

Binary fluid convection
as a
 2×2 matrix problem

Laurette Tuckerman

PMMH-ESPCI-CNRS

France

Convection due to two competing or cooperating forces

1960s: Veronis, Sani, Nield, Baines, Gill, Chandrasekhar

1970s: Caldwell, Hurle, Jakeman, Schechter, Velarde, Platten, Chavepeyer, Huppert, Moore, Gershuni, Zhukhovitskii

Codimension-two (Takens-Bogdanov) point

1980s: **Knobloch**, Platten, Legros, Proctor, Da Costa, Weiss, Guckenheimer, Bretherton, Spiegel, Brand, Le Gal, Pocheau, Croquette, Coulet, Fauve, Tirapegui, Walden, Kolodner, Passner, Surko, Rehberg, Ahlers, Deane, Toomre, Moses, Steinberg, Cross, Lücke, Fineberg, Linz, Silber, Müller, Lhost, Bensimon, Pumir, Shraiman

1990s: Clune, Rucklidge, Riecke, Schöpf, Zimmermann, Predtechensky, McCormick, Swift, Rossberg, Swinney, Barten, Kamps, Schmitz, Dominguez-Lerma, Cannell, Hollinger, Büchel, Fütterer, Jung, Bergeon, Henry, Benhadid, Huke, Bestehorn

Bergeon, Henry, BenHadid & Tuckerman,

Marangoni convection in binary mixtures with Soret effect, JFM, 1998

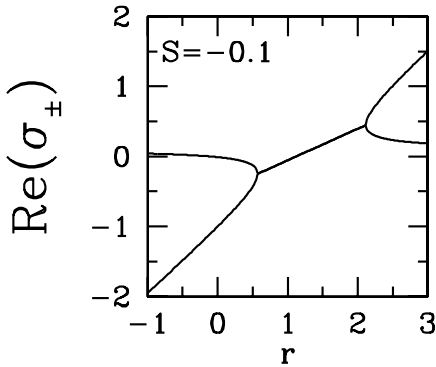
Tuckerman, Thermosolutal and binary fluid convection as a 2×2 matrix problem, Physica D, 2001

Basic idea: 2×2 matrix depending on control parameter r

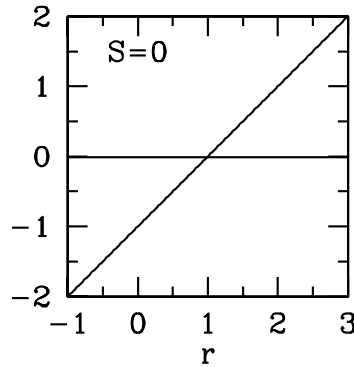
$$\begin{pmatrix} \sigma_T & \alpha \\ \beta & \sigma_C \end{pmatrix}$$

has eigenvalues

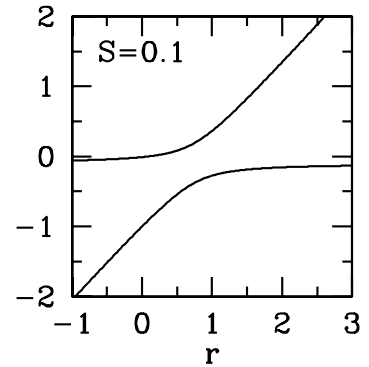
$$\sigma_{\pm} = \frac{\sigma_T + \sigma_C}{2} \pm \sqrt{\left(\frac{\sigma_T - \sigma_C}{2}\right)^2 + \alpha\beta}$$



