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Codimension-two points

The 1:2 mode interaction

System with $O(2)$ symmetry with competing wavenumbers $m = 1$ and $m = 2$
Solutions approximated as:

$$u(\theta, t) = z_1(t)e^{i\theta} + z_2(t)e^{2i\theta} + \bar{z}_1(t)e^{-i\theta} + \bar{z}_2(t)e^{-2i\theta}$$

with $z_1(t) = x_1(t) + iy_1(t) = r_1(t)e^{i\phi_1(t)}$, $z_2(t) = x_2(t) + iy_2(t) = r_2(t)e^{i\phi_2(t)}$

$O(2)$ generated by rotation by θ_0 and reflection about $\theta = 0$:

$$\begin{aligned} S_{\theta_0}(z_1, z_2) &= (e^{i\theta_0} z_1, e^{2i\theta_0} z_2) \\ \kappa(z_1, z_2) &= (\bar{z}_1, \bar{z}_2) \end{aligned}$$

Define

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} \bar{z}_1 z_2 \\ -z_1^2 \end{pmatrix}$$

Show that F is equivariant with respect to $O(2)$:

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\xrightarrow{F} \begin{pmatrix} \bar{z}_1 z_2 \\ -z_1^2 \end{pmatrix} \xrightarrow{\kappa} \begin{pmatrix} z_1 \bar{z}_2 \\ -\bar{z}_1^2 \end{pmatrix} \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\xrightarrow{\kappa} \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} z_1 \bar{z}_2 \\ -\bar{z}_1^2 \end{pmatrix} \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\xrightarrow{F} \begin{pmatrix} \bar{z}_1 z_2 \\ -z_1^2 \end{pmatrix} \xrightarrow{S_{\theta_0}} \begin{pmatrix} e^{i\theta_0} \bar{z}_1 z_2 \\ -e^{2i\theta_0} z_1^2 \end{pmatrix} \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\xrightarrow{S_{\theta_0}} \begin{pmatrix} e^{i\theta_0} z_1 \\ e^{2i\theta_0} z_2 \end{pmatrix} \xrightarrow{F} \begin{pmatrix} e^{-i\theta_0} \bar{z}_1 e^{2i\theta_0} z_2 \\ -e^{2i\theta_0} z_1^2 \end{pmatrix} \end{aligned}$$

Essentially $1 + 1 = 2$ and $2 - 1 = 1$

Dynamical system for evolution of z_1, z_2 is:

$$\begin{aligned} \dot{z}_1 &= \bar{z}_1 z_2 + z_1 (\mu_1 - \alpha_1 |z_1|^2 - \beta_1 |z_2|^2) \\ \dot{z}_2 &= -z_1^2 + z_2 (\mu_2 - \beta_2 |z_1|^2 - \alpha_2 |z_2|^2) \end{aligned}$$

Steady states

(Phase is arbitrary: $z \rightarrow x$)

$$\begin{aligned} 0 &= x_1 \left(x_2 + (\mu_1 - \alpha_1 x_1^2 - \beta_1 x_2^2) \right) \\ 0 &= -x_1^2 + x_2 \left(\mu_2 - \beta_2 x_1^2 - \alpha_2 x_2^2 \right) \end{aligned}$$

Trivial state: $x_1 = x_2 = 0$

Mode-two (“pure mode”) state: $x_1 = 0, x_2 \neq 0$:

$$x_2^2 = \mu_2 / \alpha_2$$

If $x_1 \neq 0$ then $x_2 \neq 0$! Instead, have “mixed-mode state”:

$$\begin{aligned} 0 &= x_2 + (\mu_1 - \alpha_1 x_1^2 - \beta_1 x_2^2) \\ 0 &= -x_1^2 + x_2 \left(\mu_2 - \beta_2 x_1^2 - \alpha_2 x_2^2 \right) \end{aligned}$$

(intersection of two conic sections)

Stability

Jacobian in Cartesian coordinates (even if $y = 0$, Jacobian must include y)

$$\begin{pmatrix} \mu_1 + x_2 - \alpha_1(r_1^2 + 2x_1^2) - \beta_1 r_2^2 & y_2 - 2\alpha_1 x_1 y_1 & x_1 - 2\beta_1 x_1 x_2 & y_1 - 2\beta_1 x_1 y_2 \\ -y_2 - 2\alpha_1 x_1 y_1 & \mu_1 + x_2 - 2\alpha_1(r_1^2 + 2y_1^2) - \beta_1 r_2^2 & y_1 - 2\beta_1 x_2 y_1 & -x_1 - 2\beta_1 y_1 y_2 \\ 2x_1 - 2\beta_2 x_1 x_2 & -2y_1 - 2\beta_2 x_2 y_2 & \mu_2 - \beta_2 r_1^2 - \alpha_2(r_2^2 + 2x_2^2) & -2\alpha_2 x_2 y_2 \\ 2y_1 - 2\beta_2 x_1 y_2 & 2x_1 - 2\beta_2 y_1 y_2 & 2\beta_2 x_2 y_2 & \mu_2 - \beta_2 r_1^2 - \alpha_2(r_2^2 + 2y_2^2) \end{pmatrix}$$

Trivial state:

$$\mathcal{J} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}$$

Two 2D eigenspaces. Circle pitchforks at $\mu_1 = 0$ and $\mu_2 = 0$

Mode-two state: $x_1 = y_1 = y_2 = 0$, $x_2 = \pm\sqrt{\mu_2/\alpha_2}$

$$\begin{aligned} \mathcal{J} &= \begin{pmatrix} \mu_1 + x_2 - \beta_1 x_2^2 & 0 & 0 & 0 \\ \mu_1 & \mu_1 - x_2 - \beta_1 x_2^2 & 0 & 0 \\ 0 & 0 & \mu_2 - 3\alpha_2 x_2^2 & 0 \\ 0 & 0 & 0 & \mu_2 - \alpha_2 x_2^2 \end{pmatrix} \\ &= \begin{pmatrix} \mu_1 \pm \sqrt{\frac{\mu_2}{\alpha_2}} - \beta_1 \frac{\mu_2}{\alpha_2} & 0 & 0 & 0 \\ 0 & \mu_1 \mp \sqrt{\frac{\mu_2}{\alpha_2}} - \beta_1 \frac{\mu_2}{\alpha_2} & 0 & 0 \\ 0 & 0 & -2\mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Eigenvalues $-2\mu_2$ and 0 are usual results of circle pitchfork.

Other two eigenvalues concern instability to (x_1, y_1) .

They are different because mode-two state has a phase (no CP from mode-two).

Mixed-mode branch bifurcates from trivial state at $\mu_1 = 0$

and from mode-two branch at

$$\mu_1 - \beta_1 \frac{\mu_2}{\alpha_2} \pm \sqrt{\frac{\mu_2}{\alpha_2}} = 0$$

Polar representation

$$z_1(t) = r_1(t)e^{i\phi_1(t)}, z_2(t) = r_2(t)e^{i\phi_2(t)}$$

Evolution equations:

$$\begin{aligned}(\dot{r}_1 + ir_1\dot{\phi}_1)e^{i\phi_1} &= r_1e^{-i\phi_1}r_2e^{i\phi_2} + r_1e^{i\phi_1}(\mu_1 - \alpha_1r_1^2 - \beta_1r_2^2) \\ (\dot{r}_2 + ir_2\dot{\phi}_2)e^{i\phi_2} &= -r_1e^{i\phi_1}r_1e^{i\phi_1} + r_2e^{i\phi_2}(\mu_2 - \beta_2r_1^2 - \alpha_2r_2^2)\end{aligned}$$

Dividing equations by $e^{i\phi_1}$ and by $e^{i\phi_2}$:

$$\begin{aligned}\dot{r}_1 + ir_1\dot{\phi}_1 &= r_1r_2e^{i(\phi_2 - 2\phi_1)} + r_1(\mu_1 - \alpha_1r_1^2 - \beta_1r_2^2) \\ \dot{r}_2 + ir_2\dot{\phi}_2 &= -r_1^2e^{i(2\phi_1 - \phi_2)} + r_2(\mu_2 - \beta_2r_1^2 - \alpha_2r_2^2)\end{aligned}$$

Separating real and imaginary parts and dividing imaginary parts by $r_j \neq 0$:

$$\begin{aligned}\dot{r}_1 &= r_1r_2 \cos(\phi_2 - 2\phi_1) + r_1(\mu_1 - \alpha_1r_1^2 - \beta_1r_2^2) \\ \dot{\phi}_1 &= r_2 \sin(\phi_2 - 2\phi_1) \\ \dot{r}_2 &= -r_1^2 \cos(2\phi_1 - \phi_2) + r_2(\mu_2 - \beta_2r_1^2 - \alpha_2r_2^2) \\ \dot{\phi}_2 &= -(r_1^2/r_2) \sin(2\phi_1 - \phi_2)\end{aligned}$$

Substitute $\Phi \equiv 2\phi_1 - \phi_2$:

$$\begin{aligned}\dot{r}_1 &= r_1 (r_2 \cos \Phi + \mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2) \\ \dot{r}_2 &= -r_1^2 \cos \Phi + r_2 (\mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2) \\ \dot{\phi}_1 &= -r_2 \sin \Phi \\ \dot{\phi}_2 &= -(r_1^2/r_2) \sin \Phi\end{aligned}\right\} \implies \dot{\Phi} = -(2r_2 - r_1^2/r_2) \sin \Phi$$

Suppose $\dot{r}_1 = \dot{r}_2 = \dot{\Phi} = 0$, but $r_1, r_2 \neq 0$

$$\begin{aligned}0 &= r_2 \cos \Phi + \mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2 \\ 0 &= -r_1^2 \cos \Phi + r_2 (\mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2) \\ 0 &= (2r_2 - r_1^2/r_2) \sin \Phi\end{aligned}$$

