Stability of Inviscid Taylor-Couette Flow

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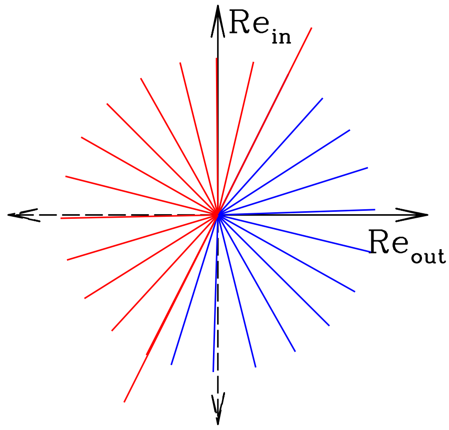


The stability boundary for inviscid Taylor-Couette flow is the well known Rayleigh line $Re_{\rm out}/Re_{\rm in} = \eta$. However, the phase diagram in the $(Re_{\rm in}, Re_{\rm out})$ plane should be centro-symmetric because there is no intrinsic difference between clockwise and counter-clockwise rotation. For example, in the figure above, the positive $Re_{\rm in}$ axis lies on the stable side (blue) of the Rayleigh line, but the negative $Re_{\rm in}$ axis lies on its unstable side (red), despite the fact that these differ only by the direction (counter-clockwise and clockwise, respectively), of inner-cylinder-only rotation and hence cannot correspond to different regimes. Clearly, the figure above is wrong or incomplete: there must be another line (or feature) which delimits the stable and unstable parts of the $(Re_{\rm in}, Re_{\rm out})$ plane.

Define the following quantities for Taylor-Couette flow. Distances are non-dimensionalized by gap and times by gap viscous times.

$$\eta \equiv \frac{r_{\rm in}}{r_{\rm out}} \qquad \mu \equiv \frac{\Omega_{\rm out}}{\Omega_{\rm in}}$$

$$r_{\rm in} = \frac{\eta}{1-\eta} \qquad r_{\rm out} = \frac{1}{1-\eta}$$

$$Re_{\rm in} \equiv \Omega_{\rm in} r_{\rm in} \qquad Re_{\rm out} \equiv \Omega_{\rm out} r_{\rm out}$$

$$C \equiv \frac{Re_{\rm out}}{Re_{\rm in}} = \frac{\Omega_{\rm out} r_{\rm out}}{\Omega_{\rm in} r_{\rm in}} = \frac{\mu}{\eta} \qquad R^2 = Re_{\rm in}^2 + Re_{\rm out}^2 = Re_{\rm in}^2(1+C^2)$$

$$A \equiv \frac{Re_{\rm out} - \eta Re_{\rm in}}{1+\eta} = \frac{Re_{\rm in}(C-\eta)}{1+\eta} = \frac{R\,(C-\eta)}{\sqrt{1+C^2}\,(1+\eta)}$$

$$B \equiv \frac{\eta(Re_{\rm in} - \eta Re_{\rm out})}{(1-\eta)^2(1+\eta)} = \frac{Re_{\rm in}\,\eta\,(1-\eta C)}{(1-\eta)^2(1+\eta)} = \frac{R\,\eta\,(1-\eta C)}{\sqrt{1+C^2}\,(1-\eta)^2(1+\eta)}$$

The Rayleigh discriminant is

$$\Phi \equiv \frac{1}{r^3}\frac{d}{dr}(ru_\theta)^2 = \frac{1}{r^3}\frac{d}{dr}\left(r(\left(Ar+\frac{B}{r}\right)\right)^2$$
$$= \frac{1}{r^3}\frac{d}{dr}\left(Ar^2+B\right)^2 = \frac{1}{r^3}2(Ar^2+B)2Ar = \frac{1}{r^2}4A(Ar^2+B)$$

The Rayleigh criterion states that laminar Couette flow is stable if $\Phi \geq 0$. This is clearly the case when $A = 0$:

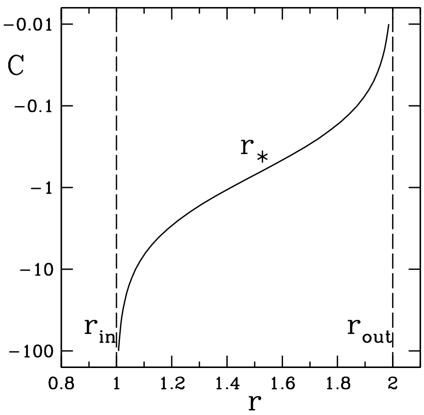
$$A = 0 \Leftrightarrow Re_{\text{out}} = \eta Re_{\text{in}} \Leftrightarrow C = \eta$$

$C = \eta$ is the classic Rayleigh stability line in the 1st and 3rd quadrants with stability for $C > \eta$.

