

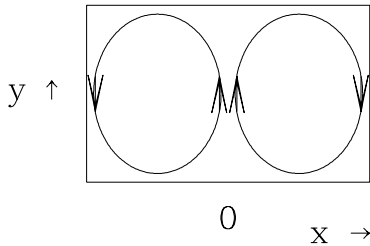
Cours : Dynamique Non-Linéaire

Laurette TUCKERMAN

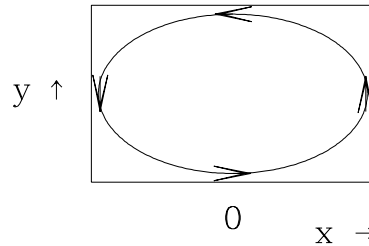
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Symmetry

Reflection Symmetry



**Symmetric 2D
vector field**



**Antisymmetric 2D
vector field**

$$\kappa \begin{pmatrix} u \\ v \end{pmatrix} (x, y) \equiv \begin{pmatrix} -u \\ v \end{pmatrix} (-x, y)$$

Group table for $\{I, \kappa\}$:

	<i>I</i>	<i>κ</i>
<i>I</i>	<i>I</i>	<i>κ</i>
<i>κ</i>	<i>κ</i>	<i>I</i>

Other reflection operators

$$(\kappa f)(x) \equiv f(-x)$$

$$\kappa a \equiv -a$$

Evolution equation: $\dot{a} = g(a)$

System has reflection symmetry $\iff g$ is *equivariant* $\iff g\kappa = \kappa g$

$$\begin{aligned} (g\kappa)(a) = (\kappa g)(a) &\iff g(-a) = -g(a) \iff g \text{ odd} \\ &\implies \dot{a} = g_1 a + g_3 a^3 = (g_1 + g_3 a^2) a \\ &\text{to cubic order} \implies \text{pitchfork bif} \end{aligned}$$

$$\kappa \begin{pmatrix} a \\ s \end{pmatrix} \equiv \begin{pmatrix} -a \\ s \end{pmatrix} \implies \kappa \begin{pmatrix} 0 \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix} \quad \text{and} \quad \kappa \begin{pmatrix} a \\ 0 \end{pmatrix} = - \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$\frac{d}{dt} \begin{pmatrix} a \\ s \end{pmatrix} = G(a, s) \equiv \begin{pmatrix} g \\ h \end{pmatrix} (a, s)$$

$$\kappa \begin{pmatrix} g \\ h \end{pmatrix} (a, s) = \begin{pmatrix} g \\ h \end{pmatrix} \kappa(a, s)$$

$$\begin{pmatrix} -g \\ h \end{pmatrix} (a, s) = \begin{pmatrix} g \\ h \end{pmatrix} (-a, s)$$

$$\begin{aligned} \begin{pmatrix} g \\ h \end{pmatrix} (a, s) &= \begin{pmatrix} g_{10}a + g_{11}as + g_{30}a^3 + g_{12}as^2 \\ h_{00} + h_{01}s + h_{20}a^2 + h_{02}s^2 + h_{21}a^2s + h_{03}s^3 \end{pmatrix} \\ &= \begin{pmatrix} (g_{10} + g_{11}s + g_{12}s^2 + g_{30}a^2)a \\ h_{00} + h_{01}s + h_{02}s^2 + h_{03}s^3 + (h_{20} + h_{21}s)a^2 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{g}(a^2, s)a \\ \tilde{h}(a^2, s) \end{pmatrix} = \tilde{g}(a^2, s) \begin{pmatrix} a \\ 0 \end{pmatrix} + \tilde{h}(a^2, s) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

invariants: $\tilde{g} = \tilde{g}\kappa$, $\tilde{h} = \tilde{h}\kappa$ **equivariant:** $(g, h)\kappa = \kappa(g, h)$

**Solutions to κ -equivariant systems are not all symmetric.
(That is why bifurcation theory is interesting.)**

**If (a, s) is a solution to a κ -equivariant system then
 $\kappa(a, s) \equiv (-a, s)$ is also a solution.**

Asymmetric solutions (a, s) , $a \neq 0$ come in pairs $(\pm a, s)$.