Mixed modes

$\Phi = 0, \pi \implies \sin \Phi = 0 \implies \dot{\phi}_1 = \dot{\phi}_2 = 0 \implies$ steady states:

$$\begin{aligned}0 &= \pm r_2 + \mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2 \\ 0 &= \mp r_1^2 + r_2 (\mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2)\end{aligned}$$

Traveling Waves

$$\begin{aligned} 0 &= \dot{\Phi} = -(2r_2 - r_1^2/r_2) \sin \Phi \\ \sin \Phi \neq 0 \implies 0 &= 2r_2^2 - r_1^2 \implies r_1^2 = 2r_2^2 \\ 0 &= \dot{\Phi} \equiv 2\dot{\phi}_1 - \dot{\phi}_2 \end{aligned}$$

Definition:

$$u(\theta, t) = u(\theta - ct, 0)$$

$$\begin{aligned} u(\theta, t) &= r_1(t)e^{i(\phi_1(t)+\theta)} + r_2(t)e^{i(\phi_2(t)+2\theta)} + \text{complex conjugate} \\ u(\theta - ct, 0) &= r_1(0)e^{i(\phi_1(0)+\theta-ct)} + r_2(0)e^{i(\phi_2(0)+2(\theta-ct))} + \text{complex conjugate} \end{aligned}$$

$$\implies \begin{cases} r_1(t) = r_1(0) \text{ and } \phi_1(t) = \phi_1(0) - ct \\ r_2(t) = r_2(0) \text{ and } \phi_2(t) = \phi_2(0) - 2ct \end{cases}$$

$$\implies 2\phi_1(t) - \phi_2(t) = 2\phi_1(0) - \phi_2(0) \implies \Phi(t) = \Phi(0)$$

$$\begin{aligned}
 0 &= 2r_2^2 - r_1^2 && \implies r_1^2 = 2r_2^2 \\
 0 &= \dot{r}_1 = r_2 \cos \Phi + \mu_1 - \alpha_1 2r_2^2 - \beta_1 r_2^2 \\
 0 &= \dot{r}_2 = -2r_2^2 \cos \Phi + r_2 (\mu_2 - \beta_2 2r_2^2 - \alpha_2 r_2^2)
 \end{aligned}$$

Add **2× blue equation** to $(1/r_2) \times$ **green equation**

$$\begin{aligned}
 0 &= 2\mu_1 + \mu_2 - (4\alpha_1 + 2\beta_1 + 2\beta_2 + \alpha_2)r_2^2 \\
 r_2^2 &= \frac{2\mu_1 + \mu_2}{4\alpha_1 + 2\beta_1 + 2\beta_2 + \alpha_2}
 \end{aligned}$$

Can also obtain:

$$\cos \Phi = \frac{\mu_1(2\alpha_2 + \beta_2) - \mu_2(2\alpha_1 + \beta_1)}{[(2\mu_1 + \mu_2)(4\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2)]^{1/2}}$$

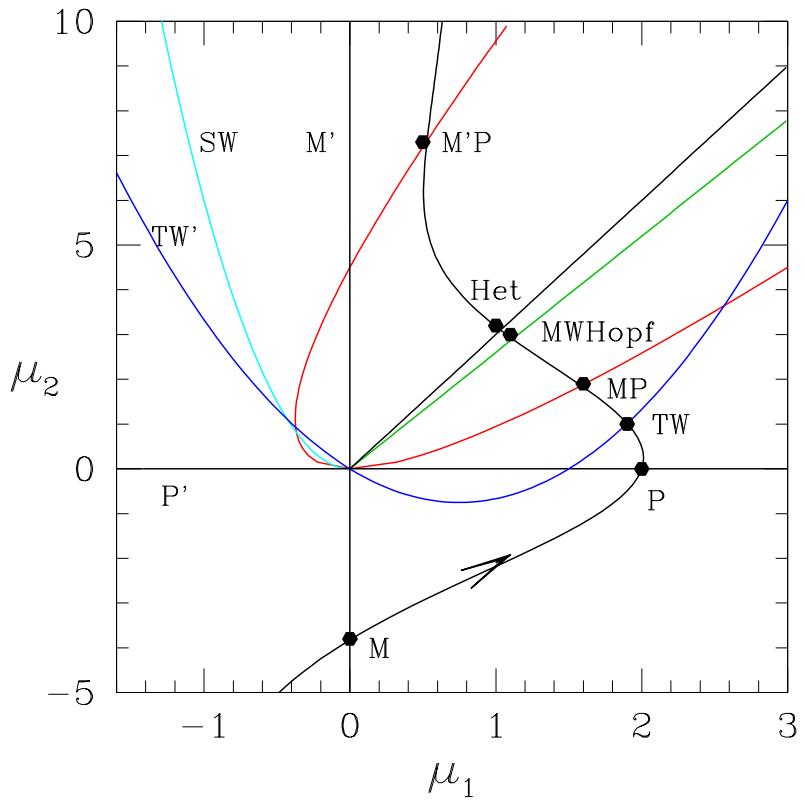
Traveling waves bifurcate from mixed mode branch when $|\cos \Phi| = 1 \iff$
 $(\mu_1(2\alpha_2 + \beta_2) - \mu_2(2\alpha_1 + \beta_1))^2 = (2\mu_1 + \mu_2)(4\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2)$

Time-dependent states

- Traveling waves via drift bif from mixed-mode branch
- Modulated waves via secondary Hopf bif from traveling waves
- Heteroclinic orbit connects two opposite-phase mode-two saddles with eigenvalues $-\lambda_- < 0 < \lambda_+$

Can prove orbit is stable if $\lambda_- > \lambda_+$, i.e. if contraction more important than expansion

$$\begin{aligned} -\left(\mu_1 - \beta_1 \frac{\mu_2}{\alpha_2} - \sqrt{\frac{\mu_2}{\alpha_2}}\right) &> \mu_1 - \beta_1 \frac{\mu_2}{\alpha_2} + \sqrt{\frac{\mu_2}{\alpha_2}} \\ \iff \beta_1 \frac{\mu_2}{\alpha_2} &> \mu_1 \end{aligned}$$



Takens-Bogdanov normal form

Meeting of Hopf and steady bifurcations

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\mu_1 x + \mu_2 y - x^3 - x^2 y\end{aligned}$$

Steady states:

$$\begin{aligned}0 &= y \\ 0 &= -\mu_1 x - x^3 \implies x = \pm\sqrt{-\mu_1}\end{aligned}$$

Jacobian:

$$\begin{aligned}\mathcal{J} &= \begin{pmatrix} 0 & 1 \\ -\mu_1 - 3x^2 - 2xy & \mu_2 - x^2 \end{pmatrix} \\ \mathcal{J}(0, 0) &= \begin{pmatrix} 0 & 1 \\ -\mu_1 & \mu_2 \end{pmatrix} \implies \lambda = \frac{\mu_2}{2} \pm \sqrt{\left(\frac{\mu_2}{2}\right)^2 - \mu_1}\end{aligned}$$

\mathcal{J} is Jordan block at codimension-two point $\mu_1 = \mu_2 = 0$

Hopf bifurcation at $\mu_2 = 0$ for $\mu_1 > 0$

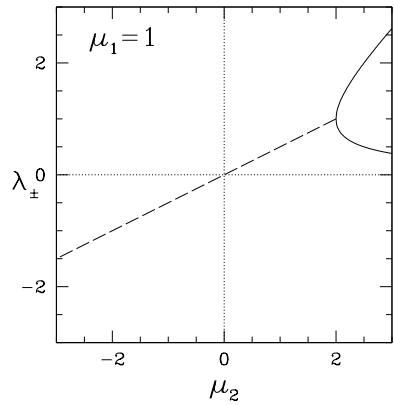
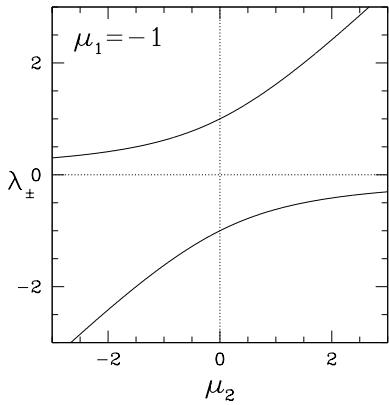
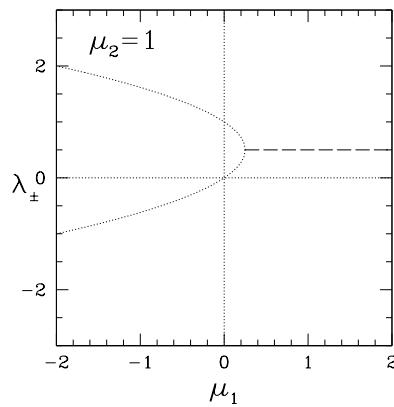
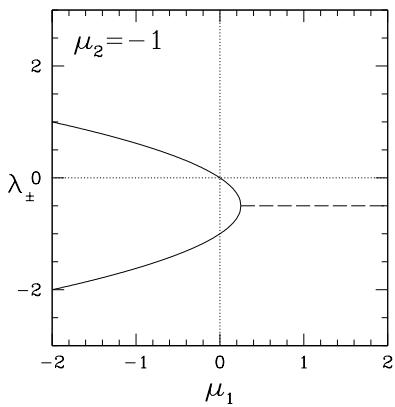
Pitchfork bifurcation at $\mu_1 = 0$