$\alpha\beta < 0$
complex coalescence



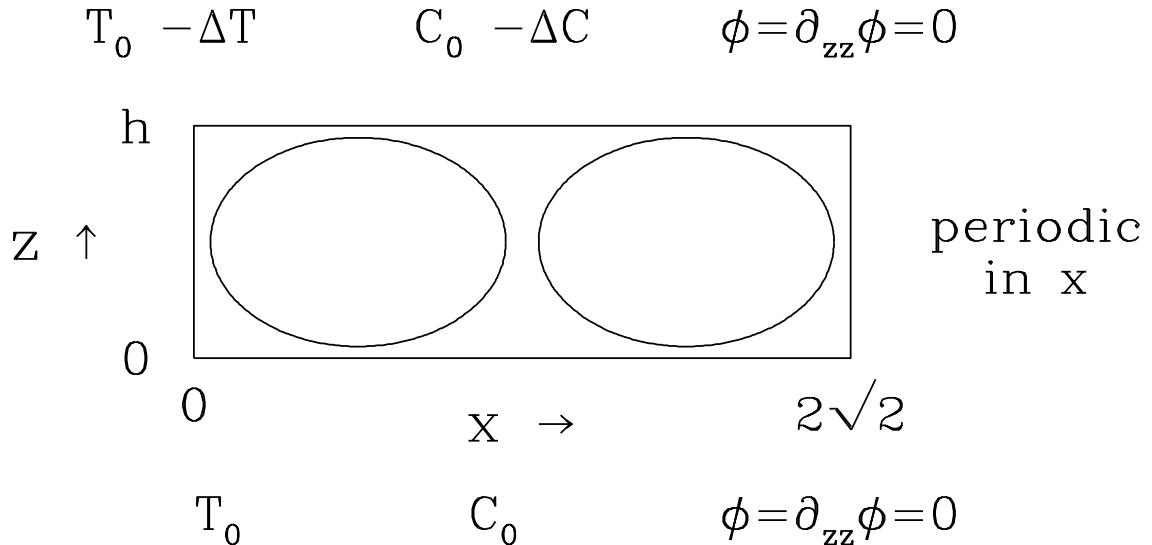
$\alpha\beta = 0$
transverse crossing



$\alpha\beta > 0$
avoided crossing

Sign of $\alpha\beta$ determines behavior of eigs where $\sigma_T(r)$ and $\sigma_C(r)$ intersect.

Simple Model: 2D Thermosolutal Problem



Vertical thermal and solutal gradients imposed at $z = 0, 1$

Boundary conditions: free-slip at $z = 0, 1$; periodic in x with length $2\sqrt{2}$

Streamfunction $\mathbf{U} = \nabla \times \psi(x, z)\mathbf{e}_y$

Density: $\rho(T, C) = \rho_0 + \rho_T(T - T_0) + \rho_C(C - C_0)$

Diffusivities: κ_T (thermal), κ_C (solutal), ν (momentum)

Conductive solution:

$$T = T_0 - z\Delta T/h, \quad C = C_0 - z\Delta C/h, \quad \mathbf{U} = \nabla \times \psi \mathbf{e}_y = \mathbf{0}$$

Four nondimensional parameters:

Fix: Lewis number $L \equiv \frac{\kappa_C}{\kappa_T} \ll 1$

Prandtl number $P \equiv \frac{\nu}{\kappa_T} \gg 1$.

Vary: Rayleigh number $R \equiv \frac{g\rho_T\Delta Th^3}{\nu\kappa_T}$

Separation ratio $S \equiv \frac{\rho_C\Delta C}{\rho_T\Delta T}$

Subtract conductive solution and nondimensionalize.

Governing Equations:

$$\partial_t \tilde{T} = \partial_x \tilde{\psi} + \mathbf{e}_y \cdot (\nabla \tilde{\psi} \times \nabla \tilde{T}) + \nabla^2 \tilde{T}$$

$$\partial_t \tilde{C} = \partial_x \tilde{\psi} + \mathbf{e}_y \cdot (\nabla \tilde{\psi} \times \nabla \tilde{C}) + L \nabla^2 \tilde{C}$$

$$\partial_t \nabla^2 \tilde{\psi} = PR \partial_x (\tilde{T} + S \tilde{C}) + \mathbf{e}_y \cdot (\nabla \tilde{\psi} \times \nabla \nabla^2 \tilde{\psi}) + P \nabla^4 \tilde{\psi}$$