**If a κ -equivariant system has a unique solution, then that
solution is symmetric.**

e.g. linear system, or Navier-Stokes equations at low Re

Linear system G which commutes with κ :

$$\begin{aligned}Gu &= \lambda u \\ \kappa Gu &= \kappa \lambda u \\ G\kappa u &= \lambda \kappa u\end{aligned}$$

$(\lambda, \kappa u)$ is also an eigenpair of G . κu could be a multiple of u :

$$\begin{aligned}\kappa u &= cu \\ \kappa^2 u &= \kappa cu \\ u &= c^2 u \\ c &= \begin{cases} 1 & \implies u \text{ is symmetric} \\ -1 & \implies u \text{ is antisymmetric} \end{cases}\end{aligned}$$

Or else if u and κu are linearly independent, then:

$$\begin{aligned}u + \kappa u &\text{ is symmetric eigenvector} \\ u - \kappa u &\text{ is antisymmetric eigenvector}\end{aligned}$$

Verify equivariance of Navier-Stokes equations

$$(\kappa f)(x) \equiv f(-x) \implies \begin{cases} (\kappa f)'(x) = -f'(-x) \\ (\kappa f)''(x) = f''(-x) \end{cases}$$

Demonstration:

$$\text{Let } \tilde{f}(x) \equiv (\kappa f)(x) \equiv f(-x)$$

$$\begin{aligned} \tilde{f}'(x) &\equiv \lim_{\Delta x \rightarrow 0} \frac{\tilde{f}(x + \Delta x) - \tilde{f}(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(-x - \Delta x) - f(-x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(-x + \Delta x) - f(-x)}{-\Delta x} = -f'(-x) \end{aligned}$$

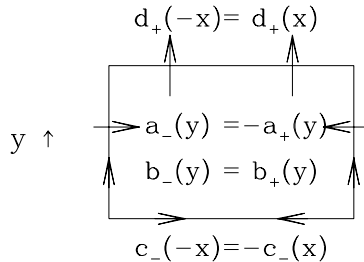
$$\left[(NS) \begin{pmatrix} u \\ v \end{pmatrix} \right] (x, y) \equiv \begin{pmatrix} -(u\partial_x + v\partial_y)u - \partial_x p + \nu(\partial_x^2 + \partial_y^2)u \\ -(u\partial_x + v\partial_y)v - \partial_y p + \nu(\partial_x^2 + \partial_y^2)v \end{pmatrix} (x, y)$$

$$\left[\kappa \begin{pmatrix} u \\ v \end{pmatrix} \right] (x, y) \equiv \begin{pmatrix} -u \\ v \end{pmatrix} (-x, y)$$

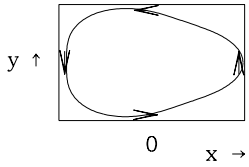
$$\left[(NS)\kappa \begin{pmatrix} u \\ v \end{pmatrix} \right] (x, y) \stackrel{?}{=} \left[\kappa(NS) \begin{pmatrix} u \\ v \end{pmatrix} \right] (x, y)$$

$$\begin{pmatrix} -[-(u\partial_x + v\partial_y)u - \partial_x p + \nu(\partial_x^2 + \partial_y^2)u] \\ -(u\partial_x + v\partial_y)v - \partial_y p + \nu(\partial_x^2 + \partial_y^2)v \end{pmatrix} (-x, y)$$

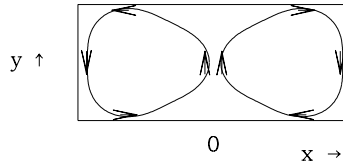
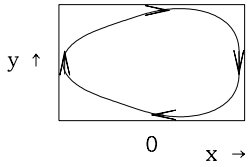
Boundary conditions & external forces determine if problem is κ -equivariant:



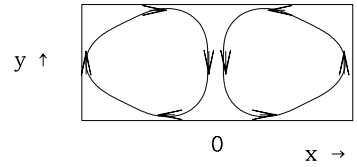
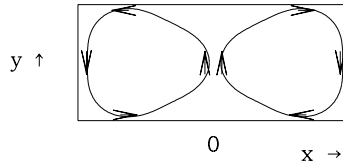
$x \rightarrow$