Real eigenvalues coalesce to form complex conjugate pair

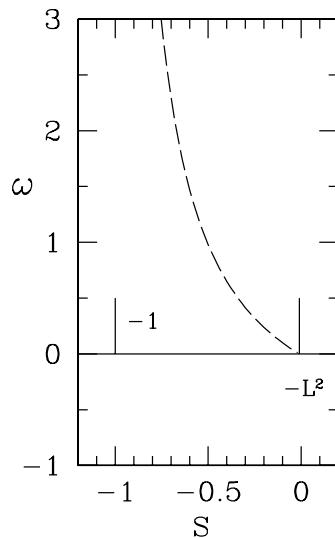
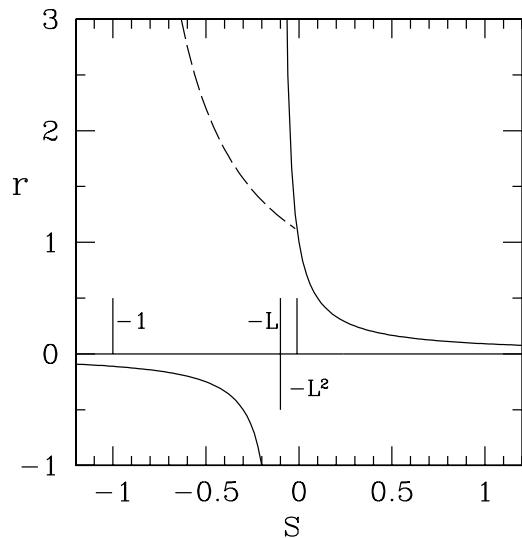
At collision, imaginary part is zero

At a nearby Hopf bifurcation, frequency is near zero

\implies period is near infinity



Takens-Bogdanov codim-two point in thermosolutal convection



Heteroclinic cycles in the French washing machine

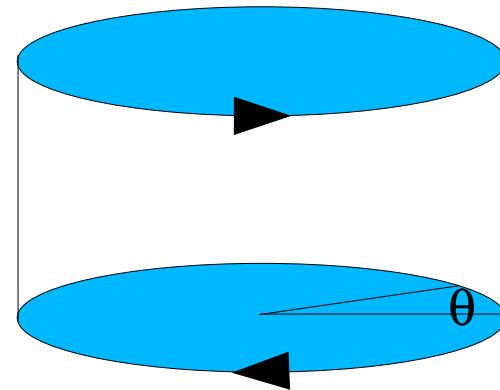
Caroline Nore

Laurette Tuckerman

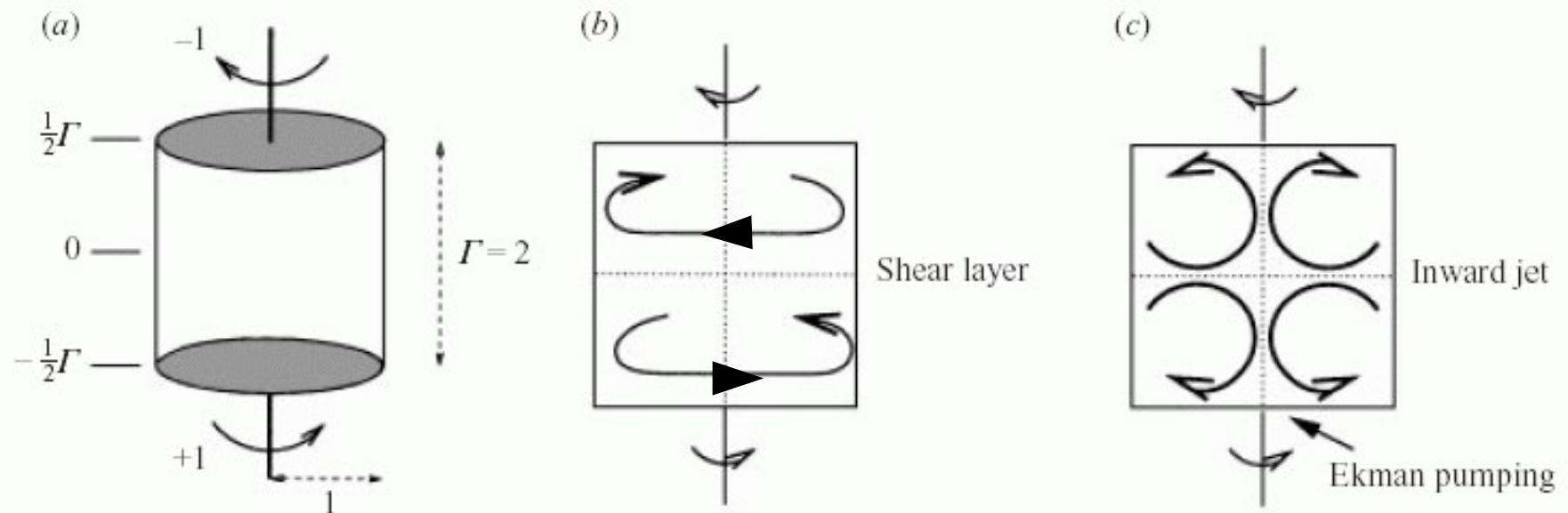
Olivier Daube

Shihe Xin

LIMSI-CNRS, France



The French Washing Machine (von Karman flow)



Symmetry Group:
Rotations in θ and
Combined reflection in z and θ
Rot/Ref don't commute $\Rightarrow O(2)$

*Douady, Brachet, Couder, Fauve et al
Le Gal et al, Rabaud et al, Daviaud et al.
Gelfgat et al, Lopez & Marques et al*

Numerical Methods

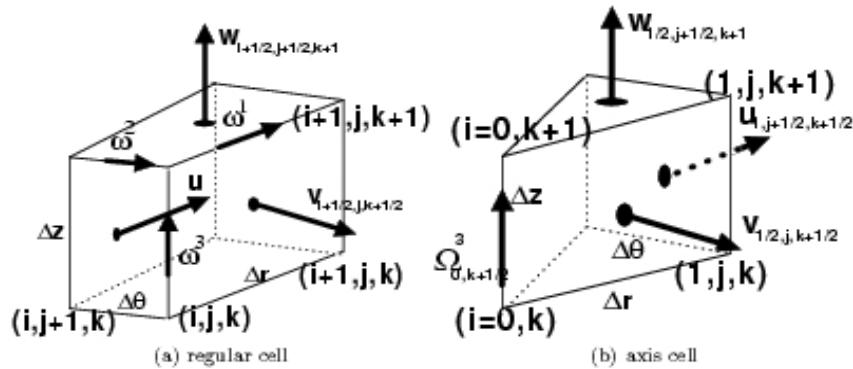


Figure 1: The MAC grid

Time-integration code for Navier-Stokes eqns by Daube

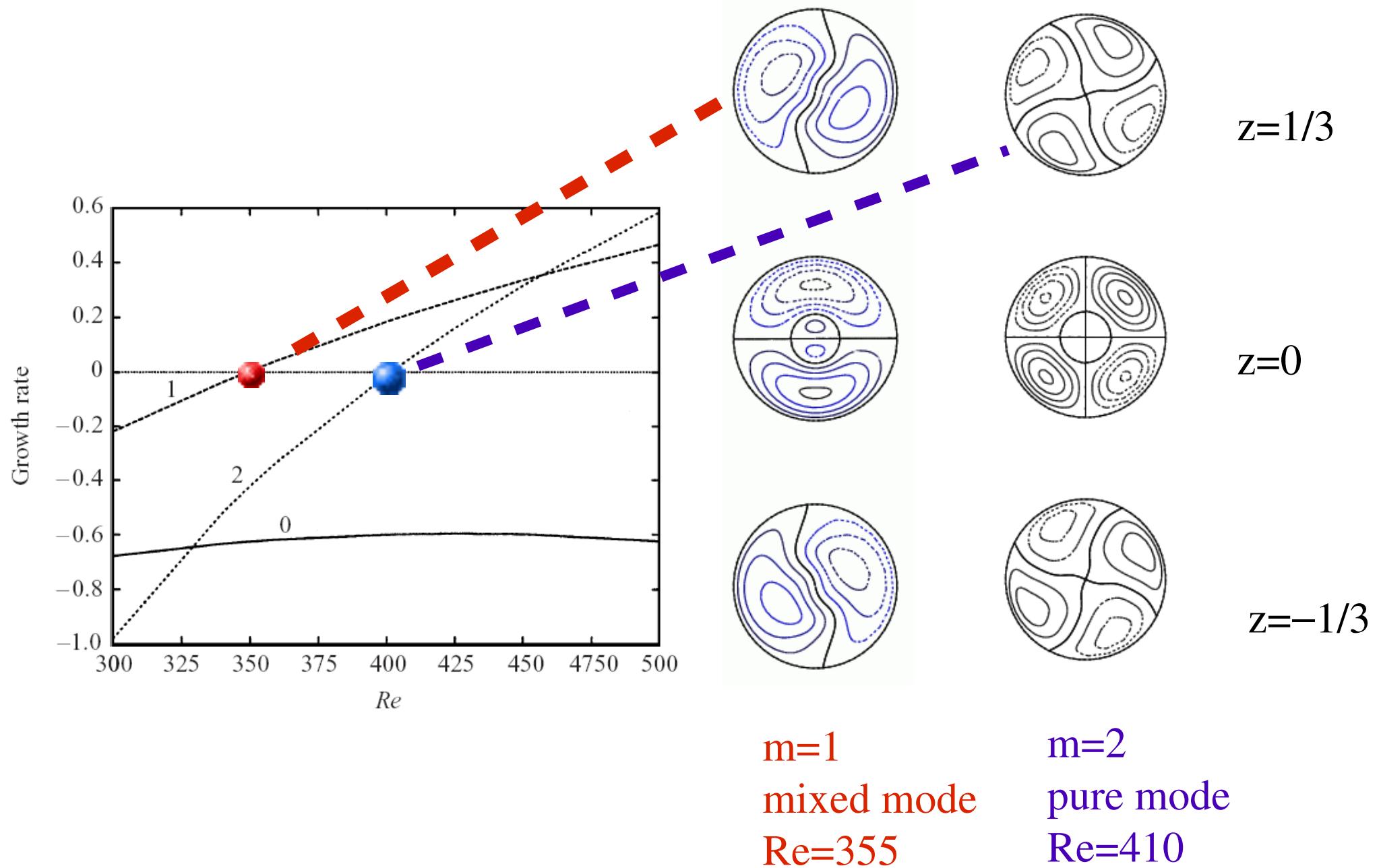
Spatial: finite differences in (r,z) , Fourier in θ

