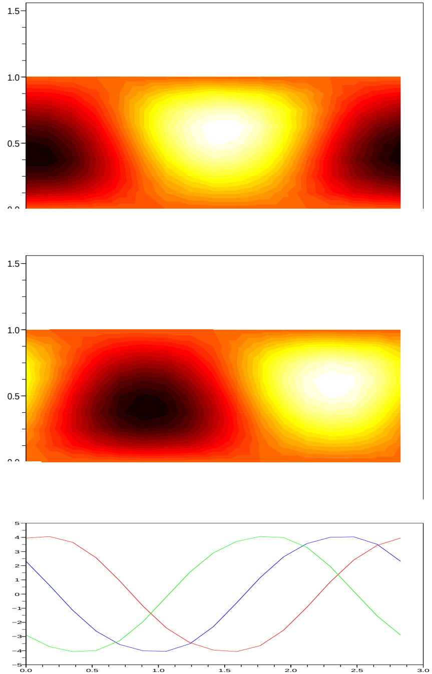
Standing waves in Rayleigh-Bénard convection in a cylinder with $\Gamma = R/H = 1.47$ and $Pr = 1$ at at $Ra = 26\,000$. From Borońska & Tuckerman, J. Fluid Mech. 559, 279 (2006).



Travelling wave in Rayleigh-Bénard convection in a cylinder with $\Gamma = R/H = 1.47$ and $Pr = 1$ at $Ra = 26\,000$. From Borońska & Tuckerman, J. Fluid Mech. 559, 279 (2006).



Standing waves



Travelling waves

Thermosolutal convection with $S = -0.1$, $L = 0.1$, $Pr = 10$, $r \equiv Ra/Ra_c = 1.3$.

Standing vs. Travelling Waves

Jacobian in $(r_+, r_-, \phi_+, \phi_-)$ coordinates:

$$\begin{pmatrix} \mu + a_r r_-^2 + b_r(r_+^2 + r_-^2) + 2b_r r_+^2 & 2(a_r + b_r) r_- r_+ & 0 & 0 \\ 2(a_r + b_r) r_- r_+ & \mu + a_r r_+^2 + b_r(r_+^2 + r_-^2) + 2b_r r_-^2 & 0 & 0 \\ 2b_i r_+ & 2(a_i + b_i) r_- & 0 & 0 \\ -2(a_i + b_i) r_+ & -2b_i r_+ & 0 & 0 \end{pmatrix}$$

Block lower-triangular:

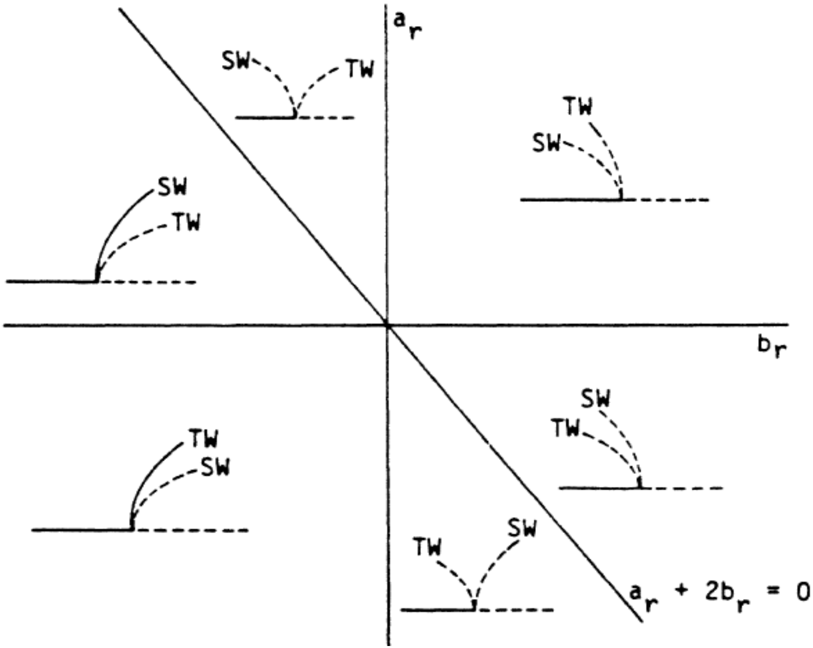
$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\left. \begin{array}{r} AX = \lambda X \\ CX + DY = \lambda Y \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} X = 0 \\ (\lambda, Y) \text{ eig of } D \end{array} \right\} \text{ or } \left\{ \begin{array}{l} (\lambda, X) \text{ eig of } A \\ Y = (\lambda I - D)^{-1} C X \end{array} \right\}$$