However, Φ changes sign if

$$Ar^2 + B = 0 \Leftrightarrow r^2 = r_*^2 \equiv -\frac{B}{A} = \frac{\eta^2}{(1-\eta)^2}\frac{C-\frac{1}{\eta}}{C-\eta}$$

for r_* in the interval $[r_{\text{in}}, r_{\text{out}}]$. For $C \leq 0$, we have $r_* \in [r_{\text{in}}, r_{\text{out}}]$, so an interval of instability exists. At the endpoints of the 2nd quadrant, we have $r_* = r_{\text{out}}$ for $C = 0$ and $r_* = r_{\text{in}}$ for $C = -\infty$.



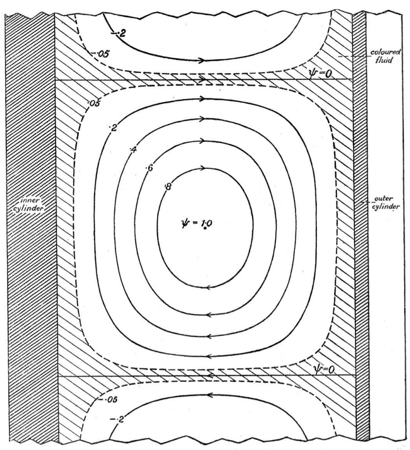


Fig. 5. Stream lines of motion after instability has set in, μ positive.

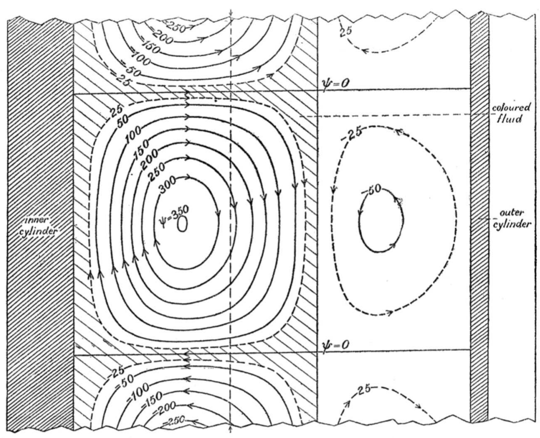
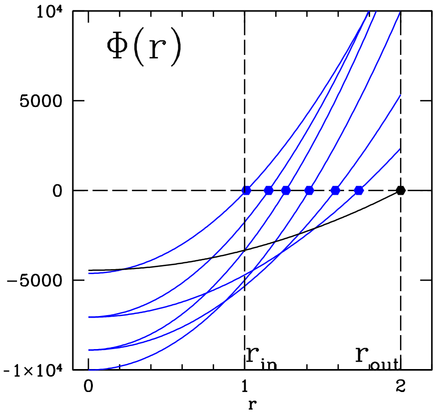
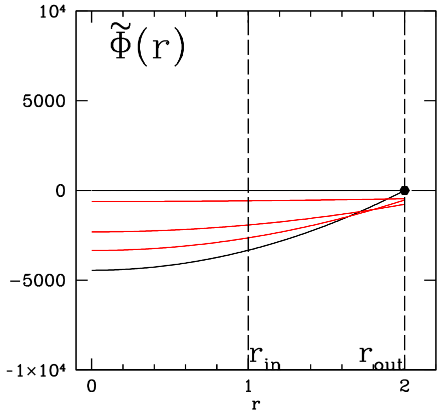


Fig. 7. Stream lines of motion after instability has set in, μ = −1·5.

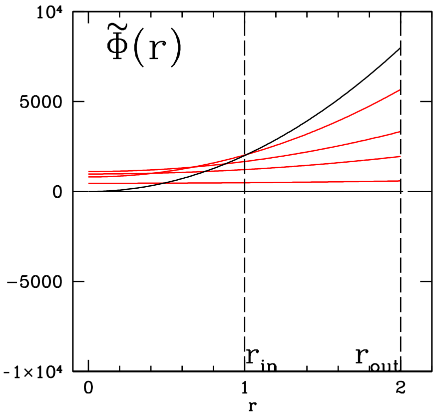
The figure above shows sketches of flow in the corotating and counter-rotating regimes Taylor's famous 1923 paper. The corotating vortices fill up the entire domain, whereas the counter-rotating spiral vortices are localized near the inner cylinder, where Φ is negative.