Linear Analysis:

$$\begin{Bmatrix} \tilde{T} \\ \tilde{C} \\ \tilde{\psi} \end{Bmatrix} (x, z, t) = \begin{Bmatrix} T \cos(kx) \\ C \cos(kx) \\ \psi \sin(kx) \end{Bmatrix} \sin(\pi z) e^{(k^2 + \pi^2)\sigma t}$$

$$k = k_c = \pi/\sqrt{2}, \quad \gamma^2 \equiv k^2 + \pi^2, \quad r \equiv Rk^2/\gamma^6$$

$$P = \infty \quad \implies \quad \psi = -r(T + SC)\gamma^2/k$$

$$2 \times 2 \text{ system:} \quad \sigma \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r - 1 & rS \\ r & rS - L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}$$

“thermal eigenvalue” $\sigma_T = r - 1$ has threshold $r_T = 1$

“solutal eigenvalue” $\sigma_C = rS - L$ $r_C = L/S$

Eigenvalues intersect at $r_{\text{int}} = \frac{1-S}{1-L}$ with coupling Sr^2

Eigenvalues:

$$\sigma_{\pm} = \frac{(r-1) + (rS-L)}{2} \pm \sqrt{\left(\frac{(r-1) - (rS-L)}{2}\right)^2 + Sr^2}$$
$$\equiv f_L \pm \sqrt{g_L}$$

complex coalescence ($S < 0$), cross transversely ($S = 0$), or avoided crossing ($S > 0$)

Inverting:

$$r = \frac{\sigma_{\pm}^2 + \sigma_{\pm}(1+L) + L}{\sigma_{\pm}(1+S) + (S+L)}$$

Each value of σ corresponds to a unique value of r .

In particular, unless $S = -L$, \exists exactly one pitchfork bifurcation $\sigma = 0$ at

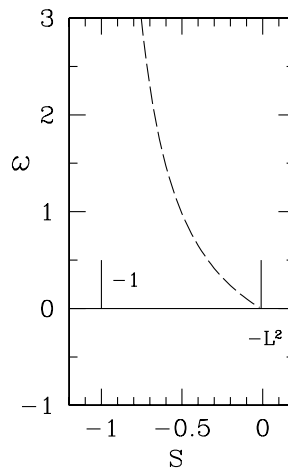
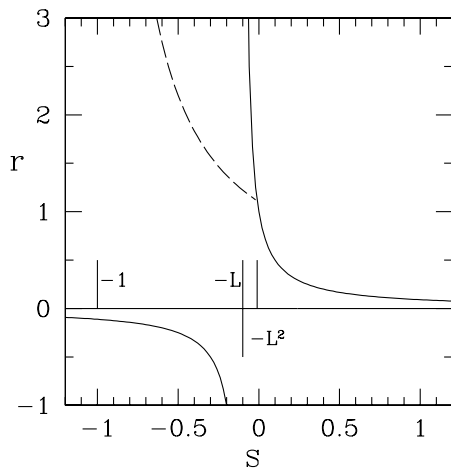
$$r = r_{PF} = \frac{L}{S+L}$$