Temporal: 2nd order backward difference formula

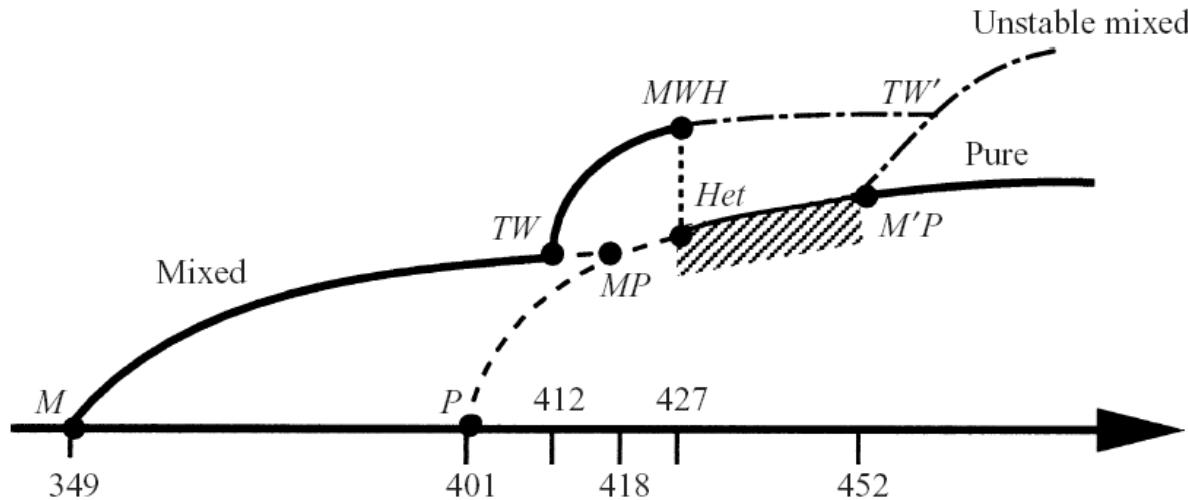
Adaptations:

- Steady state solving via Newton for axisymmetric flows
- Linear stability about axisymmetric and 3D flows

Linear Stability of Basic Axisymmetric Flow



Bifurcation Diagram for 1:2 mode interaction



Normal Form

$$\dot{z}_1 = \overline{z_1} z_2 + z_1 (\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2)$$

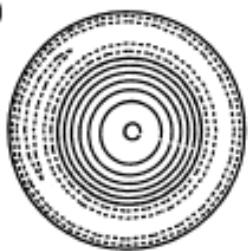
$$\dot{z}_2 = \pm z_1^2 + z_2 (\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2)$$

↑ quadratic terms

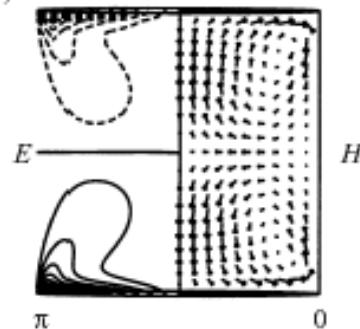
Armbruster, Guckenheimer & Holmes; Proctor & Jones (1988)

Mixed Mode (from m=1 eigenvector)

(a)



(c)



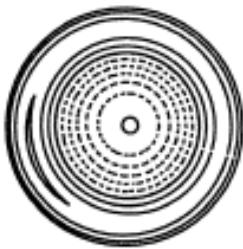
E

H



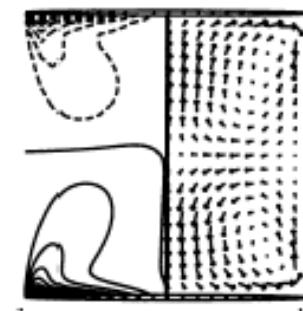
π

H

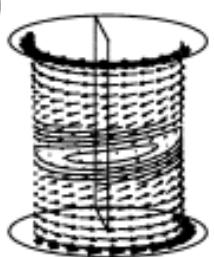


$\frac{3}{2}\pi$

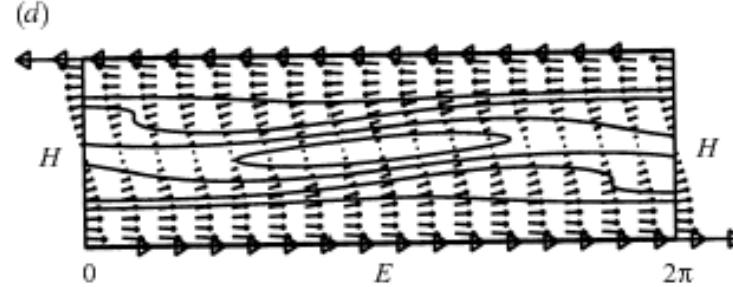
$\frac{1}{2}\pi$



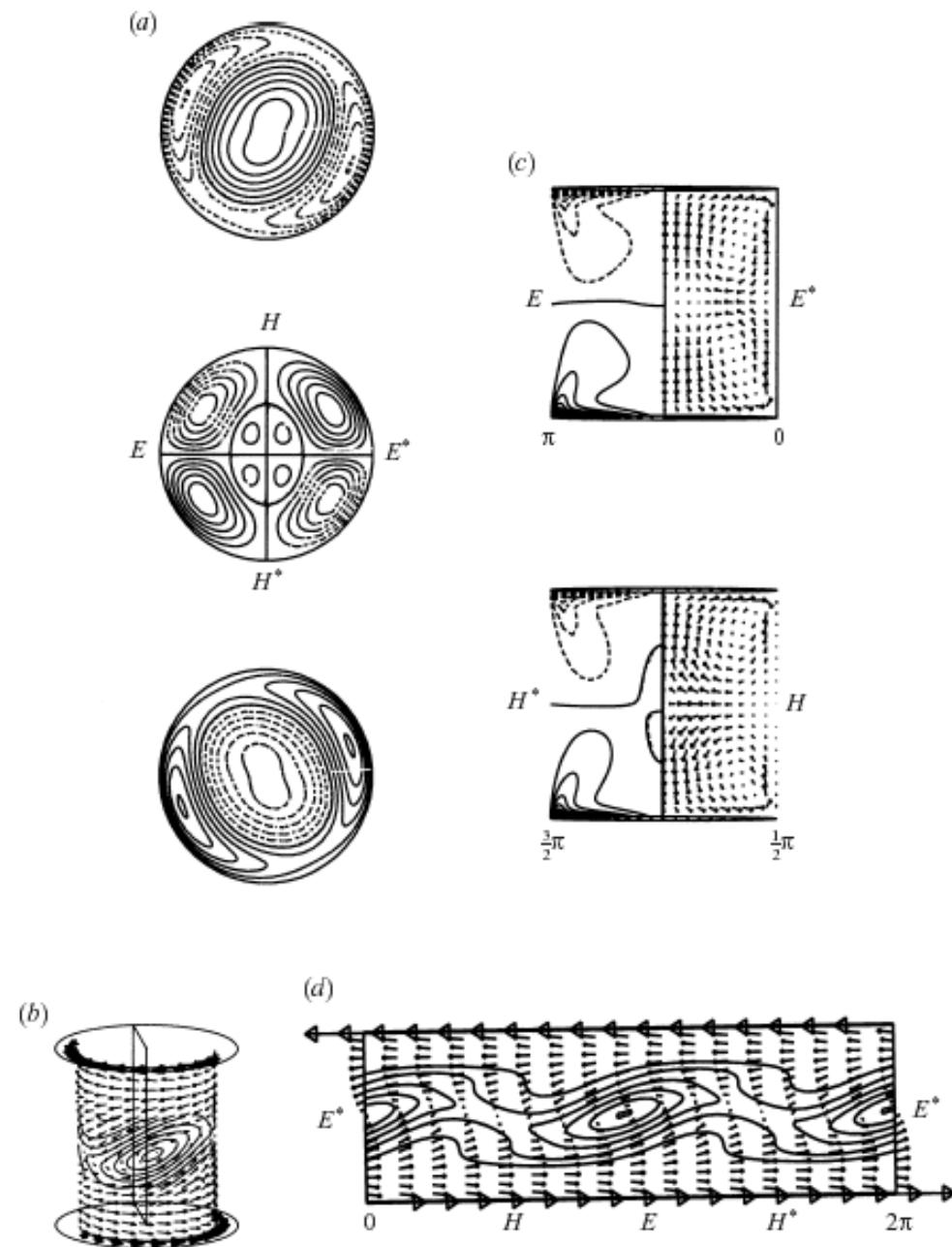
(b)