Directions $\phi_\pm$ are neutral. For eigs in $r_\pm$ directions, use

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \lambda_\pm = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$$

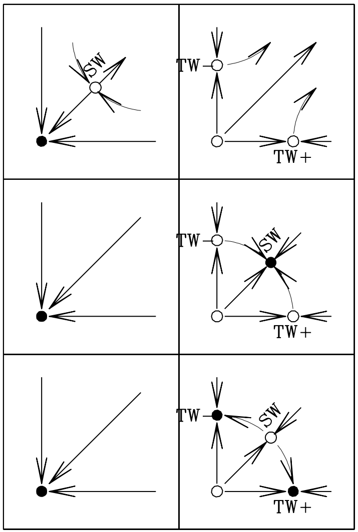
Standing vs. Travelling Waves



Stability and branching direction of standing waves (SW) and travelling waves (TW) in (a_r, b_r) parameter plane. Either standing or traveling waves are stable, or neither are stable, depending on nonlinear coefficients a, b. *(Knobloch, Phys. Rev. A* **34***, 1538 (1986).)*

Standing vs. Travelling Waves

origin :	μ along r_+	μ along r_-
TW^+ :	-2μ along r_+	$-a_r\mu/b_r$ along r_-
TW^- :	-2μ along r_-	$-a_r\mu/b_r$ along r_+
SW :	-2μ along (r_+, r_-)	$2a_r\mu/(a_r + 2b_r)$ perp to (r_+, r_-)



Symmetries of Standing and Travelling Waves

Time translation: $T_{t_0} u(t) \equiv u(t + t_0)$
Group of all time translations: S^1
Symmetry group of homogeneous stationary state: $O(2) \times S^1$.

Travelling wave symmetries:

$$(T_{t_0} S_{\omega t_0} u)(\theta, t) \equiv u(\theta + \omega t_0, t + t_0) = u(\theta, t)$$

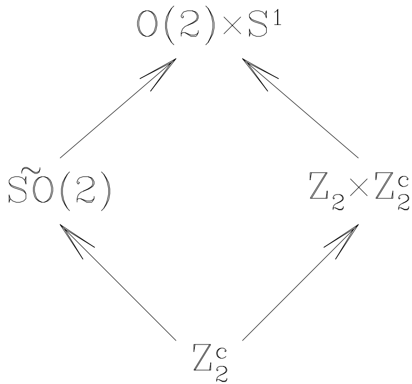
Group $\widetilde{SO}(2)$ **(isomorphic to** $SO(2)$**)**

Standing wave symmetries (arbitrary symmetry axis at $\theta = 0$**)**

$$\begin{aligned} (\kappa u)(\theta, t) &\equiv u(-\theta, t) = u(\theta, t) && Z_2 \\ (T_{\pi/\omega} S_\pi u)(\theta, t) &\equiv u(\theta + \pi, t + \pi/\omega) = u(\theta, t) && Z_2^c \end{aligned}$$

Group $Z_2 \times Z_2^c$

Lattice of Isotropy Subgroups



Steady-state mode interactions

Bifurcations to two wavenumbers, m and n.

$$w(\rho,\theta) = \frac{1}{2}\left(z_m(\rho)e^{im\theta} + z_n(\rho)e^{in\theta} + \bar{z}_m(\rho)e^{-im\theta} + \bar{z}_n(\rho)e^{-in\theta}\right)$$

$$\begin{aligned} S_{\theta_0}(z_m, z_n) &= (e^{im\theta_0}\, z_m, e^{in\theta_0}\, z_n) \\ \kappa(z_m, z_n) &= (\bar{z}_m, \bar{z}_n) \end{aligned}$$

$$f(z_m, z_n) = \begin{pmatrix} f_m \\ f_n \end{pmatrix}(z_m, z_n) = \begin{pmatrix} f_{mpqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \\ f_{npqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \end{pmatrix}$$