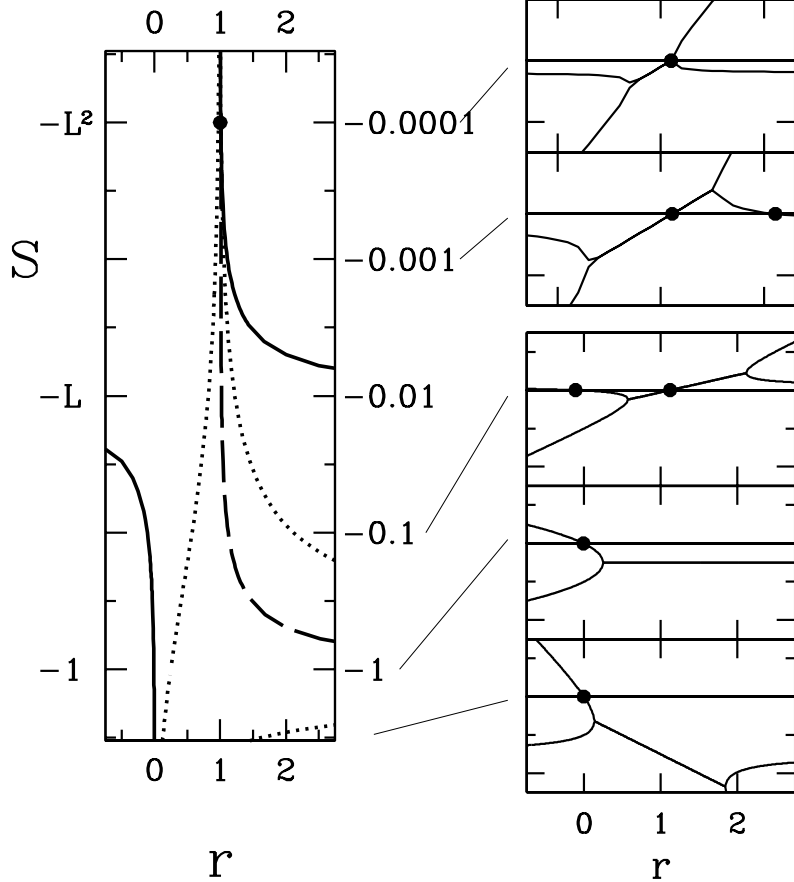
If $S < 0$, eigenvalues are complex over the range where $g_L < 0$:

$$\frac{1-L}{(1+\sqrt{-S})^2} \equiv r_{c-} < r < r_{c+} \equiv \frac{1-L}{(1-\sqrt{-S})^2}$$

There is a Hopf bifurcation if $f_L = 0$ in this range, i.e. if $-1 < S < -L^2$

$$r_H = \frac{1+L}{1+S} \rightarrow \begin{cases} \infty \\ r_{PF} \end{cases} \quad \omega_H^2 = -\frac{S+L^2}{1+S} \rightarrow \begin{cases} \infty \\ 0 \end{cases} \quad \text{as } S \rightarrow \begin{cases} -1 \\ -L^2 \end{cases}$$





$$-L^2 < S$$

● codim-two (linear)

$$-L < S < -L^2$$

$$r_{\text{PF}} \rightarrow \pm\infty$$

$$-1 < S < -L$$

$$r_{\text{H}} \rightarrow \infty$$

$$S < -1$$

Lorenz Model: including nonlinear interactions

$$\begin{aligned}
 J[\psi, \nabla^2 \psi] &= J[\psi, -\gamma^2 \psi] \\
 &= \partial_x \psi \partial_z(-\gamma^2 \psi) - \partial_x(-\gamma^2 \psi) \partial_z \psi = 0
 \end{aligned}$$

$$\begin{aligned}
 J[\psi, T] &= \hat{\psi} \hat{T} [\partial_x(\sin qx \sin \pi z) \partial_z(\cos qx \sin \pi z) \\
 &\quad - \partial_x(\cos qx \sin \pi z) \partial_z(\sin qx \sin \pi z)] \\
 &= \hat{\psi} \hat{T} q\pi [\cos qx \sin \pi z \cos qx \cos \pi z \\
 &\quad + \sin qx \sin \pi z \sin qx \cos \pi z] \\
 &\quad + \hat{\psi} \hat{T} q\pi (\cos^2 qx + \sin^2 qx) \sin \pi z \cos \pi z \\
 &= \hat{\psi} \hat{T} \frac{q\pi}{2} \sin 2\pi z
 \end{aligned}$$

↑ ↑

functions

↑ ↑

scalars

$$\psi(x, z, t) = \hat{\psi}(t) \sin qx \sin \pi z$$

$$T(x, z, t) = \hat{T}_1(t) \cos qx \sin \pi z + \hat{T}_2(t) \sin 2\pi z$$

$$\begin{aligned} J[\psi, T_2] &= \hat{\psi} \hat{T}_2 [\partial_x(\sin qx \sin \pi z) \partial_z(\sin 2\pi z) \\ &\quad - \partial_x(\sin 2\pi z) \partial_z(\sin qx \sin \pi z)] \\ &= \hat{\psi} \hat{T}_2 q 2\pi \cos qx \sin \pi z \cos 2\pi z \\ &= \hat{\psi} \hat{T}_2 q \pi \cos qx (\sin \pi z + \sin 3\pi z) \end{aligned}$$