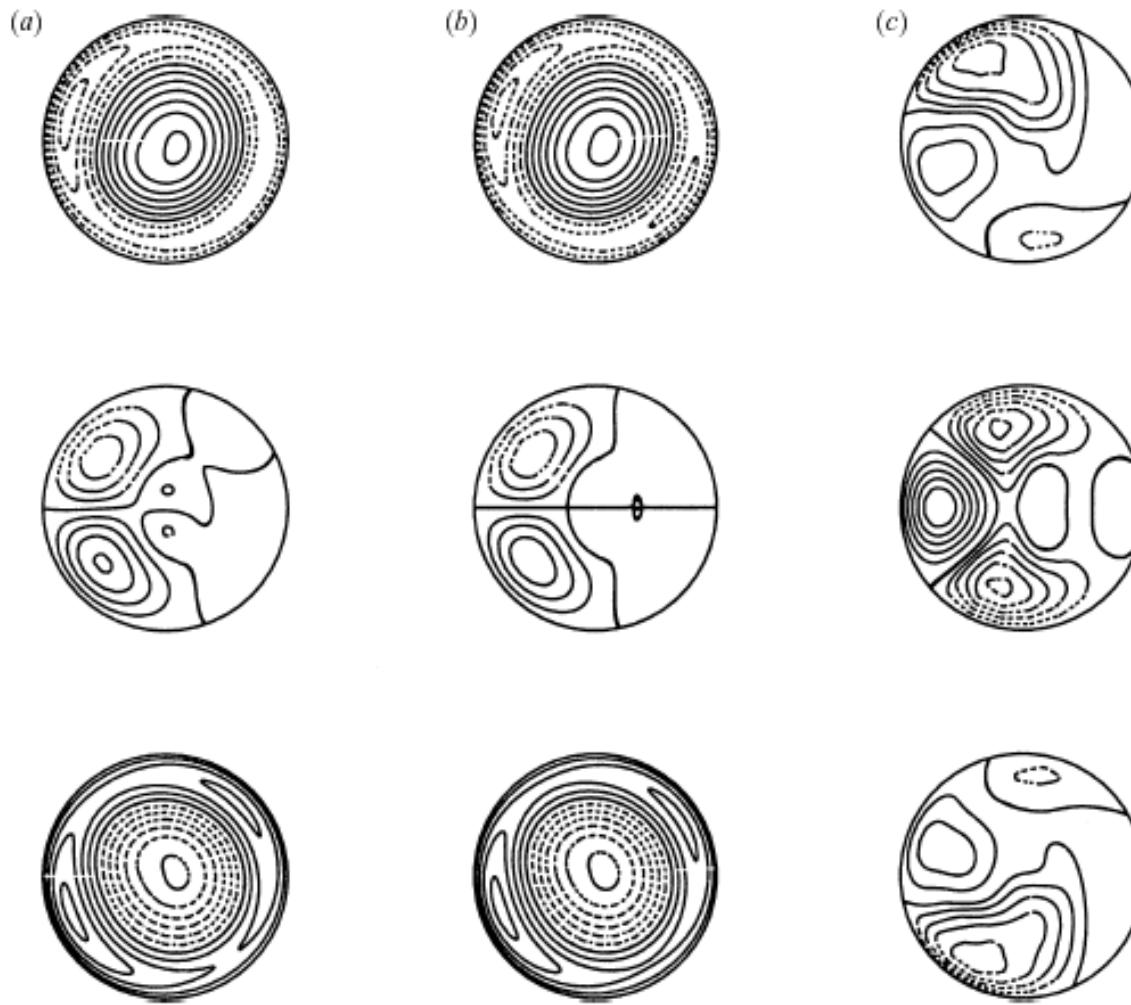
(d)



Pure Mode (from $m=2$ eigenvector)



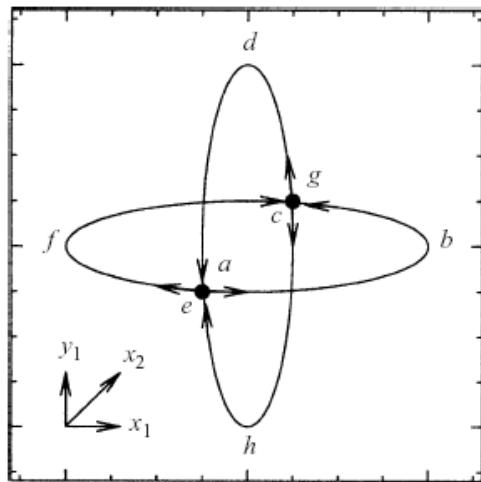
Travelling Waves ($Re=415$)



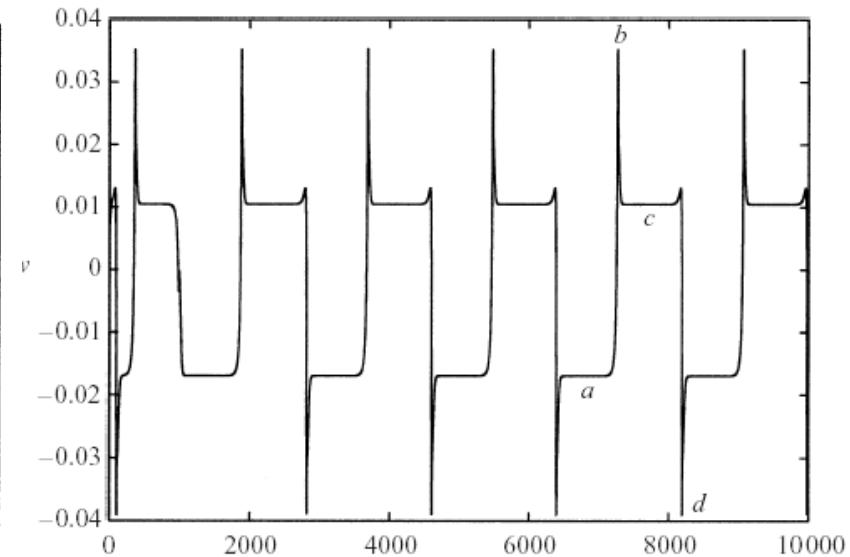
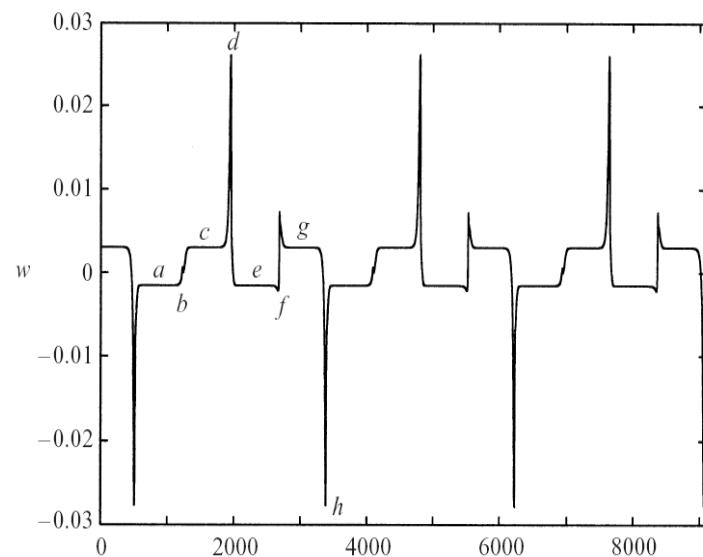
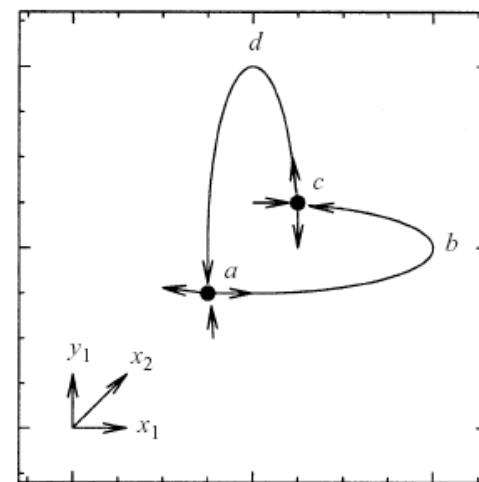
$TW =$ Mixed Mode + Eigenvector
 Reflection-Symmetric Antisymmetric