$\kappa f = f\kappa \Longrightarrow f$ **real.**

$$\begin{aligned} S_\theta f(z_m, z_n) &= \begin{pmatrix} e^{im\theta} f_{mpqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \\ e^{in\theta} f_{npqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \end{pmatrix} \\ f S_\theta(z_m, z_n) &= \begin{pmatrix} f_{mpqrs}(e^{im\theta} z_m)^p (e^{in\theta} z_n)^q (e^{-im\theta} \bar{z}_m)^r (e^{-in\theta} \bar{z}_n)^s \\ f_{npqrs}(e^{im\theta} z_m)^p (e^{in\theta} z_n)^q (e^{-im\theta} \bar{z}_m)^r (e^{-in\theta} \bar{z}_n)^s \end{pmatrix} \end{aligned}$$

$$S_\theta f = f S_\theta \Longrightarrow$$

$$f_{mpqrs} = 0 \ \text{ or } \ m = mp + nq - mr - ns$$
$$f_{npqrs} = 0 \ \text{ or } \ n = mp + nq - mr - ns$$

All invariants are products and sums of:

$$|z_m|^2, |z_n|^2, \ \text{ and } \ \Delta \equiv z_m^n \bar{z}_n^m + \bar{z}_m^n z_n^m$$

All equivariants are sums of:

$$\begin{pmatrix} z_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_n \end{pmatrix}, \begin{pmatrix} \bar{z}_m^{n-1} z_n^m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_m^n \bar{z}_n^{m-1} \end{pmatrix}$$

with coefficients which are invariants.
Most general equivariant evolution equation is:

$$\frac{d}{dt}\begin{pmatrix} z_m \\ z_n \end{pmatrix} = a\begin{pmatrix} z_m \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ z_n \end{pmatrix} + c\begin{pmatrix} \bar{z}_m^{n-1} z_n^m \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ z_m^n \bar{z}_n^{m-1} \end{pmatrix}$$

where a, b, c, d are functions of $(|z_m|^2, |z_n|^2, \Delta)$.

Assume $m+n-1>3$ and truncate to cubic order.
The most general set of equivariant equations is independent of m, n:

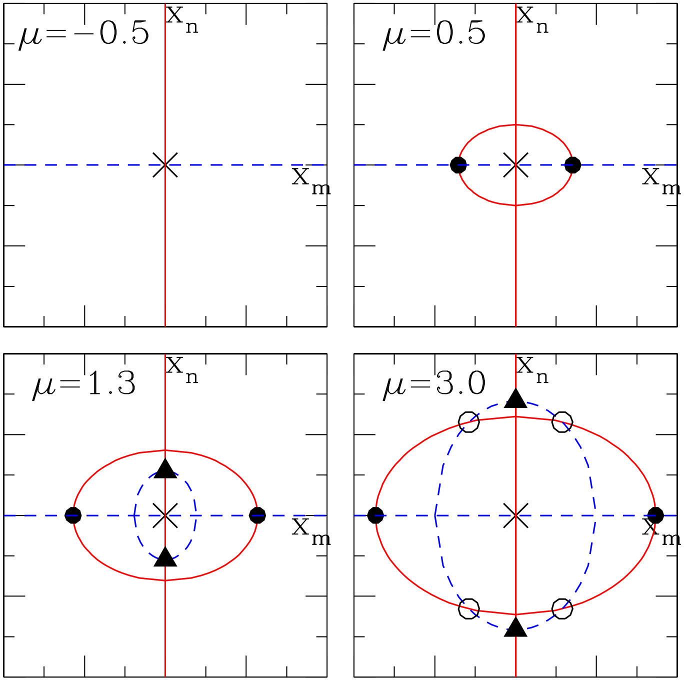
$$\dot{z}_m = (a_0 + a_m|z_m|^2 + a_n|z_n|^2)z_m$$
$$\dot{z}_n = (b_0 + b_m|z_m|^2 + b_n|z_n|^2)z_n$$

a, b **real** $\Longrightarrow$ **phases play no role** $\Longrightarrow$ **replace** (z_m, z_n) **by** (x_m, x_n).

Steady states:

$$x_m = 0 \quad \text{or} \quad a_0 + a_m x_m^2 + a_n x_n^2 = 0 \quad \text{and}$$
$$x_n = 0 \quad \text{or} \quad b_0 + b_m x_m^2 + b_n x_n^2 = 0$$

