Polar Coordinates

Laurette S. Tuckerman

http://blog.espci.fr/laurette

Functions of two Cartesian coordinates (x, y) can be written in the basis of *monomials* $x^k y^n$. This is a *tensor product*, in that we have chosen a basis $X_K \equiv \{x^k, k = 0 \ldots K\}$ for functions of x, a basis $Y_N \equiv \{y^n, n = 0 \ldots N\}$ for functions in y, and our basis for functions of (x, y) is $X_K \times Y_M = \{x^k y^n, k \leq K, n \leq N\}$, consisting of all possible products of a function from X_K and a function from Y_N. In polar coordinates (r, θ), the azimuthal direction is periodic, and so functions of θ are written as series of trigonometric functions $\cos(m\theta)$, $\sin(m\theta)$ or, more compactly, $\exp(im\theta)$. The dependence in the radial direction can be described using polynomials r^j. However, a simple tensor product basis $R_J \times \Theta_M = \{r^j e^{im\theta}, j \leq J, |m| \leq M\}$ cannot be used in polar coordinates, because the radial and azimuthal directions are linked. In fact, most (specifically 3/4) of the functions in $R_J \times \Theta_M$ are not smooth, meaning not infinitely differentiable.

Consider the simple innocuous-seeming function r. In polar coordinates this function describes a cone, which is continuous but certainly not differentiable at $r = 0$, as shown on the left in figure 1. The source of this singularity is that r is taken to be positive in polar coordinates, with $r = +\sqrt{x^2 + y^2}$. Another example is $\cos(\theta)$. (It will not be necessary to distinguish between $e^{im\theta}$, $\cos(m\theta)$, and $\sin(m\theta)$.) Since θ takes on all values in $[0, 2\pi]$ around any circle of arbitrarily small radius (and is not actually defined at the origin), $\cos(\theta)$ takes on all values in $[-1, 1]$, and is thus also singular when considered as a function of (r, θ); it is actually discontinuous, as shown on the right in figure 1. We will now derive in several equivalent ways the conditions which (j, m) must satisfy in order for $r^j e^{im\theta}$ to be smooth.

Figure 1: Left: The function $r = \sqrt{x^2 + y^2}$ describes a cone in polar coordinates, whose cross-section with the x-axis is shown above. Right: The function $\cos(\theta) = x/r$ is undefined in polar coordinates at the origin. Its cross-section with the x-axis, shown above, is discontinuous.

1 Conversion of Cartesian to polar coordinates

Since we know that the monomials $x^k y^n$ form a complete basis for smooth functions in the plane, we proceed by converting these to polar coordinates.

$$\begin{aligned}
x^k y^n &= (r\cos\theta)^k (r\sin\theta)^n \\
&= r^{k+n}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^k \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^n \\
&= \left(\frac{r}{4}\right)^{k+n}(-i)^n \sum_{k'=0}^{k}\binom{k}{k'} e^{ik'\theta}e^{-i(k-k')\theta}\sum_{n'=0}^{n}\binom{n}{n'} e^{in'\theta}(-1)^{n-n'}e^{-i(n-n')\theta} \\
&= \left(\frac{r}{4}\right)^{k+n}(-i)^n \sum_{k'=0}^{k}\sum_{n'=0}^{n}\binom{k}{k'}\binom{n}{n'}(-1)^{n-n'}e^{i(2k'-k+2n'-n)\theta} \qquad (1)
\end{aligned}$$

We see from this sum that the wavenumbers $2k'-k+2n'-n$ permitted in the trigonometric series multiplying r^{k+n} are constrained to be between $-(k+n)$ and $k+n$ and that they differ from $k+n$ by the even integer $2(k'+n')$. Defining $j \equiv k+n$ and $m \equiv 2k'-k+2n'-n$, we obtain as a smooth expansion for a function of r and θ:

$$f(r,\theta) = \sum_{j=0}^{\infty} \sum_{\substack{m=-j \\ j+m \text{ even}}}^{j} f_{jm} r^j e^{im\theta} = \sum_{m=-\infty}^{\infty} \sum_{\substack{j=|m| \\ j+m \text{ even}}}^{\infty} f_{jm} r^j e^{im\theta} \qquad (2)$$

It can be shown that $f_{j,-m} = f^*_{j,m}$ so that (2) is real.