Including $\hat{T}_3(t) \cos qx \sin 3\pi z \implies$ new terms \implies
Closure problem for nonlinear equations

Lorenz (1963) proposed stopping at T_2 .

$$\begin{aligned}
\partial_t \hat{\psi} &= Pr(q\hat{T}_1/\gamma^2 - \gamma^2 \hat{\psi}) & \sin qx \sin \pi z \\
\partial_t \hat{T}_1 + q\pi \hat{\psi} \hat{T}_2 &= Ra q \hat{\psi} - \gamma^2 \hat{T}_1 & \cos qx \sin \pi z \\
\partial_t \hat{T}_2 + \frac{q\pi}{2} \hat{\psi} \hat{T}_1 &= -(2\pi)^2 \hat{T}_2 & \sin 2\pi z
\end{aligned}$$

Define:

$$X \equiv \frac{\pi q}{\sqrt{2}\gamma^2} \hat{\psi}, \quad Y \equiv \frac{\pi q^2}{\sqrt{2}\gamma^6} \hat{T}_1, \quad Z \equiv \frac{\pi q^2}{\sqrt{2}\gamma^6} \hat{T}_2,$$

$$\tau \equiv \gamma^2 t, \quad r \equiv \frac{q^2}{\gamma^6} Ra, \quad b \equiv \frac{4\pi^2}{\gamma^2} = \frac{8}{3}, \quad \sigma \equiv Pr$$

Famous Lorenz Model:

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = -XZ + rX - Y$$

$$\dot{Z} = XY - bZ$$

Do the same for C

Nonlinear interactions generate terms of form $\left\{ \begin{matrix} T_2 \\ C_2 \end{matrix} \right\} \sin(2\pi z)$

Truncating at this order yields:

$$\partial_t \begin{pmatrix} T \\ C \\ \psi \\ T_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} -\gamma^2 & 0 & -k & 0 & 0 \\ 0 & -L\gamma^2 & -k & 0 & 0 \\ -P\gamma^4 r/k & -PS\gamma^4 r/k & -P\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & -4\pi^2 & 0 \\ 0 & 0 & 0 & 0 & -L4\pi^2 \end{pmatrix} \begin{pmatrix} T \\ C \\ \psi \\ T_2 \\ C_2 \end{pmatrix}$$

$$+ \pi k \psi \begin{pmatrix} -T_2 \\ -C_2 \\ 0 \\ \frac{1}{2}T \\ \frac{1}{2}C \end{pmatrix}$$

Veronis (1965)

Steady states obey strange eigenproblem

After eliminating ψ , T_2 , C_2 , steady states obey:

$$\begin{pmatrix} r - 1 & rS \\ Lr & L(rS - L) \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} = \frac{1}{2} \left(\frac{r\gamma}{2} \right)^2 (T + SC)^2 \begin{pmatrix} T \\ C \end{pmatrix}$$

i.e. an eigenvalue problem for the square amplitude or energy:

$$E \equiv \frac{1}{2} \left(\frac{r\gamma}{2} \right)^2 (T + SC)^2$$

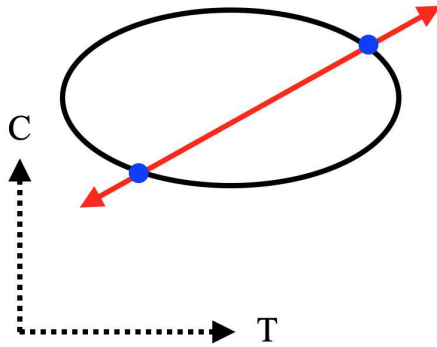
Solutions are:

$$E = \frac{(r - 1) + L(rS - L)}{2} \pm \sqrt{\left(\frac{(r - 1) - L(rS - L)}{2} \right)^2 + SLr^2}$$
$$\equiv f_{NL}(r) \pm \sqrt{g_{NL}(r)}$$