Two types of heteroclinic cycles

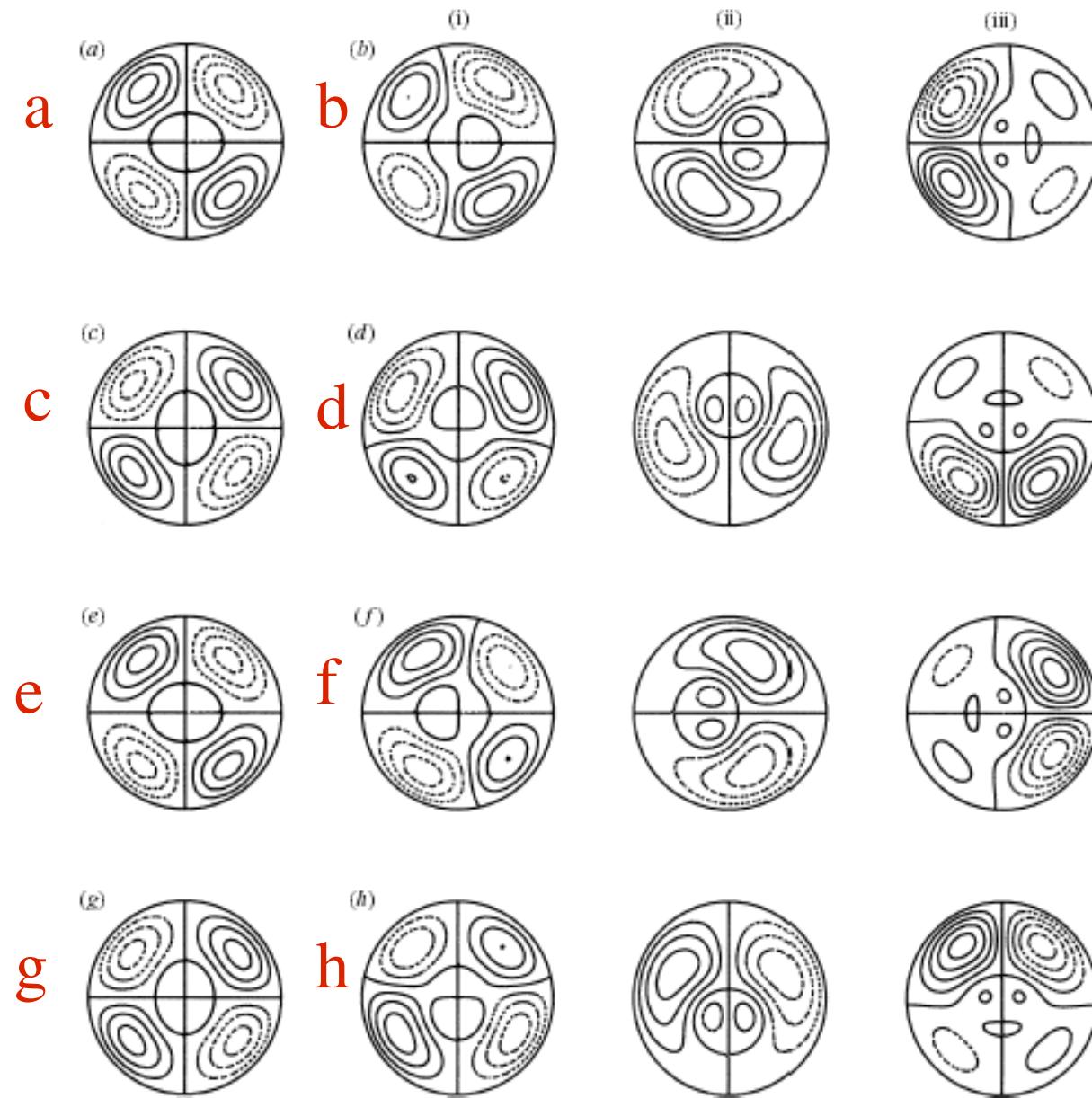
4 plateaus



2 plateaus

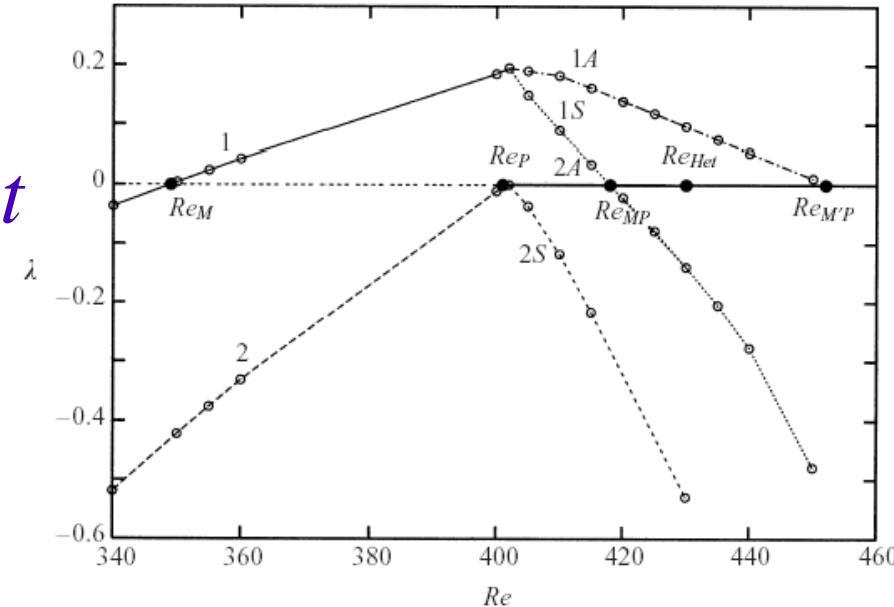


Heteroclinic Cycle ($Re=430$)

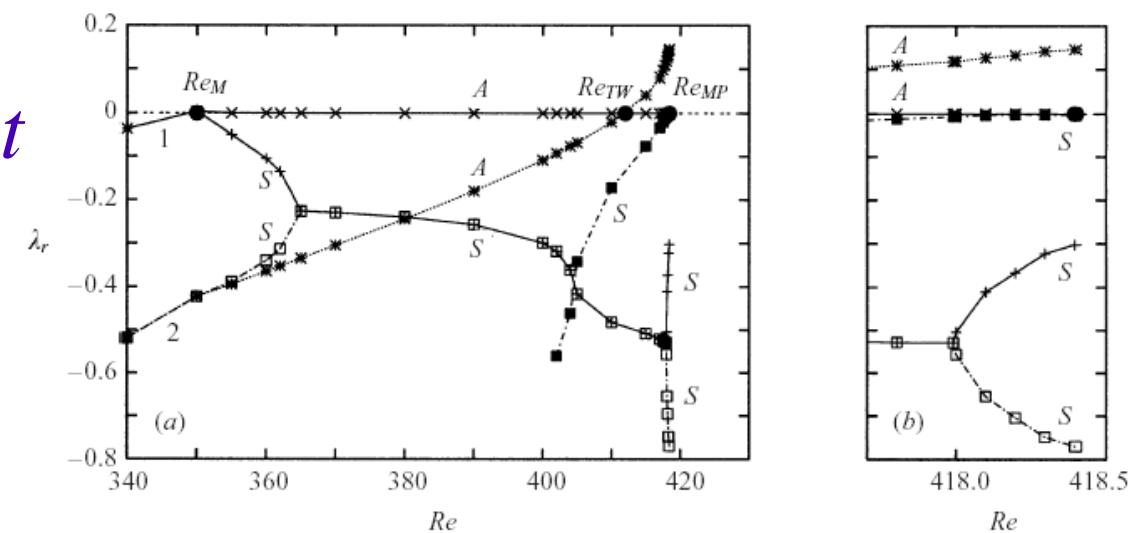


Linear stability analysis about nonaxisymmetric flows

*Eigenvalues about
pure mode*



*Eigenvalues about
mixed mode*



Conclusion

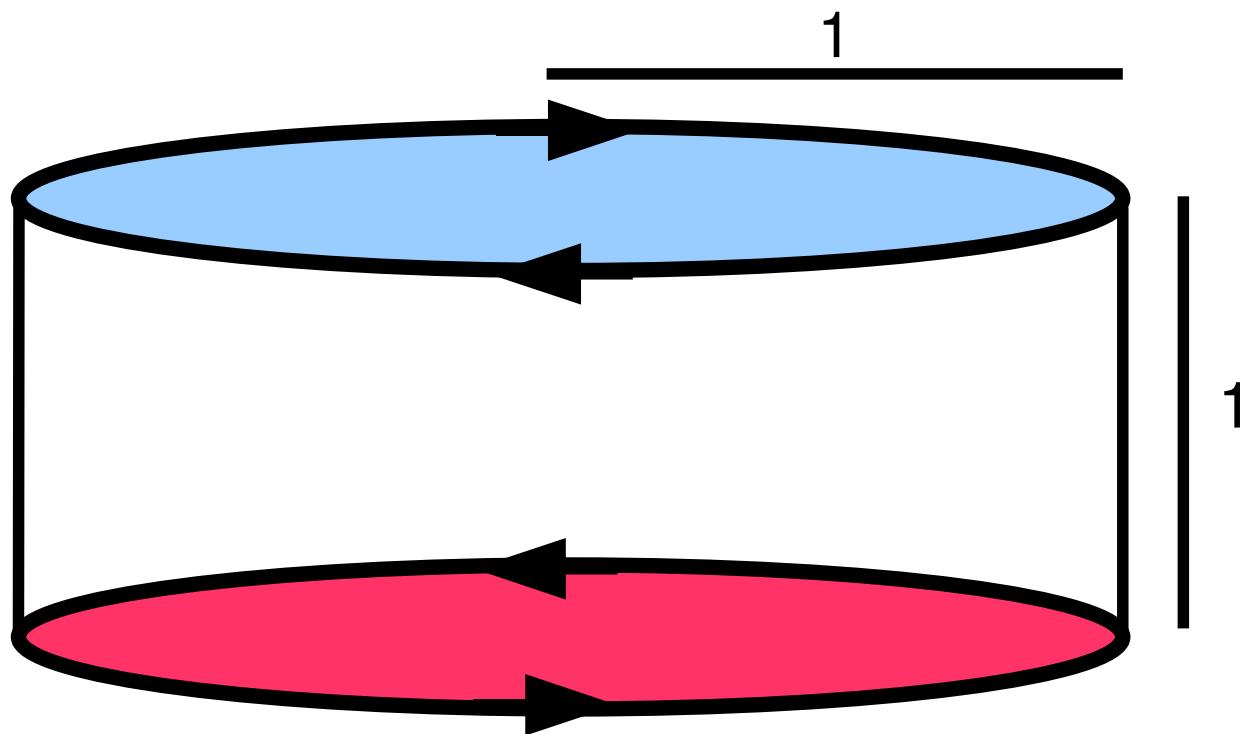
*counter-rotating von Karman flow with diameter=height
is almost perfect realisation of 1:2 mode interaction*

- *steady states (mixed and pure modes)*
- *travelling waves*
- *robust heteroclinic cycles of two kinds*
- *possible Kelvin-Helmholtz instability mechanism*

Convection + counter-rotation (Rayleigh-Bénard + von Kármán)