We may also derive this result without using complex variables: For n odd,

$$\begin{aligned}
x^k y^n &= (r\cos\theta)^k (r\sin\theta)^n \\
&= r^{k+n}\sum_{k'=0}^{[k/2]} c_{kk'}\cos((k-2k')\theta)\sum_{n'=0}^{[n/2]} s_{nn'}\sin((n-2n')\theta) \\
&= r^{k+n}\sum_{k'=0}^{[k/2]}\sum_{n'=0}^{[n/2]}\frac{c_{kk'}s_{nn'}}{2}\big(\sin((k-2k'+n-2n')\theta)+\sin((-k+2k'+n-2n')\theta)\big) \qquad (3)
\end{aligned}$$

where $[k/2]$ means the integer part of $k/2$. For n even,

$$\begin{aligned}
x^k y^n &= (r\cos\theta)^k (r\sin\theta)^n \\
&= r^{k+n} \sum_{k'=0}^{[k/2]} c_{kk'} \cos((k-2k')\theta) \sum_{n'=0}^{[n/2]} {s'}_{nn'} \cos((n-2n')\theta) \\
&= r^{k+n} \sum_{k'=0}^{[k/2]} \sum_{n'=0}^{[n/2]} \frac{c_{kk'} {s'}_{nn'}}{2} \big(\cos((k-2k'+n-2n')\theta) + \cos((k-2k'-n+2n')\theta)\big) \qquad (4)
\end{aligned}$$

The wavenumbers $k-2k'+n-2n'$, $|-k+2k'+n-2n'|$ have a maximum value of $k+n$ and have the same parity as $k+n$ (which is the same as that of $k-n$). We obtain as a smooth expansion for a function of r and θ:

$$\begin{aligned}
f(r,\theta) &= \sum_{j=0}^{\infty} \sum_{\substack{m=0 \\ j+m \text{ even}}}^{j} r^j \left(f^c_{jm}\cos(m\theta) + f^s_{jm}\sin(m\theta)\right) \qquad (5) \\
&= \sum_{m=0}^{\infty} \sum_{\substack{j=m \\ j+m \text{ even}}}^{\infty} r^j \left(f^c_{jm}\cos(m\theta) + f^s_{jm}\sin(m\theta)\right) \qquad (6)
\end{aligned}$$

2 Action with differential operators

The monomials $x^k y^n$ are easily differentiated. For example:

$$\frac{\partial}{\partial x}(x^k y^n) = k x^{k-1} y^n \qquad (7)$$

This function is not singular at $x=0$, since, even when $k-1$ is a negative exponent, the coefficient k is zero. The analogous result does not hold for arbitrary functions $r^j e^{im\theta}$. For example,

$$\begin{aligned}
\nabla\left(r^j e^{im\theta}\right) &= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right)\left(r^j e^{im\theta}\right) \\
&= (\mathbf{e}_r j + \mathbf{e}_\theta i m) r^{j-1} e^{im\theta} \qquad (8)
\end{aligned}$$

whose θ component is singular at $r=0$ for $j=0$, $m\neq 0$. (Indeed, we have already seen that the azimuthal wavenumber m must not exceed the radial exponent j.)

$$
\begin{aligned}
\Delta\left(r^j e^{im\theta}\right) &= \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(r^j e^{im\theta}\right)\\
&= \left(j(j-1)+j-m^2\right) r^{j-2}e^{im\theta}\\
&= \left(j^2-m^2\right) r^{j-2}e^{im\theta}
\end{aligned}
\tag{9}
$$