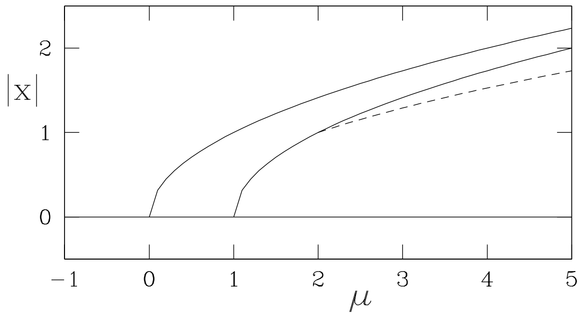
$$\text{origin}: \quad x_m = 0,\ x_n = 0$$
$$\text{pure } m \text{ modes}: \quad x_m^2 = -a_0/a_m,\ x_n = 0$$
$$\text{pure } n \text{ modes}: \quad x_m = 0,\ x_n^2 = -b_0/b_n$$
$$\text{mixed modes}: \quad x_m^2 = \frac{a_0 b_n - b_0 a_n}{b_m a_n - a_m b_n},$$
$$x_n^2 = \frac{a_0 b_m - b_0 a_m}{-(b_m a_n - a_m b_n)}$$



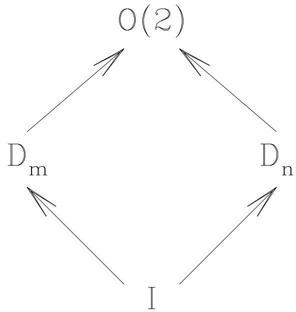
$$a_0 = \mu \qquad b_0 = \mu - 1 \qquad a_m = b_n = -1 \qquad a_n = b_m = -2$$

$$x_m = 0 \quad \text{or} \quad \mu - x_m^2 - 2x_n^2 = 0 \quad \text{and}$$

$$x_n = 0 \quad \text{or} \quad (\mu - 1) - 2x_m^2 - x_n^2 = 0$$



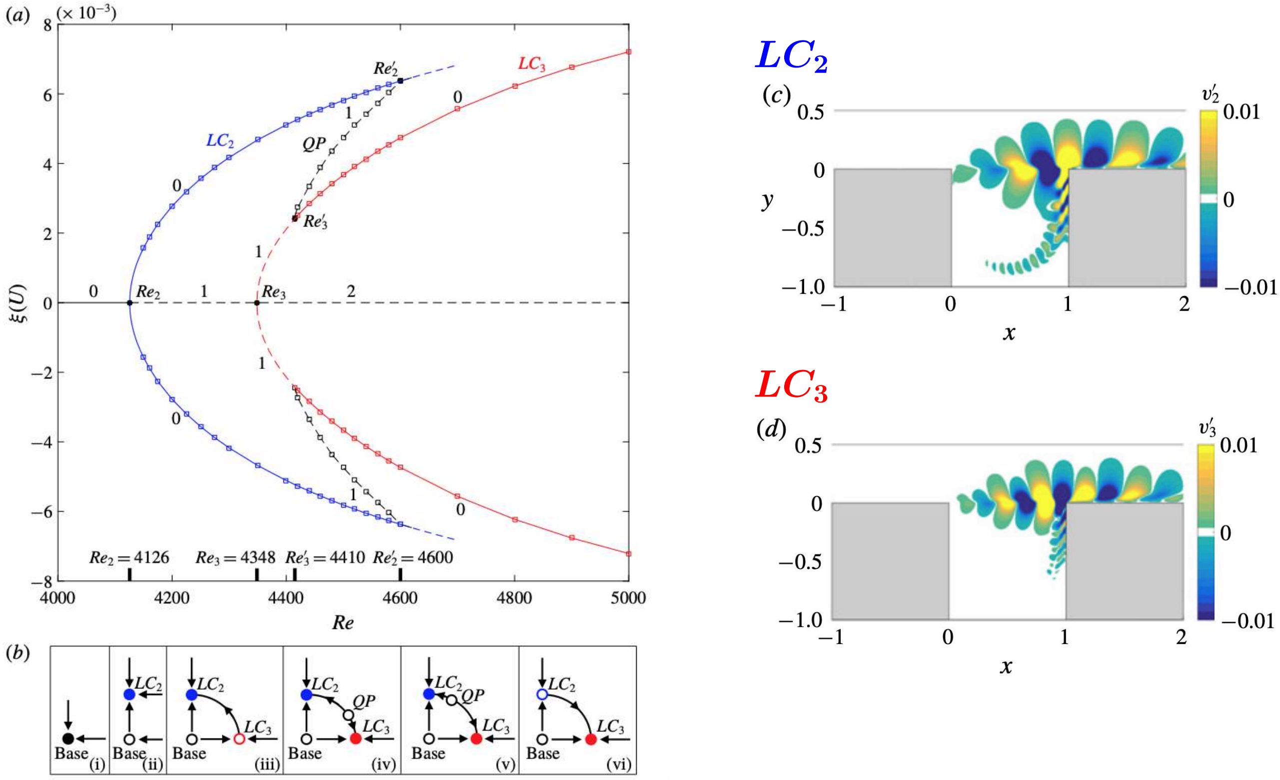
Bifurcation diagram for (x_m, x_n). $|x| \equiv \sqrt{(x_m^2 + x_n^2)}$ as a function of μ. Solid curves: pure modes. Dashed curve: mixed mode.



