A brief introduction to linear stability analysis in fluid dynamics

Some basic definitions

The Navier-Stokes equations are:

$$\begin{aligned} \partial_t \mathbf{U} &= -(\mathbf{U}\cdot\nabla)\mathbf{U} - \nabla P + \frac{1}{R}\nabla^2\mathbf{U} & \text{(1a)} \\ \nabla\cdot\mathbf{U} &= 0 & \text{(1b)} \\ \mathbf{U} &= \mathbf{U}_b \qquad \text{on a boundary set B} & \text{(1c)} \end{aligned}$$

A base flow is a steady solution to these equations, where (1a)) is replaced by

$$0 = -(\mathbf{U}\cdot\nabla)\mathbf{U} - \nabla P + \frac{1}{R}\nabla^2\mathbf{U} \tag{2}$$

Linear stability analysis starts by *linearizing* about this base flow

$$\begin{aligned} \partial_t \mathbf{u} &= -(\mathbf{U}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{U} - \nabla p + \tfrac{1}{R}\nabla^2\mathbf{u} & \text{(3a)} \\ \nabla\cdot\mathbf{u} &= 0 & \text{(3b)} \\ \mathbf{u} &= 0 \qquad \text{on a boundary set B} & \text{(3c)} \end{aligned}$$

Equation (3a) is linear and *homogeneous* in time (which means here that no time is singled out, that every time is equivalent to every other time). The boundary conditions (3c) are also *homogeneous* (which, confusingly, means something different here, i.e. that the boundary values have been set to zero). This means that the entire problem is linear, because the constant term $\mathbf{U}_b$ and the quadratic term $(\mathbf{U}\cdot\nabla)\mathbf{U}$ are not present. The solution to such a problem is exponential or trigonometric in time. Thus we can write $\mathbf{u}$ in the form $\mathbf{u}(\mathbf{x},t) = \exp(\lambda t)\tilde{\mathbf{u}}(\mathbf{x})$ and replace (3a) by

$$\lambda\tilde{\mathbf{u}} = -(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} - (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} - \nabla p + \tfrac{1}{R}\nabla^2\tilde{\mathbf{u}} \tag{4}$$

where $\lambda = \sigma + i\omega$ may be real or complex. Similar concepts can be introduced for time-periodic base states.

Turning now to the geometry and the spatial dependence of problems (1) and (3), a base state may depend on one, two, or three spatial directions, i.e. $\mathbf{U} = \mathbf{U}(x_1)$ or $\mathbf{U}(x_1,x_2)$ or $\mathbf{U}(x_1,x_2,x_3)$. It may also have one, two, or three components, i.e. $\mathbf{U} = U_1\mathbf{e}_1$ or $\mathbf{U} = U_1\mathbf{e}_1 + U_2\mathbf{e}_2$ or $\mathbf{U} = U_1\mathbf{e}_1 + U_2\mathbf{e}_2 + U_3\mathbf{e}_3$. The difficulty of solving a problem is primarily determined by the number of independent variables x_1, x_2, x_3, and hence when we describe a problem as one, two, or three dimensional, we will be referring to the number of independent variables.

For example, in the Taylor-Couette problem of flow between two infinitely long differentially rotating concentric cylinders, the base state is $\mathbf{U} = (Ar + B/r)\mathbf{e}_\theta$ i.e. it depends on r and is oriented in the θ direction. If axial throughflow is added to this problem, the base state becomes $\mathbf{U} = (Ar+B/r)\mathbf{e}_\theta + C(1-r^2)\mathbf{e}_z$. The flow is still one-dimensional (it depends only on r), but has two components. The flow past a circular cylinder depends on x and y, and also has components in both directions. Since the governing equations (1a)-(1b) are the same for all fluid-dynamical problems, the type of base flow depends on the boundary conditions (1c).

If the domain is infinite or periodic in a direction (so that there are no distinguished points in that direction), then the base flow will be *homogeneous* in that direction, i.e. it will not vary in that direction. If there are boundaries in a direction, then a non-zero solution will have a non-trivial dependence in that direction.

The linear stability problem depends on both the geometry, i.e. the boundary conditions (3a), and also on the base flow $\mathbf{U}$. Solutions, i.e. eigenvectors, will have a non-trivial dependence in the directions in which there are finite boundaries or on which the base flow depends. In other *homogeneous* directions, the spatial dependence of the eigenvectors must be exponential or trigonometric. If one seeks an eigenvector which is spatially bounded, then the spatial dependence is assumed to be trigonometric. For example, eigenvectors of the Taylor-Couette problem are of the form $\mathbf{u}(r)\exp(i(kz+m\theta))$, where either or both of k, m may be zero, while eigenvectors of the cylinder wake problem are of the form $\mathbf{u}(x,y)\exp(i\beta z)$.