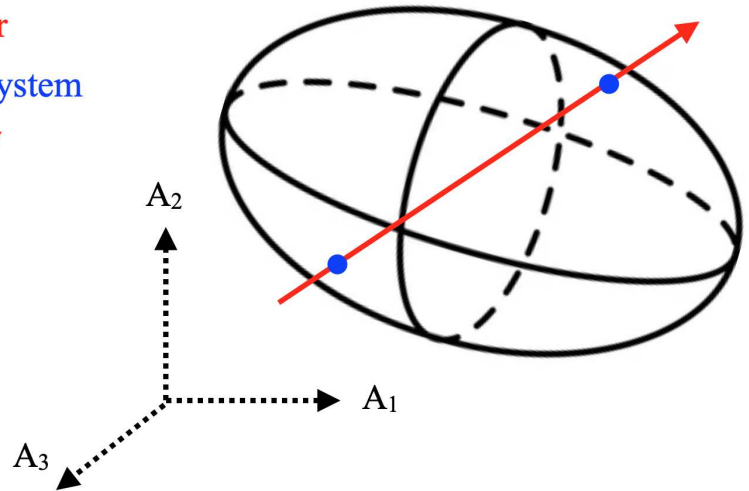
ellipse: $E = \frac{1}{2} \left(\frac{r\gamma}{2} \right)^2 (T + SC)^2$

arrow: corresponding eigenvector

blue dots: solution to nonlinear system

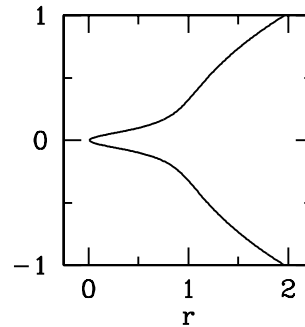
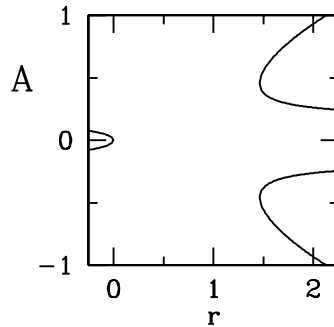
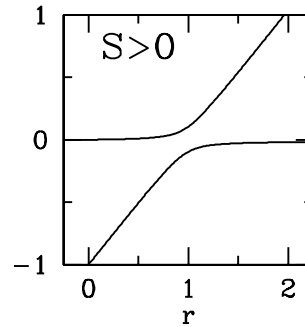
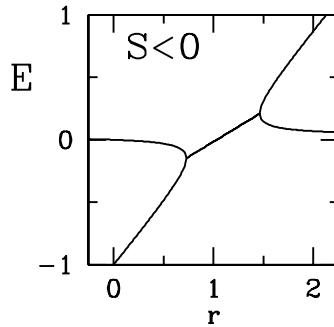


ellipsoid: eigenvalue hypersurface



Nonlinear system of n equations in n unknowns \Leftrightarrow

$$\left\{ \begin{array}{l} 1 \text{ equation defining the eigenvalue} \\ n - 1 \text{ equations defining the eigenvector} \end{array} \right.$$



$S < 0$

Creation of two pairs of real solutions
 $\pm \sqrt{E_{\pm}}$ via saddle-node bifurcations

$S > 0$

change of curvature between
 weak solutal (“Soret”) and strong
 thermal (“Rayleigh”) convective regimes

Exact analogy between **growth rate** and **energy** eigenproblems

- Translation from **growth rate** to **energy** via:

$$\sigma \rightarrow E$$

$$L \rightarrow L^2$$

$$S \rightarrow LS$$

- **Complex or negative σ indicate oscillating or decaying transients**
→ **Only real and positive values of E are meaningful**

- Velocity streamfunction obeys $\psi = -r(T + SC)\gamma^2/k$ for:
 $P = \infty$ in growth rate problem → **all P in energy problem**

- Coupling has sign of S :

$$Sr^2 \text{ for growth-rate matrix} \rightarrow L Sr^2 \text{ for energy matrix}$$

