Bifurcation Analysis of the Eckhaus Instability

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The Eckhaus instability [1] is the only purely two-dimensional instability occurring in convection [2], Taylor-Couette flow [3], liquid crystals [4], and other pattern-forming systems. Despite its widespread occurrence, the Eckhaus instability has not been analyzed in bifurcation-theoretic terms. It can be studied analytically by averaging over the "depth" [5], leading to the classic one-dimensional Ginzburg-Landau equation:

$$\frac{\partial A}{\partial t} = \mu A + \frac{\partial^2 A}{\partial x^2} - |A|^2 A \,. \tag{1}$$

where A describes a modulation of a roll structure w, i.e. $w = A(x,t)\, e^{i\chi_c x}$ + c.c. In order to perform a bifurcation-theoretic analysis, it is necessary to discretize the spectrum of (1). Imposing 2π–periodicity on w, we arrive at the boundary condition:

$$A(x+2\pi,\, t)\, e^{i\chi_c x} = A(x,t) \tag{2}$$

Here $\chi_c = q_c L/2\pi$, where q_c is the critical wavenumber and L is the length of the container. All variables have been scaled to $L/2\pi$, so that in convection, for instance, $\mu \sim (L/2\pi)^2\, (Ra - Ra_c)\, /Ra_c$.

As is well known, the trivial solution $A = 0$ loses stability at values $\mu = Q^2$ via supercritical pitchfork bifurcations to pure mode states $A = \sqrt{\mu - Q^2}\, e^{iQx}$. (Although the O(2) symmetry of the problem actually leads to "circles" of solutions parametrized by phase ϕ, we can restrict the analysis to $\phi = 0$.) Our analysis differs from the classic one in that our boundary conditions (2) lead to the requirement that $\chi_c + Q$ be integer. In the generic case, where χ_c is neither integer nor half-integer, the allowed wavenumbers Q_n can be listed in order of appearance of the associated branches A_n. For example, if $\chi_c = N - 1/4$, N integer,

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we have the allowed wavenumbers $Q_0 = 1/4$, $Q_1 = -3/4$, $Q_2 = 5/4$, $Q_3 = -7/4$, etc. For μ in the range $Q_n^2 < \mu < Q_{n+1}^2$, the trivial state has n positive eigenvalues, or unstable directions: we say that it has instability index n, as in Fig. 1.

In [6], we study the stability of the pure mode states by linearizing (1) about A_n. It can be verified that the eigenvalues are:

$$\sigma_{n0}(\mu) = -2(\mu - Q_n{}^2)$$
$$\sigma_{nk\pm}(\mu) = -(k^2 + \mu - Q_n{}^2) \pm \sqrt{(\mu - Q_n{}^2)^2 + (2kQ_n)^2}.$$

Here, boundary condition (2) constrains k to be a (positive) integer. The only possibility for instability resides with σ_{nk+}, which is positive for $\mu < \mu_{nk}$, where:

$$\mu_{nk} \equiv 3Q_n^2 - \frac{1}{2}k^2.$$

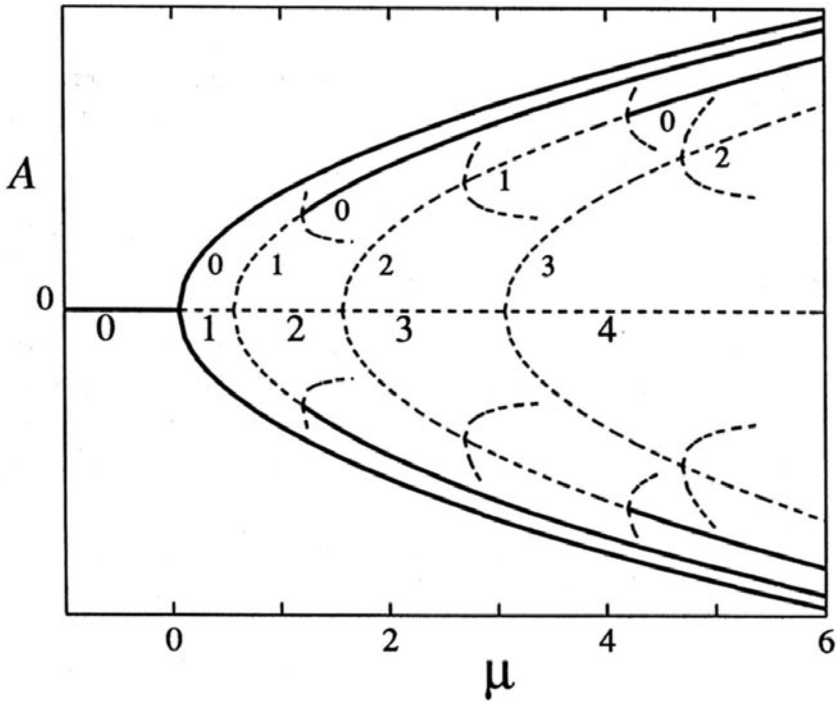

Fig. 1. Bifurcation diagram. Amplitude A of the trivial state, the pure-mode states A_n, and the mixed-mode states as a function of μ. Solid and dashed curves represent stable and unstable steady states, respectively. Each branch is labeled by its instability index n.