This result will not be singular if we require that $m = \pm j$ for $j = 0$ or $j = 1$. The functions whose laplacians are not singular are $f_{00} = r^0 e^{i0\theta} = 1$ or $f_{1,\pm 1} = re^{\pm i\theta} \sim x, y$ or any $r^j e^{im\theta}$ for $j \geq 2$. Taking the laplacian again, we obtain:

$$
\nabla^4\left(r^j e^{im\theta}\right) = (j^2-m^2)((j-2)^2-m^2)\, r^{j-4}e^{im\theta} \tag{10}
$$

This result will not be singular if we require that $m = \pm j$ or $m = \pm(j-2)$ for $j \leq 4$. Continuing in this way, we obtain that, for the function to be infinitely differentiable, we require $m = \pm j$, $m = \pm(j-2)$, $m = \pm(j-4)$, or $\cdots$ so that, as before:

$$
f(r,\theta) = \sum_{j=0}^{\infty} \sum_{\substack{m=-j \\ j+m \text{ even}}}^{j} f_{jm} r^j e^{im\theta} = \sum_{m=-\infty}^{\infty} \sum_{\substack{j=|m| \\ j+m \text{ even}}}^{\infty} f_{jm} r^j e^{im\theta} \tag{11}
$$

3 Expansions of Bessel functions

The eigenfunctions of the Laplacian in polar coordinates are products of Bessel functions in radius and trigonometric or exponential or linear functions in θ. If θ covers the entire circle, rather than a wedge, then the functions of θ are the trigonometric functions $e^{im\theta}$, with m integer. If the domain is a circle, rather than an annulus, then the appropriate Bessel functions, which are nonsingular at the origin, are $J_m(\lambda r)$ or $I_m(\lambda r)$, where the possible values of λ are determined by the boundary conditions at the outer boundary of the circle.

The Bessel functions $J_m(\lambda r)$ and $I_m(\lambda r)$ can be expanded as power series:

$$
\left\{ \begin{array}{c} J_m \\ I_m \end{array} \right\}(\lambda r) = \left(\frac{\lambda r}{2}\right)^m \sum_{j=0}^{\infty} \frac{(\mp\lambda r/2)^{2j}}{j!\,\Gamma(m+j+1)} \tag{12}
$$

This power series obeys the same relations between radial exponent $m+2j$ and angular wavenumber m as seen in the previous sections.

4 Rapid oscillation

Trigonometric functions with wavenumber m contain m oscillations. As we approach the origin, these oscillations become compressed over a circle of decreasing circumference. The requirement that the radial function multiplying $e^{im\theta}$ have r^m as its lowest order term forces the amplitude of these oscillations to be quickly damped as r approaches zero. This is illustrated in figure 2.