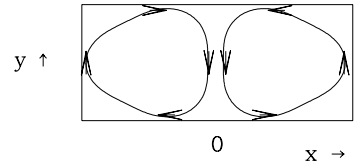
$\kappa \Downarrow$



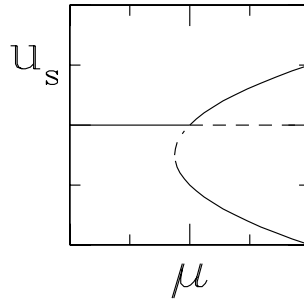
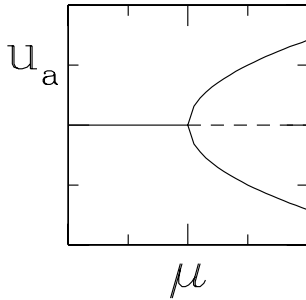
$\kappa \Downarrow$



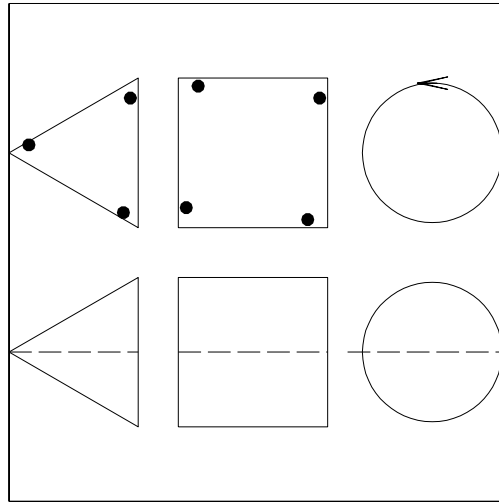
$\kappa \Downarrow$



Not equivalent flows



Rotations and reflections of the plane



Z_2	Z_3	Z_4	\dots	$SO(2)$
D_2	D_3	D_4	\dots	$O(2)$

Natural representation of $O(2)$ on (x, y) uses $z = x + iy$

$$S_\theta z \equiv e^{i\theta} z$$

$$\kappa z \equiv \bar{z}$$

$$\kappa S_\theta z = \kappa(e^{i\theta} z) = e^{-i\theta} \bar{z}$$

$$S_\theta \kappa z = S_\theta \bar{z} = e^{i\theta} \bar{z}$$

$$\kappa S_\theta z = S_{-\theta} \kappa z$$

$$(S_{\theta_0} w)(\rho, \theta) \equiv w(\rho, \theta + \theta_0)$$

$$(\kappa w)(\rho, \theta) \equiv w(\rho, -\theta)$$

Circle Pitchfork

$$f(z, \bar{z}) = \sum_{m,n} f_{mn} z^m \bar{z}^n$$

$$\kappa f(z, \bar{z}) = \overline{f(z, \bar{z})} = \bar{f}_{mn} \bar{z}^m z^n$$

f real

$$f(\kappa(z, \bar{z})) = f_{mn} \bar{z}^m z^n$$

$$S_\theta f(z, \bar{z}) = e^{i\theta} f(z, \bar{z})$$

$$f(S_\theta(z, \bar{z}))$$

$$= f_{mn} (e^{i\theta} z)^m \overline{(e^{i\theta} z)}^n$$

$$= e^{i\theta} f_{mn} z^m \bar{z}^n$$

$$= f_{mn} e^{im\theta} z^m e^{-in\theta} \bar{z}^n$$

$$m - n = 1 \text{ or } f_{mn} = 0$$

$$f(z, \bar{z}) = f_{10}z + f_{21}z^2\bar{z} + f_{32}z^3\bar{z}^2 + \dots$$

$$= (f_{10} + f_{21}|z|^2 + f_{32}|z|^4 + \dots)z$$

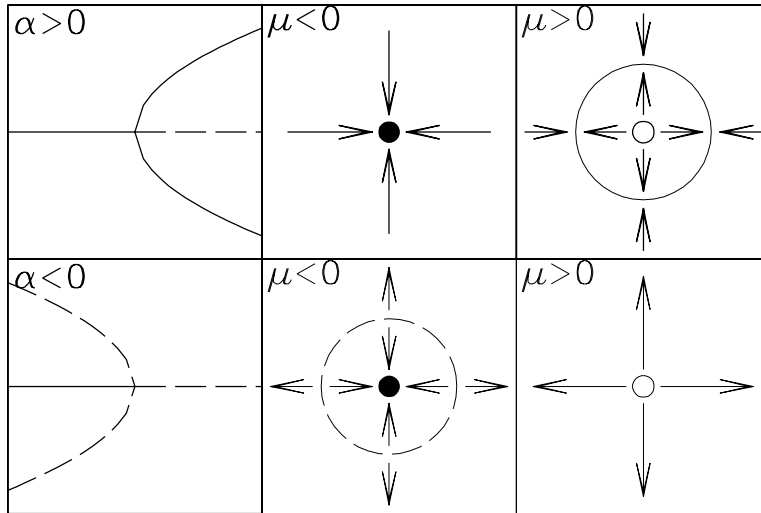
$$= \tilde{f}(|z|^2)z$$

$|z|^2$ invariant

z equivariant

Circle Pitchfork

$$\dot{z} = (\mu - \alpha|z|^2)z$$



Circle Pitchfork

$$\dot{z} = (\mu - |z|^2)z$$

Cartesian Form:

$$\dot{x} + i\dot{y} = \mu(x + iy) - (x^2 + y^2)(x + iy)$$

$$\dot{x} = \mu x - (x^2 + y^2)x$$