When branch A_n is created at $\mu = \mu_n \equiv Q_n^2$, the eigenvalues σ_{nk+} are positive for $1 \le k \le n$; by continuity of spectra, A_n "inherits" these n unstable directions from its parent state $A = 0$ at the moment of bifurcation. Bifurcations of state A_n take place at $\mu = \mu_{nk}$, at which the instability index of A_n decreases by one. These are pitchfork bifurcations, since they break the D_n symmetry of the parent state A_n. A center manifold reduction shows these bifurcations to be subcritical: i.e. the coefficient of the cubic term in the evolution equation for the amplitude of the bifurcating eigenvector is positive at the bifurcation. Thus, new "mixed-mode" branches exist for $\mu > \mu_{nk}$ as shown in Fig. 1.

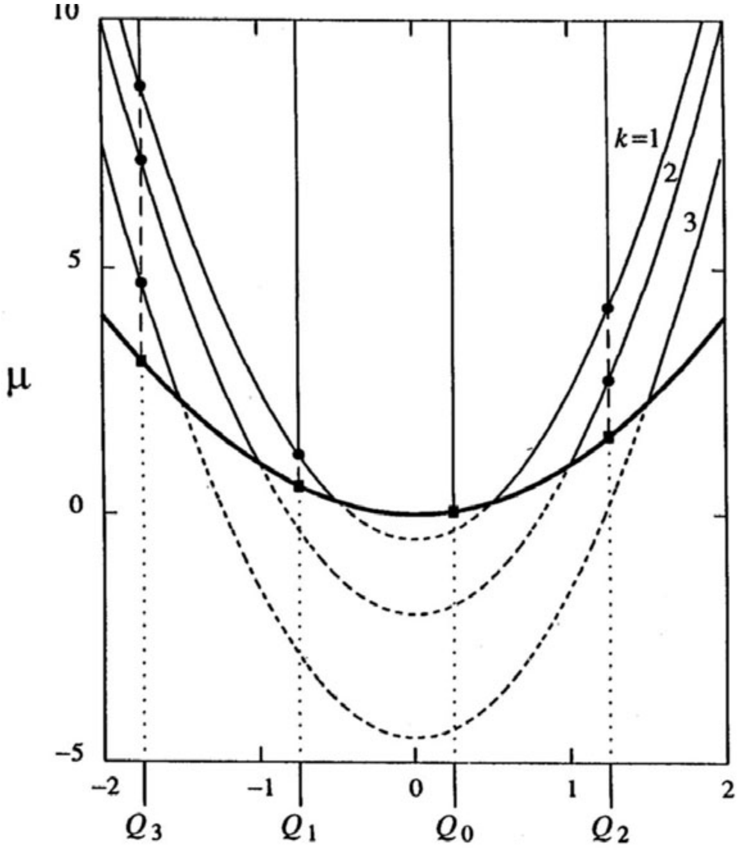

Fig. 2. Eckhaus parabolas $\mu^{\mathrm{E}}(Q,k) \equiv 3Q^2 - k^2/2$ and existence parabola $\mu = Q^2$. The bifurcations at μ_n and μ_{nk} are represented by squares and circles, respectively, along the vertical lines of allowed wavenumbers Q_n. (The typical case shown here has allowed wavenumbers $Q_0 = 1/4$, $Q_1 = -3/4$, $Q_2 = 5/4$, $Q_3 = -7/4$, etc.) The dotted, dashed, and solid portions of the lines denote non-existent, unstable, and stable regimes. The dashed portion of each Eckhaus parabola is "irrelevant", i.e. falls below the existence parabola. The $k = 1$ parabola separates the stable and unstable regimes, except for branch A_0: for a finite geometry, stabilization occurs at $\mu = 3Q^2 - 1/2$ rather than $\mu = 3Q^2$. We emphasize that a change in the length L of the container corresponds only to a *translation* of the vertical lines and not to a change of scale.

Figure 2 displays all of the primary and secondary bifurcations as intersections of the vertical lines $Q = Q_n$ with the "existence parabola" $\mu = Q^2$ and the "Eckhaus parabolas" $\mu^E(Q,k) \equiv 3Q^2 - k^2/2$. Branch A_n undergoes n secondary bifurcations: exactly the number necessary to render A_n stable after the final bifurcation at $\mu_{n1} = 3Q_n^2 - 1/2$. This is a crucial difference, already noted in [7, 3], between the finite-length and classic infinite-length treatment: in the infinite case, where k can be taken arbitrarily close to 0, a branch with wavenumber Q will only be stable for $\mu > \sup_k \, [3Q^2 - k^2/2] = 3Q^2$.

A wide class of wavelength-changing transitions in less symmetric systems, such as spherical Couette flow [8] and cylindrical convection [9], can be interpreted as perturbations of the "ideal" Eckhaus diagram. The curvature of these domains breaks the translation and reflection symmetry, transforming the pitchfork bifurcations of the ideal case to imperfect bifurcations. In future work, we will show that the subcritical secondary bifurcations result from mode-interactions between pairs of pure-mode branches.

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