- Each value of σ and of E is achieved exactly once:

$$r = \frac{\sigma^2 + \sigma(1 + L) + L}{\sigma(1 + S) + (S + L)} \longrightarrow r = \frac{E^2 + E(1 + L^2) + L^2}{E(1 + LS) + L(S + L)}$$

- At the pitchfork bifurcation, both σ and E necessarily vanish:

$$r_{\text{PF}} = \frac{L}{S + L} \longrightarrow r_{\text{PF}} = \frac{L^2}{LS + L^2} = \frac{L}{S + L}$$

- Transition from complex to positive real E is a saddle-node:

$$r_{c+} = \frac{1 - L}{(1 - \sqrt{-S})^2} \longrightarrow r_{\text{SN}} = \frac{1 - L^2}{(1 - \sqrt{-LS})^2}$$

- Codimension-two bifurcations at $f = g = 0$:

$$\left\{ \begin{array}{l} S_* = -L^2 \\ r_* = \frac{1}{1-L} \end{array} \right\} \text{Takens-Bogdanov} \longrightarrow \left\{ \begin{array}{l} S_* = -L^3 \\ r_* = \frac{1}{1-L^2} \end{array} \right\} \text{degenerate PF} \\ \text{(PF/Hopf)} \qquad \qquad \qquad \text{(PF/SN)}$$

Bifurcations corresponding to $\left\{ \begin{array}{l} \text{growth rate } \sigma \\ \text{energy } E \end{array} \right\} = f \pm \sqrt{g}$

S	f	g	σ	E
	$\pm\sqrt{g}$	+	pitchfork	pitchfork
-	0	-	Hopf	
-	+	0		saddle-node
0	0	0	Takens-Bogdanov (Hopf/pitchfork)	degenerate pitchfork (super/subcritical)

Lorenz-like Hybrid Model:

$$\frac{1}{\gamma^2} \frac{d}{dt} \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r - 1 & rS \\ r & rS - L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} - \frac{1}{2} \left(\frac{r\gamma}{2} \right)^2 (T + SC)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1/L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}$$

$S = 0$ is an *organizing center*.

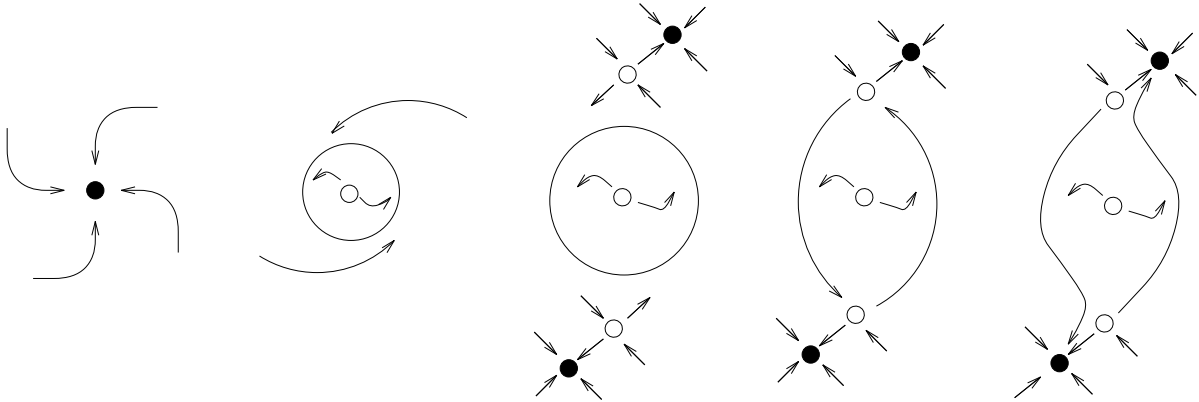
Reproduces:

- **linear stability diagram**
- **nonlinear steady states**
- **limit cycle created by Hopf, annihilated by global bifurcation**

Assumptions:

- $P = \infty$
- truncate C_3, T_3, C_4, T_4 , etc. despite slow damping of C_3 , etc.