Tuckerman with Bordja, Cruz Navarro, Martin Witkowski, Barkley

$\text{Pr}=1$ $\Gamma=1$ axisymmetric



Symmetry: R_π

$$z \rightarrow z_{\text{mid}} - z$$

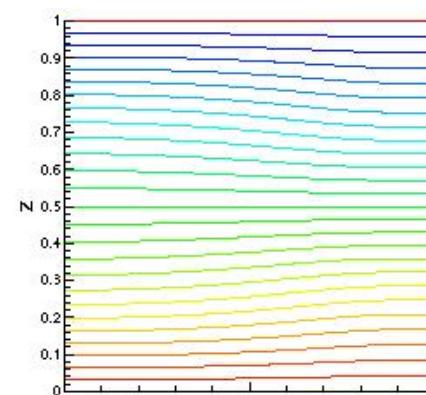
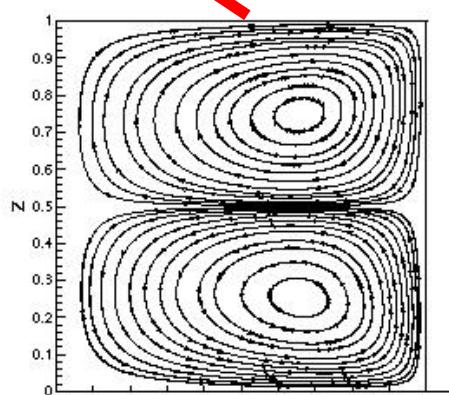
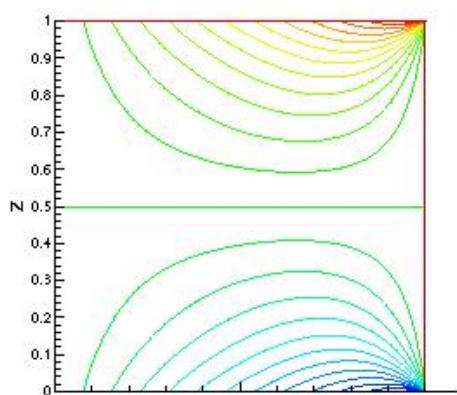
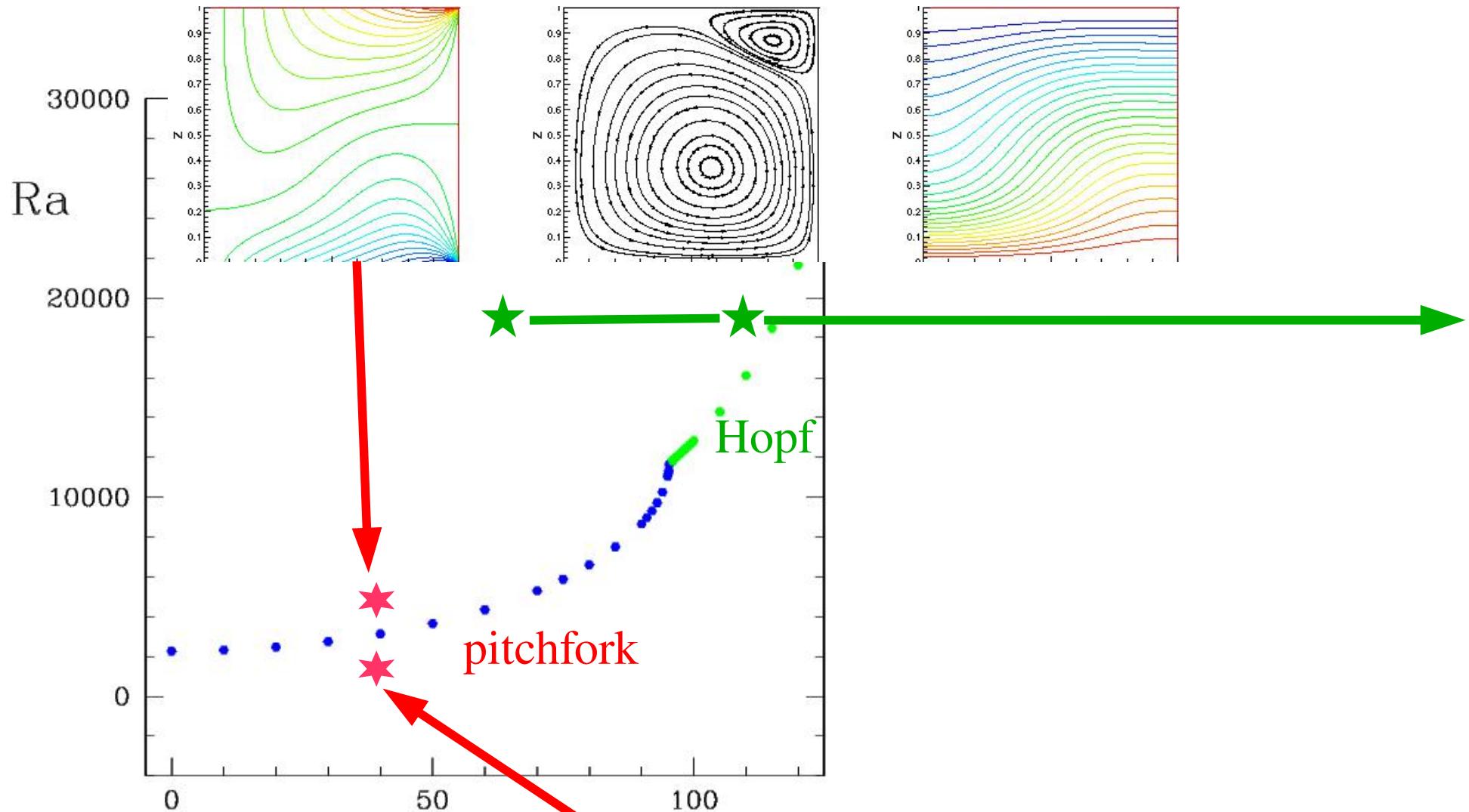
$$T \rightarrow T_{\text{mid}} - T$$

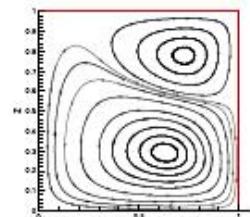
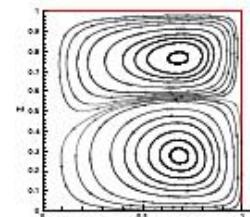
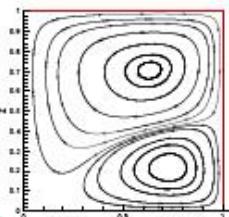
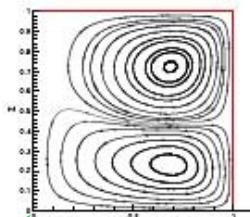
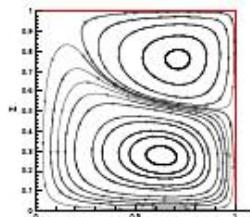
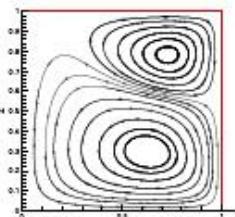
$$\theta \rightarrow \theta_{\text{mid}} - \theta$$

$$u_z \rightarrow -u_z$$

$$u_\theta \rightarrow -u_\theta$$

(Nore et al. 2003)





$t = 0$

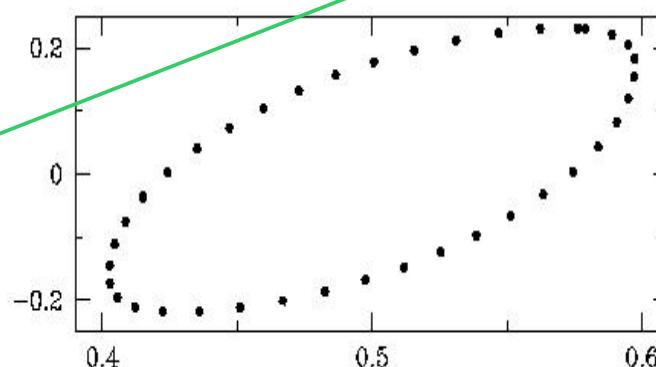
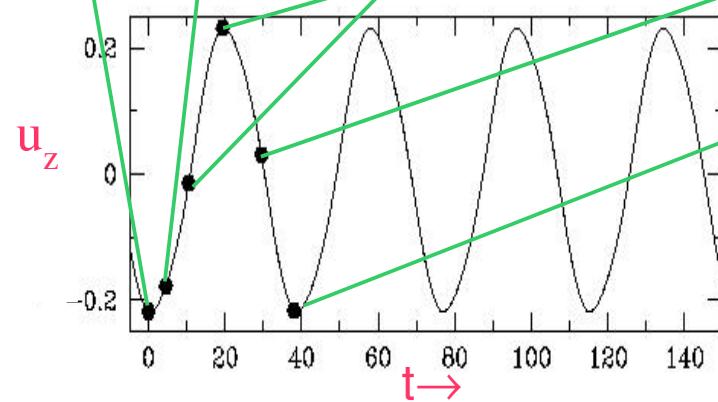
$t = 4.5 \approx \tau/8$

$t = 10.5 \approx \tau/4$

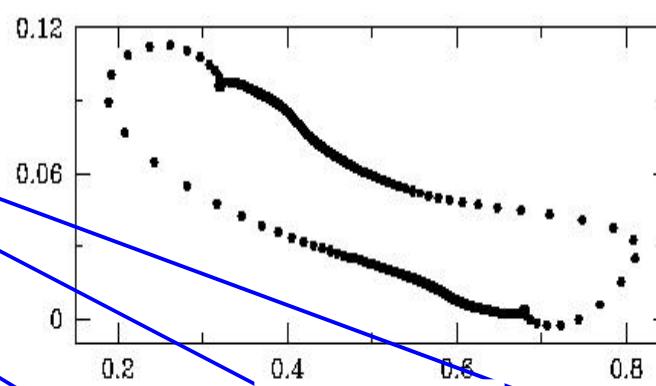
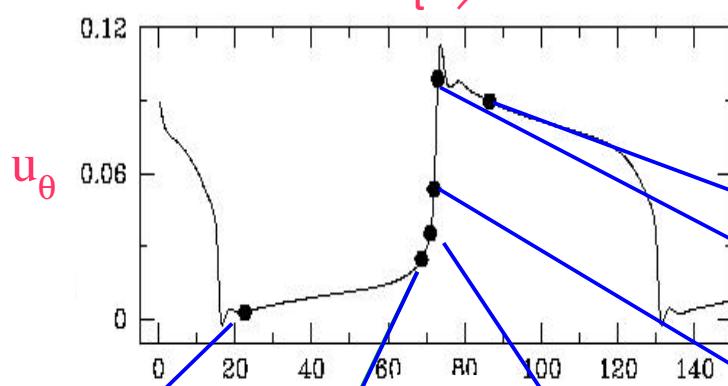
$t = 19.4 \approx \tau/2$