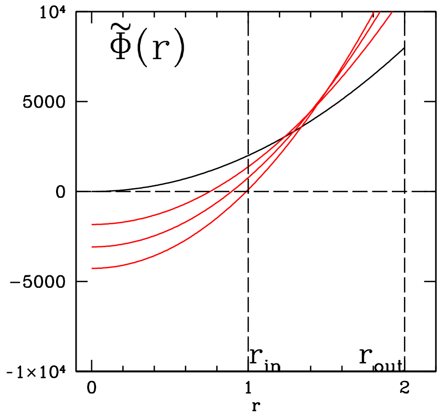
$C = -64, -4, -2, -1, -0.5, -0.25, 0$



$C = 0, 0.1, 0.2, 0.4, 0.5 (= \eta)$



$C = 0.5(= \eta), 0.6, 0.8, 1.0, 1.4, 2(= 1/\eta)$



$C = 2, 4, 8, 64$

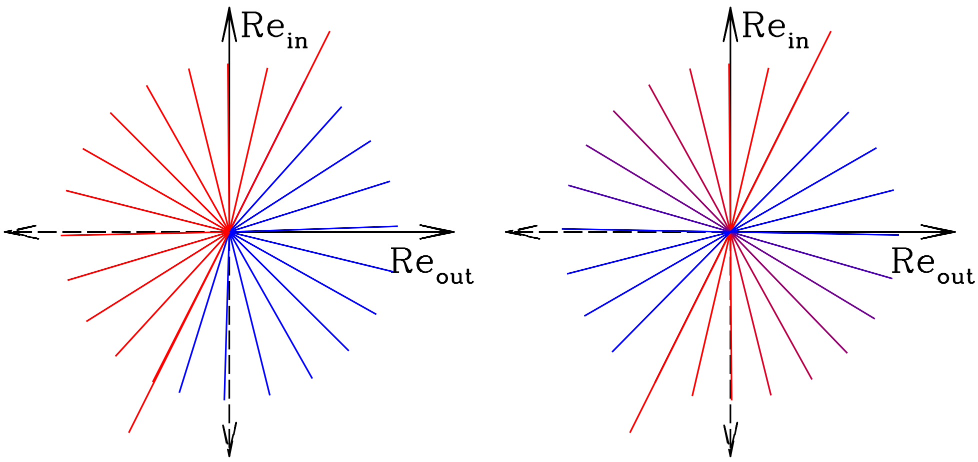
The figure above shows $\tilde{\Phi}(r) \equiv 4r^2 \Phi(r)$ for $\eta = 0.5$. We set $Re_{\rm in}^2 + Re_{\rm out}^2 = 100^2$ and scan over positive and negative values of C.

• For $C \leq 0$ (e.g. $Re_{\rm in} \geq 0$ and $Re_{\rm out} \leq 0$, $r_* \in [r_{\rm in}, r_{\rm out}]$ and these values are shown as dots. The flow is stable where $\Phi \geq 0$ and hence to the right of these dots. The left-most curve represents $C = -\infty$, i.e. the negative half-$Re_{\rm out}$ axis, where $r_* = r_{\rm in}$ and the entire interval is stable. The black curve is $C = 0$, i.e. the $Re_{\rm in}$ axis, for which $\Phi < 0$ throughout the interval, since $r_* = r_{\rm out}$ and so none of the interval is stable.

• For $0 \leq C \leq \eta$, $\Phi < 0$ everywhere in $[r_{\rm in}, r_{\rm out}]$ and so the flow is unstable. The black curves are $C = 0$ and $C = \eta$.

• For $\eta \leq C \leq 1/\eta$, $\Phi > 0$ for all r and so the flow is stable. The black curves are $C = \eta$ and $C = 1/\eta$.

• For $C \geq \frac{1}{\eta}$, there is a negative region of Φ, but for $r \in [0, r_{\rm in}]$, which is not in the domain, so the flow is stable in the entire $[r_{\rm in}, r_{\rm out}]$ interval. The black curve is $C = 1/\eta$.



The figure on the left illustrates a change in stability at the Rayleigh line, which does not respect the physically necessary centro-symmetry of the $(Re_{\rm out}, Re_{\rm in})$ plane. Red is used to denote the unstable parameters and blue the stable parameters.

The figure on the right shows the gradual change in stability that occurs in the 2nd and 4th quadrants, i.e. when $Re_{\rm in}$ and $Re_{\rm out}$ are of opposite signs. This is illustrated by the gradual change of color from red ($Re_{\rm out} = 0$, entire $[r_{\rm in}, r_{\rm out}]$ interval unstable) to blue ($Re_{\rm in} = 0$, entire $[r_{\rm in}, r_{\rm out}]$ interval stable). Thus, there are three important lines: the usual Rayleigh line $Re_{\rm out} = \eta Re_{\rm in}$ and the $Re_{\rm in}$ and $Re_{\rm out}$ axes, at which the Rayleigh discriminant changes character. Moving counter-clockwise from the positive $Re_{\rm out}$ axis, Φ changes from negative everywhere (blue) to positive everywhere (red, upon crossing the positive Rayleigh line) to positive and negative intervals (purple, crossing the positive $Re_{\rm in}$ axis), to negative everywhere (blue, crossing the negative $Re_{\rm out}$ axis). The same sequence is followed in the 3rd and 4th quadrants.