Going one step further, if instead of being homogeneous, the problem is periodic in time or space via either a force, the boundary conditions, or the base state, then the associated linear stability problem is called a *Floquet problem.* If the problem is periodic in time with period T, then its solution has temporal dependence $\exp(\lambda t)f(t)$, where f is a periodic function with period T called the Floquet function, λ is called the Floquet exponent, and $\exp(\lambda T)$ is called the Floquet multiplier. If the linear problem is spatially periodic in, for example, z with wavelength L_z then its solution has spatial dependence $\exp(i\beta z)f(z)$ where f has periodicity length L_z and we have assumed that the Floquet exponent $i\beta$ is imaginary so that the solution is bounded in z.

Concerning numerical methods, the base state can be found by time integration of (1a), (1b), (1c) or by using Newton's method on (2), (1b), (1c). The linear stability problem can be solved by time integration of (3a), (3b), (3c) or by finding the eigenvalues of (4), (3b), (3c) via the power method, Arnoldi's method, or the RG algorithm. The choice and feasibility of the numerical methods depends on the size of the problem, and changes with time as computers become more powerful.

Some basic history

The first linear stability analyses in fluid dynamics probably date from the early 1900s and were carried out by researchers like Rayleigh, Taylor, and later, Chandrasekhar, and, of course, many others. These analyses treated a one-dimensional analytically determined base flow and, correspondingly, an eigenvector which was inhomogeneous in the same dimension and homogeneous or trigonometric in other directions.

In the 1980s, increased computing power led to the possibility of calculating two-dimensional base flows numerically, in cases such as the flow past a circular cylinder. The corresponding eigenvectors are necessarily also two-dimensional and homogeneous or trigonometric in the third direction.

At around the same time, techniques were developed for approximating certain two-dimensional base flows, called *weakly non-parallel*, as a sequence of one-dimensional flows, each located at different values, called *stations*, of a second dimension, then studying the stability of each one-dimensional flow separately and assembling and interpreting the results, leading to classifications such as absolute and convective, local and global. The weakly non-parallel approach led to important advances in understanding, but is not applicable to strongly two-dimensional or three-dimensional flows.

In the 2000s, the numerical computation of three-dimensional base states and of their stability became feasible. At around the same time, some researchers in the weakly non-parallel community turned to the full and exact computation of two- or three-dimensional base states and their stability. These researchers used the term “global” to distinguish this from the weakly non-parallel approach. This nomenclature has a number of problems:

–It leads to historical inaccuracy. New researchers often date the advent of two- or three-dimensional stability analysis to the time at which the term “global” began to be used, and mistakenly attribute its development to the authors who initiated the use of this term.

–It leads to confusion. New researchers often believe that “global stability analysis” comprises a specialized theory, rather than referring merely to the numerical solution of equations (1) and (3).

–It privileges the weakly non-parallel approach, describing full and exact analysis as a special or competing technique. It is as though the WKB approach to solving differential equations was treated as fundamental, and other non-WKB techniques were given a special name.

–It unnecessarily overloads the word “global”, which already has other uses in fluid dynamics. For example, it is used to describe a type of bifurcation that does not involve eigenvalue crossing as well as an oceanographic circulation.

The computation and linear stability analysis of a two-dimensional or three-dimensional base state should be called precisely what they are, i.e. 2D or 3D, full, exact, numerical. These terms are descriptive and logical and do not lead to confusion or historical inaccuracy.

Bibliography

These pages contain a very partial bibliography of numerical computation of two- and three-dimensional base flows and their linear stability from the 1980s through the early 2000s.

2D Base State, 2D Eigenvector

L.E. Eriksson & A. Rizzi, Computer-aided analysis of the convergence to steady state of discrete approximations to the Euler equations, Journal of Computational Physics, 57, 90-128 (1985).

L.S. Tuckerman & P.S. Marcus, *Formation of Taylor vortices in spherical Couette flow*, in Lecture Notes in Physics (No. 218): Proc. of the Ninth Int'l. Conf. on Numerical Methods in Fluid Dynamics, ed. by Soubbarameyer & J.P. Boujot, (Springer, New York, 1985), p. 552–556.

C.P. Jackson, A finite-element study of the onset of vortex shedding in flow past variously shaped bodies, Journal of Fluid Mechanics 182, 23-45 (1987).

A. Zebib, Stability of viscous flow past a circular cylinder, Journal of Engineering Mathematics 21, 155-165 (1987).

I. Goldhirsch, S.A. Orszag & B.K. Maulik, An efficient method for computing leading eigenvalues and eigenvectors of large asymmetric matrices, Journal of Scientific Computing 2, 33-58 (1987).

P.S. Marcus & L.S. Tuckerman, Simulation of flow between concentric rotating spheres. Part 1. Steady states, Journal of Fluid Mechanics 185, 1-30 (1987).

P.S. Marcus & L.S. Tuckerman, Simulation of flow between concentric rotating spheres. Part 2. Transitions, Journal of Fluid Mechanics 185, 31-65 (1987).