$t = 29.5 \approx 3\tau/4$

$t = 38 \approx \tau$

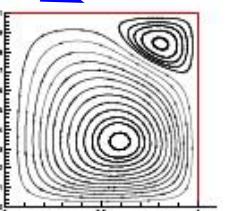
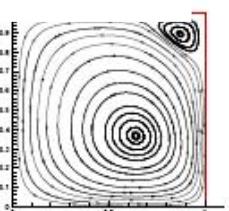
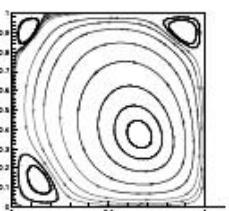
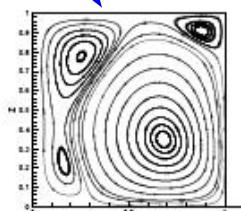
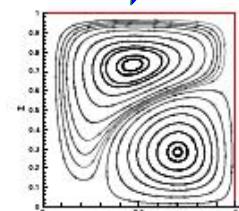
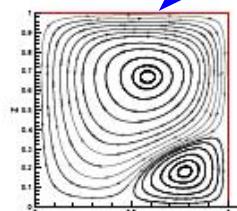


Re=110
near Hopf bif
sinusoidal
cycle



Re=63

near saddle-node
 u_θ near-heteroclinic
cycle



$t = 22.6 = t_0$

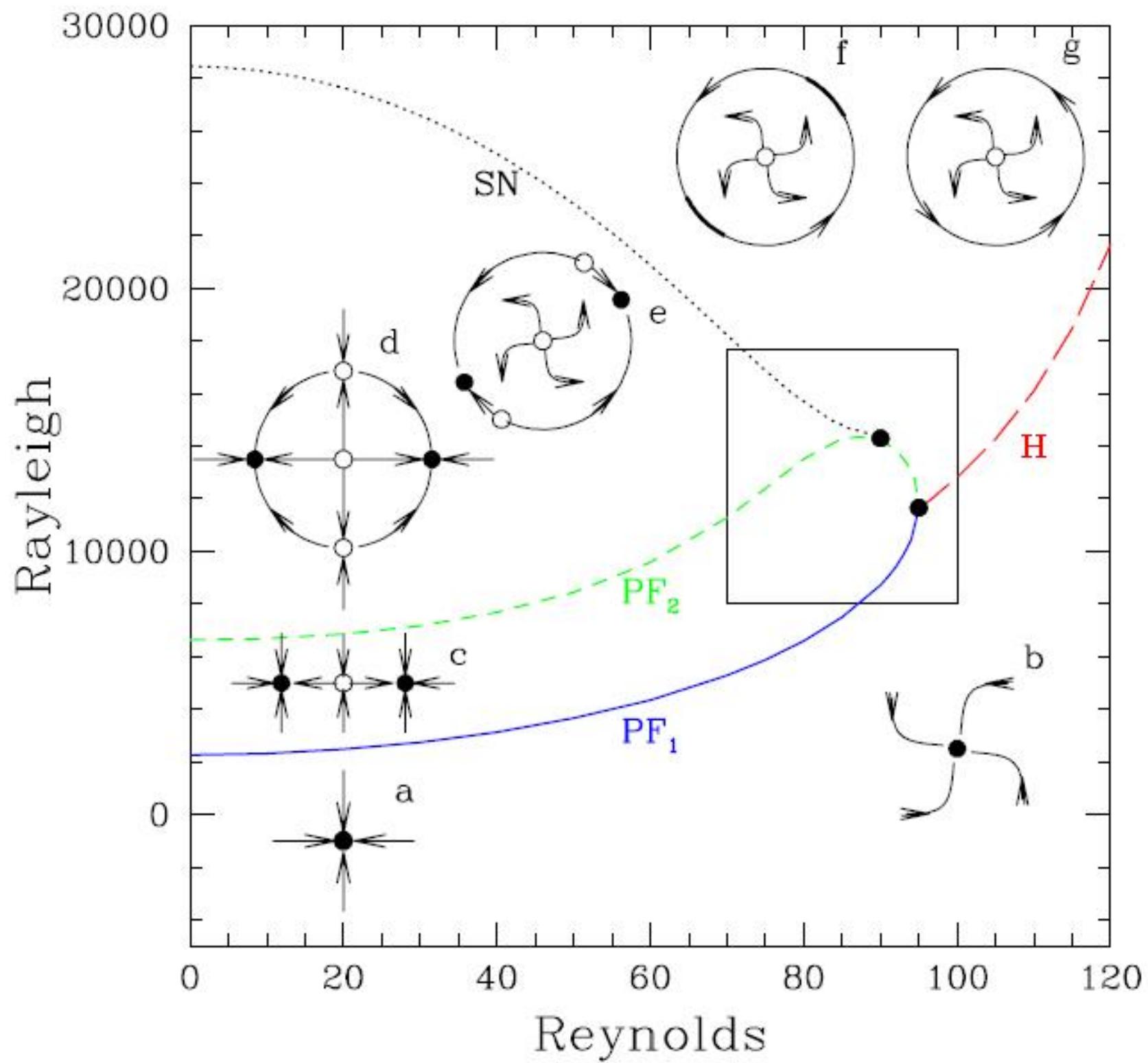
$t = 68.7$

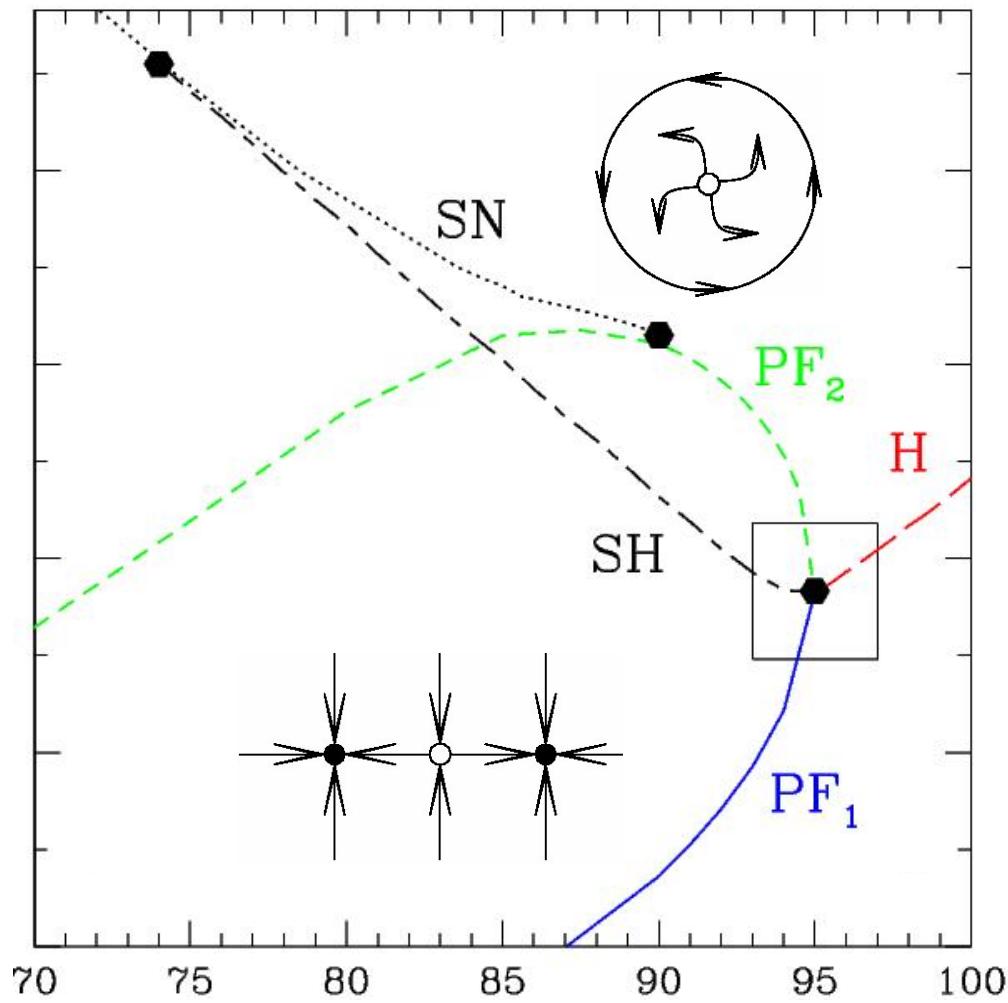
$t = 71.0$

$t = 72.0$

$t = 73.0$

$t = 86.5 \approx t_0 + \tau/2$





SH: subcritical
secondary Hopf

G: gluing

SNP: saddle-node
of periodic orbits

PF₁: pitchfork

PF₂: pitchfork

H: Hopf

$$\frac{H}{\mu_1}$$

Takens-Bogdanov normal form:

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = \mu_1 x - \mu_2 y - \delta x^3 - x^2 y$$

Eigenvalues: from real to complex