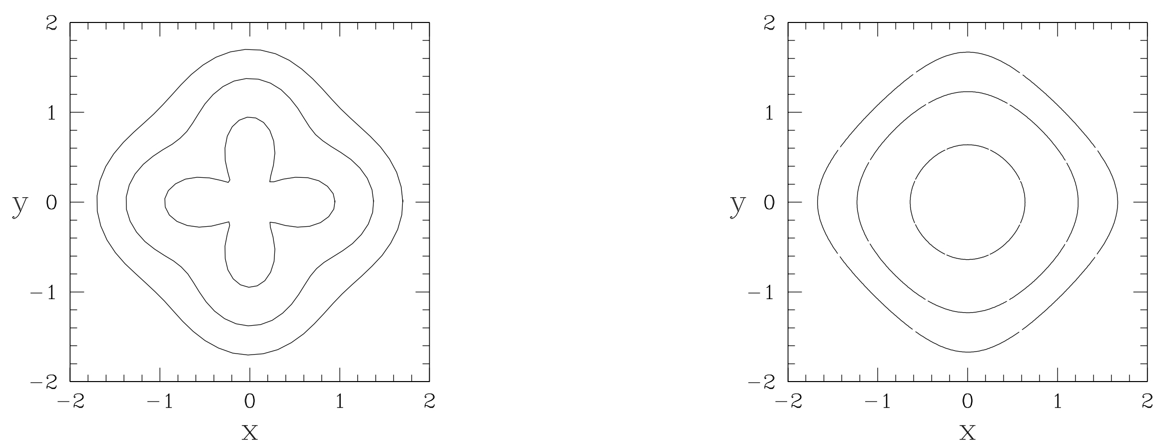

Figure 2: Left: Contour plot of $f = r^2 + \frac{1}{2}\cos(4\theta)$. The radial extent of the oscillations of the contours remain constant as $r \to 0$, leading to a discontinuity at the center. Right: Contour plot of $f = r^2 - \frac{1}{10}r^4\cos(4\theta)$. The radial extent of the oscillations of the contours are sufficiently damped as $r \to 0$ for f to be, not only continuous, but analytic at the origin.

5 Pole problem on a sphere

An analogous problem exists at the poles in spherical coordinates (ρ, ξ, ϕ), with the longitudinal ϕ playing the role of the polar θ and the latitude ξ playing the role of the polar r. There, the basis functions are spherical harmonics $Y_l^m(\xi, \phi) \sim P_{lm}(\cos(\xi))e^{im\phi}$, where P_l^m are associated Legendre functions and $|m| \le l$. The inequality $|m| \le l$ governing the spherical harmonics is analogous to the requirement $|m| \le j$ that exists for polar coordinates. Indeed, using the substitutions

$$\begin{aligned} r &= \rho\sin(\xi) \\ z &= \rho\cos(\xi) \end{aligned} \tag{13}$$

and expressing the associated Legendre functions in cylindrical coordinates as

$$\begin{aligned} P_0^0 &\sim 1 \\ P_1^0 &\sim z \\ P_1^{\pm 1} &\sim r \\ P_2^0 &\sim (2z^2 - r^2) \\ P_2^{\pm 1} &\sim z\,r \\ P_2^{\pm 2} &\sim r^2 \end{aligned} \tag{14}$$

we see that P_l^m includes only powers of r which are greater than m.

6 Boundary conditions for Laplace's and Poisson's equations

Suppose we wish to solve:

$$\Delta f \equiv (\frac{1}{r}\partial_r r \partial_r - \frac{1}{r^2}\partial_\theta^2) f = g \tag{15}$$

in polar coordinates on the disk $r \leq 1$. We decompose f and g into Fourier series in θ, with coefficient functions multiplying $e^{im\theta}$. The equations for each m decouple. For fixed m, we write f, g as polynomials in r, beginning with the term r^m which insures regularity at the origin:

$$\begin{aligned} f(r)e^{im\theta} &= (f_m r^m + f_{m+2} r^{m+2} + f_{m+4} r^{m+4} + f_{m+6} r^{m+6} + \ldots) e^{im\theta} \\ g(r)e^{im\theta} &= (g_m r^m + g_{m+2} r^{m+2} + g_{m+4} r^{m+4} + g_{m+6} r^{m+6} + \ldots) e^{im\theta} \end{aligned} \tag{16}$$

In terms of the expansions (6), equation (15) becomes:

$$\begin{aligned} (m^2 - m^2) f_m\, r^{m-2} + (m+2)^2 f_{m+2}\, r^m \quad & + (m+4)^2 f_{m+4}\, r^{m+2} + (m+6)^2 f_{m+6}\, r^{m+6} + \ldots \\ = g_m\, r^m \quad & + g_{m+2}\, r^{m+2} + g_{m+4}\, r^{m+4} + g_{m+6}\, r^{m+6} + \ldots \end{aligned} \tag{17}$$

which is written for each power in r as:

$$\begin{aligned} (m+2)^2 f_{m+2} &= g_m \\ (m+4)^2 f_{m+4} &= g_{m+2} \\ (m+6)^2 f_{m+6} &= g_{m+4} \end{aligned} \tag{18}$$