K.N. Christodoulou & L.E. Scriven, Finding leading modes of a viscous free surface flow: an asymmetric generalized eigenproblem, Journal of Scientific Computing 3, 355-406 (1988).

D.S. Riley & K.H. Winters, Modal exchange mechanisms in Lapwood convection, Journal of Fluid Mechanics 204, 325–358 (1989).

P.M. Gresho, D. K. Gartling, J.R. Torczynski, K.A. Cliffe, K.H. Winters, T. J. Garratt, A. Spence, J. W. Goodrich, Is the steady viscous incompressible two-dimensional flow over a backward-facing step at Re=800 stable?, International Journal for Numerical Methods in Fluids 17, 501–541 (1993).

N. Ramanan, G. M. Homsy, Linear stability of lid-driven cavity flow, Physics of Fluids 6, 2690 (1994).

W.S. Edwards, L.S. Tuckerman, R.A. Friesner & D. Sorensen, Krylov Methods for the Incompressible Navier-Stokes Equations, J. Comput. Phys. 110, 82-102 (1994).

A. Yu. Gelfgat & I. Tanasawa, Numerical Analysis of Oscillatory Instability of Buoyancy Convection with the Galerkin Spectral Method, Numerical Heat Transfer. Part A: Applications, 25, 527–648 (1994).

C.K. Mamun & L.S. Tuckerman, Asymmetry and Hopf bifurcation in spherical Couette flow, Physics of Fluids 7, 80-91 (1995).

M.F. Schatz, D. Barkley, & H.L. Swinney, Instabilities in spatially periodic channel flow, Physics of Fluids 7, 344-358 (1995).

T. Dauxois, S. Fauve & L.S. Tuckerman, Stability of periodic arrays of vortices, Physics of Fluids 8, 487-495 (1996).

S. Xin, P. Le Quéré & L.S. Tuckerman, Bifurcation analysis of double-diffusive convection with opposing horizontal thermal and solutal gradients, Physics of Fluids 10, 850-858 (1998).

A. Bergeon, D. Henry, H. BenHadid, & L.S. Tuckerman, Marangoni convection in binary mixtures with Soret effect, Journal of Fluid Mechanics 375, 143-177 (1998).

K.A. Cliffe, A. Spence, S.J. Tavener, The numerical analysis of bifurcation problems with application to fluid mechanics, Acta Numerica 39-131 (2000) (Review article)

L.S. Tuckerman & D. Barkley, Bifurcation analysis for timesteppers, in Numerical Methods for Bifurcation Problems and Large-Scale Dynamical Systems, ed. by E. Doedel & L.S. Tuckerman (Springer, New York, 2000), p. 452–466. (Review article)

2D Base State, 3D Eigenvector (trigonometric in third direction)

R.T. Pierrehumbert & S.E. Widnall, The two- and three-dimensional instabilities of a spatially periodic shear layer, Journal of Fluid Mechanics 114, 59-82 (1982).

C. Amon & A.T. Patera, Numerical calculation of stable three-dimensional tertiary states in grooved-channel flow, Physics of Fluids A 1, 2005 (1989).

R. Natarajan & A. Acrivos, The instability of the steady flow past spheres and disks, Journal of Fluid Mechanics 254, 323–344 (1993).

C.K. Aidun, N.G. Triantafillopoulos, J.D. Benson, Global stability of a lid-driven cavity with throughflow: Flow visualization studies, Physics of Fluids A 3, 2091 (1991).

N. Ramanan, G. M. Homsy, Linear stability of lid-driven cavity flow, Physics of Fluids 6, 2690 (1994).

D. Barkley & R.D. Henderson, Three-dimensional Floquet stability analysis of the wake of a circular cylinder Journal of Fluid Mechanics, 322, 215-241 (1996).

R.D. Henderson & D. Barkley, Secondary instability in the wake of a circular cylinder, Physics of Fluids 8, 1683-1685 (1996).

D. Barkley & L.S. Tuckerman, Stability analysis of perturbed plane Couette flow, Physics of Fluids 11, 1187-1195 (1999).

A. Yu. Gelfgat, P.Z. Bar-Yoseph, A. Solan, Three-dimensional instability of axisymmetric flow in a rotating lid-cylinder container, Journal of Fluid Mechanics, 438, 363-377 (2001).

D. Barkley, M.G.M. Gomes & R.D. Henderson, Three-dimensional instability in flow over a backward-facing step, Journal of Fluid Mechanics 473, 167-190 (2002).

C. Nore, L.S. Tuckerman, O. Daube & S. Xin, The 1:2 mode interaction in exactly counter-rotating von Kármán swirling flow, Journal of Fluid Mechanics 477, 51-88 (2003).

C. Nore, M. Tartar, O. Daube & L.S. Tuckerman, Survey of instability thresholds of flow between exactly counter-rotating disks, Journal of Fluid Mechanics 511, 45-65 (2004).