Lorenz-like Hybrid Model for $S < 0$:



r

Hopf

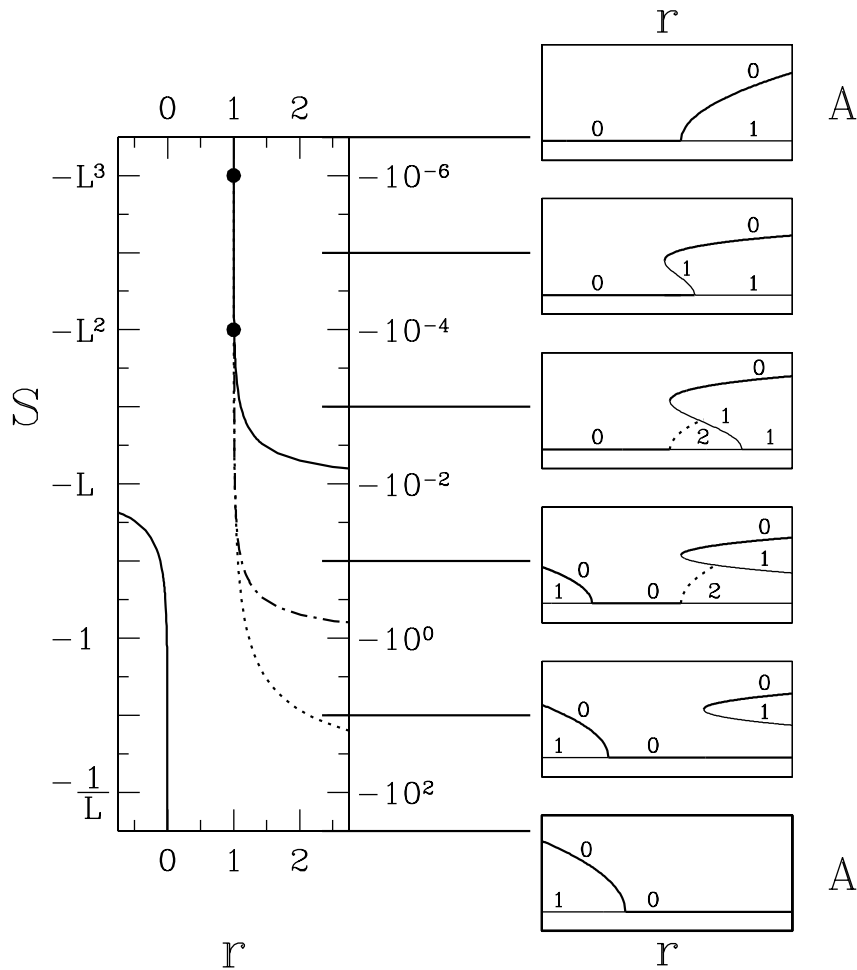
**Saddle-
Node**

**Heteroclinic
(Global)**

$$r_H = \frac{1+L}{1+S}$$

$$r_{SN} = \frac{1-L^2}{(1-\sqrt{-LS})^2}$$

r_G



A

$$-L^3 < S$$

● codim-two (nonlinear)

$$-L^2 < S < -L^3$$

● codim-two (linear)

$$-L < S < -L^2$$

$$r_{\text{PF}} \rightarrow \pm\infty$$

$$-1 < S < -L$$

$$r_{\text{H}} \rightarrow \infty$$

$$-1/L < S < -1$$

$$r_{\text{SN}} \rightarrow \infty$$

$$S < -1/L$$

A

Conclusions

- **Organizing center $S = 0$: complex coalesce / avoided crossing.**
Complementary to Takens-Bogdanov point S_* .
- **Simple 2×2 matrix model captures all bifurcations.**
Yields analytic formulas.
- **Steady-state problem is *also* eigenproblem.**
Nonlinear problem reduced to eigenproblem + one scalar equation.
- **Analogy between growth-rate and steady-state problems.**
Exact, quantitative correspondence.
- **Hybrid model also captures limit cycle and global bifurcation.**