We see that f_m is not determined by the equation and another condition must be added. The Dirichlet boundary condition $f(r = 1) = a$ determines f_m:

$$f_m + f_{m+2} + f_{m+4} + f_{m+6} + \ldots = a \tag{19}$$

The Neumann boundary condition $\partial_r f(r=1) = a$

$$mf_m + (m+2)f_{m+2} + (m+4)f_{m+4} + (m+6)f_{m+6} + \ldots = a \tag{20}$$

determines f_m if and only if $m > 0$. For the axisymmetric component, $m = 0$, the Neumann boundary condition neither determines f_m nor is automatically compatible with the conditions (6) derived from the differential equation. Substituting (6) and $m = 0$ into (20), we obtain:

$$\frac{g_0}{2} + \frac{g_2}{4} + \frac{g_4}{6} + \ldots = a \tag{21}$$

which can be written in terms of the function g as:

$$\int_{r=0}^{1} r\; dr g(r) = a \tag{22}$$

This is the well-known compatibility condition for a general domain:

$$\int dv\; g = \int dv\; \Delta f = \int ds\; \hat{\mathbf{n}} \cdot \nabla f \tag{23}$$

which is written for the disk $r \leq 1$ as:

$$\int_{r=0}^{1} r\; dr \int_{\theta=0}^{2\pi} d\theta\; g(r,\theta) = \int_{\theta=0}^{2\pi} d\theta \partial_r f \tag{24}$$

(Note that for $m \neq 0$, integrals over θ yield zero.) Thus, for $m = 0$, the Neumann boundary condition is a condition on g, not on f, and determination of f_0 requires an additional condition, such as a Dirichlet condition applied at some r value.

Consider specifically Laplace's equation and homogeneous Neumann boundary conditions. For the non-axisymmetric case, $m > 0$, conditions (6) with $g_m = g_{m+2} = \ldots = 0$ and Neumann boundary condition (20) with $a = 0$ allows us to conclude that $f = 0$. For the axisymmetric case, $m = 0$, we may conclude from (6) only, i.e. from Laplace's equation itself, that the solution is $f = f_0$, i.e. that f is constant.

7 Functions on an annulus

The considerations above apply to a disk which contains the origin $r = 0$. An annulus $r_{\text{in}} \leq r_{\text{out}}$ does not contain a coordinate singularity and functions may be expanded using all products of r^j and $e^{im\theta}$:

$$f(r,\theta) = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} f_{jm} r^j e^{im\theta} = \sum_{m=-\infty}^{\infty} \sum_{j=0}^{\infty} f_{jm} r^j e^{im\theta} \tag{25}$$

A disk has only one geometric boundary and only one boundary condition must be imposed in order to solve Laplace's or Poisson's equation. (Regularity can be said to supply an additional boundary condition at the origin.) But on an annulus, boundary conditions must be imposed at both the inner and outer radii.

8 Vector fields

The conclusions above apply to scalar fields, as well as to the u^z component of a vector field in cylindrical coordinates. For components u^r and u^θ of a vector field $\mathbf{u}$, different rules apply. We have

$$\begin{aligned} u^r &= \cos(\theta)u^x + \sin(\theta)u^y \\ u^\theta &= \sin(\theta)u^x - \cos(\theta)u^y \end{aligned} \tag{26}$$

Expanding u^x and u^y according to (2) leads to

$$u^{r,\theta} = \sum_{j=0}^{\infty} \sum_{\substack{m=-|j+1| \\ j+m \text{ odd}}}^{|j+1|} u_{jm}^{r,\theta} r^j e^{im\theta} = \sum_{m=-\infty}^{\infty} \sum_{\substack{j=|m-1| \\ j+m \text{ odd}}}^{\infty} u_{jm}^{r,\theta} r^j e^{im\theta} \tag{27}$$