Lattice of isotropy subgroups for (m, n) mode interaction.

Example of shear-driven cavity flow

Y. Bengana, J.-Ch. Loiseau, J.-Ch. Robinet and L. S. Tuckerman



Case $(m, n) = (1, 2)$**:**

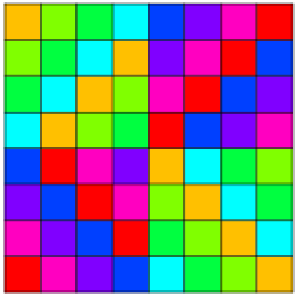
$$\begin{aligned}\dot{z}_1 &= c_0 \bar{z}_1 z_2 + (a_0 + a_1 |z_1|^2 + a_2 |z_2|^2) z_1 \\ \dot{z}_2 &= d_0 z_1^2 + (b_0 + b_1 |z_1|^2 + b_2 |z_2|^2) z_2\end{aligned}$$

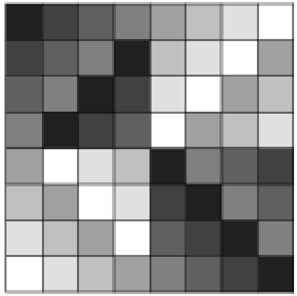
Case $(m, n) = (1, 3)$**:**

$$\begin{aligned}\dot{z}_1 &= c_0 \bar{z}_1^2 z_3 + (a_0 + a_1 |z_1|^2 + a_3 |z_3|^2) z_1 \\ \dot{z}_3 &= d_0 z_1^3 + (b_0 + b_1 |z_1|^2 + b_3 |z_3|^2) z_3\end{aligned}$$

$\Longrightarrow$ **Interesting dynamics like heteroclinic orbits!**

Group Table for D_4, symmetries of a square ($\rho \equiv S_{\pi/2}$)





From Mathworld, by Eric Weisstein, Wolfram Research.

e	e	ρ	ρ^2	ρ^3	κ	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$
e	e	ρ	ρ^2	ρ^3	κ	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$
ρ	ρ	ρ^2	ρ^3	e	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$	κ
ρ^2	ρ^2	ρ^3	e	ρ	$\kappa\rho^2$	$\kappa\rho^3$	κ	$\kappa\rho$
ρ^3	ρ^3	e	ρ	ρ^2	$\kappa\rho^3$	κ	$\kappa\rho$	$\kappa\rho^2$
κ	κ	$\kappa\rho^3$	$\kappa\rho^2$	$\kappa\rho$	e	ρ^3	ρ^2	ρ
$\kappa\rho$	$\kappa\rho$	κ	$\kappa\rho^3$	$\kappa\rho^2$	ρ	e	ρ^3	ρ^2
$\kappa\rho^2$	$\kappa\rho^2$	$\kappa\rho$	κ	$\kappa\rho^3$	ρ^2	ρ	e	ρ^3
$\kappa\rho^3$	$\kappa\rho^3$	$\kappa\rho^2$	$\kappa\rho$	κ	ρ^3	ρ^2	ρ	e

Quotient Groups

One one-element subgroup: $\{e\}$

Five two-element subgroups: $\{e, \rho^2\}$, $\{e, \kappa\}$, $\{e, \kappa\rho\}$, $\{e, \kappa\rho^2\}$, $\{e, \kappa\rho^3\}$

Two four-element subgroups: $\{e, \rho, \rho^2, \rho^3\}$, $\{e, \rho^2, \kappa, \kappa\rho^2\}$

$\{e, \rho, \rho^2, \rho^3\}$ **is isomorphic to** Z_4

$\{e, \rho^2, \kappa, \kappa\rho^2\}$ **is isomorphic to** $Z_2 \times Z_2$

Normal subgroup: $gng^{-1} \in N$ **for all elements** $g \in \Gamma, n \in N$

$N \equiv \{e, \rho, \rho^2, \rho^3\}$ **is normal subgroup of** $\Gamma \equiv D_4$

Can form quotient group Γ/N **isomorphic to** Z_2