The vector Laplacian couples u^r and u^θ: Hence the vector Poisson equation

$$\begin{bmatrix} \Delta - \frac{1}{r^2} & -\frac{2}{r^2}\partial_\theta & 0 \\ \frac{2}{r^2}\partial_\theta & \Delta - \frac{1}{r^2} & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} u^r \\ u^\theta \\ u^z \end{bmatrix} = \begin{bmatrix} g^r \\ g^\theta \\ g^z \end{bmatrix} \tag{28}$$

must be transformed in some way in order to solve for (u^r, u^θ). One possibility is to diagonalize the two-by-two block. If we define

$$\begin{aligned} u^\pm &= u^r \pm i u^\theta \\ g^\pm &= g^r \pm i g^\theta \end{aligned} \tag{29}$$

$$\begin{aligned} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \begin{bmatrix} \Delta - \frac{1}{r^2} & -\frac{2}{r^2}\partial_\theta \\ \frac{2}{r^2}\partial_\theta & \Delta - \frac{1}{r^2} \end{bmatrix} \frac{1}{2} \begin{bmatrix} I & I \\ -iI & I \end{bmatrix} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \begin{bmatrix} u^r \\ u^\theta \end{bmatrix} &= \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \begin{bmatrix} g^r \\ g^\theta \end{bmatrix} \\ \begin{bmatrix} \Delta - \frac{1}{r^2} + \frac{2i}{r^2}\partial_\theta & 0 \\ 0 & \Delta - \frac{1}{r^2} - \frac{2i}{r^2}\partial_\theta \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix} &= \begin{bmatrix} g^+ \\ g^- \end{bmatrix} \end{aligned} \tag{30}$$

For the m^{th} Fourier component, this becomes

$$\begin{bmatrix} \Delta - \frac{1}{r^2} - \frac{2m^2}{r^2} & 0 \\ 0 & \Delta - \frac{1}{r^2} + \frac{2m^2}{r^2} \end{bmatrix} \begin{bmatrix} u_m^+ \\ u_m^- \end{bmatrix} = \begin{bmatrix} g_m^+ \\ g_m^- \end{bmatrix} \tag{31}$$

A second possibility presents itself if the solution $\mathbf{u}$ is required to be solenoidal, i.e. divergence-free, as is the case for an incompressible velocity field or a magnetic field. (The right-hand-side $\mathbf{g}$ must then also be divergence-free.) We may then substitute for u^θ:

$$-\frac{1}{r}\partial_\theta u^\theta = \left(\frac{1}{r}\partial_r r u^r + \partial_z u^z\right) \tag{32}$$

(This subsitution does not work for $m = 0$, but for this component the vector Poisson equation is decoupled.) We thus obtain the equation to be solved for u^r,

$$\begin{aligned} g^r &= (\Delta - \frac{1}{r^2})u^r - \frac{2}{r^2}\partial_\theta u^\theta = (\Delta - \frac{1}{r^2})u^r + \frac{2}{r}\left(\frac{1}{r}\partial_r r u^r + \partial_z u^z\right) \\ &= (\Delta - \frac{1}{r^2} + \frac{2}{r^2}\partial_r r)u^r + \frac{2}{r}\partial_z u^z \end{aligned} \tag{33}$$

making the vector Poisson equation upper triangular:

$$\begin{bmatrix} \frac{1}{r}\partial_\theta & \frac{1}{r}\partial_r r & \partial_z \\ 0 & \Delta - \frac{1}{r^2} + \frac{2}{r^2}\partial_r r & \frac{2}{r}\partial_z \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} u^\theta \\ u^r \\ u^z \end{bmatrix} = \begin{bmatrix} g^\theta \\ g^r \\ g^z \end{bmatrix} \tag{34}$$

References

Codes for solving partial differential equations in polar or cylindrical coordinates deal with the singularity at the origin in various ways. (Surprisingly many treatments are wrong.) For two correct treatments, see, for example:

L.S. Tuckerman, *Divergence-free velocity fields in nonperiodic geometries*, J. Comput. Phys. **80**, 403–441 (1989).

T. Matsushima & P.S. Marcus, *A spectral method for polar coordinates*, J. Comput. Phys. **120**, 365–374 (1995).