$$\dot{y} = \mu y - (x^2 + y^2)y$$

Polar Form:

$$(\dot{r} + ir\dot{\theta})e^{i\theta} = (\mu - r^2)re^{i\theta}$$

$$\dot{r} = \mu r - r^3$$

$$\dot{\theta} = 0$$

Subcritical Form:

$$\dot{z} = (\mu + |z|^2)z$$

Stability of origin (use Cartesian coordinates):

$$J(x=0, y=0) = \begin{pmatrix} \mu - (3x^2 + y^2) & -2xy \\ -2xy & \mu - (x^2 + 3y^2) \end{pmatrix} \Big|_{(0,0)} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$

double eigenvalue μ .

Stability of states on circle (use polar coordinates):

$$J(r = \sqrt{\mu}, \theta) = \begin{pmatrix} \mu - 3r^2 & 0 \\ 0 & 0 \end{pmatrix} \Big|_{\sqrt{\mu}, \theta} = \begin{pmatrix} -2\mu & 0 \\ 0 & 0 \end{pmatrix}$$

eigenvalue -2μ along r and marginal eigenvalue 0 along θ .

$$\frac{dw}{dt}(\theta) = \mathcal{F}(w)(\theta)$$

\mathcal{F} indep of θ

$$0 = \mathcal{F}(W)$$

W a steady solution on circle

$$0 = \frac{d\mathcal{F}(W)}{d\theta} = \frac{\delta\mathcal{F}}{\delta W} \frac{\partial W}{\partial \theta}$$

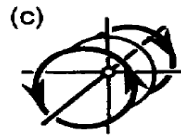
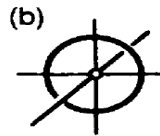
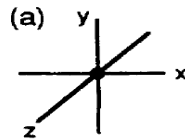
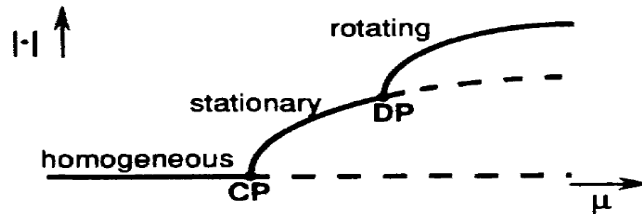
Jacobian or Frechet derivative
marginal eigenvector

Drift Pitchfork

$$\dot{r} = (\mu - r^2)r$$

$$\dot{\theta} = \zeta$$

$$\dot{\zeta} = (r^2 - 1 - \zeta^2)\zeta$$



$$\mu < 0$$

$$0 \leq \mu \leq 1$$

$$1 < \mu$$

$$r = 0$$

$$r = \sqrt{\mu}$$

$$r = \sqrt{\mu}$$

$$\theta = \theta_0$$

$$\theta = \theta_0 + \zeta t$$

$$\zeta = 0$$

$$\zeta = 0$$

$$\zeta = \pm\sqrt{\mu - 1}$$

Drift Pitchfork

Speed at onset is slow: $\zeta = \sqrt{\mu - \mu_{DP}} = \sqrt{\mu - 1}$

Motion along the circle: group orbit

Symmetry-breaking variable: ζ

At $\mu = \mu_{DP} = 1$, Jacobian contains a Jordan block

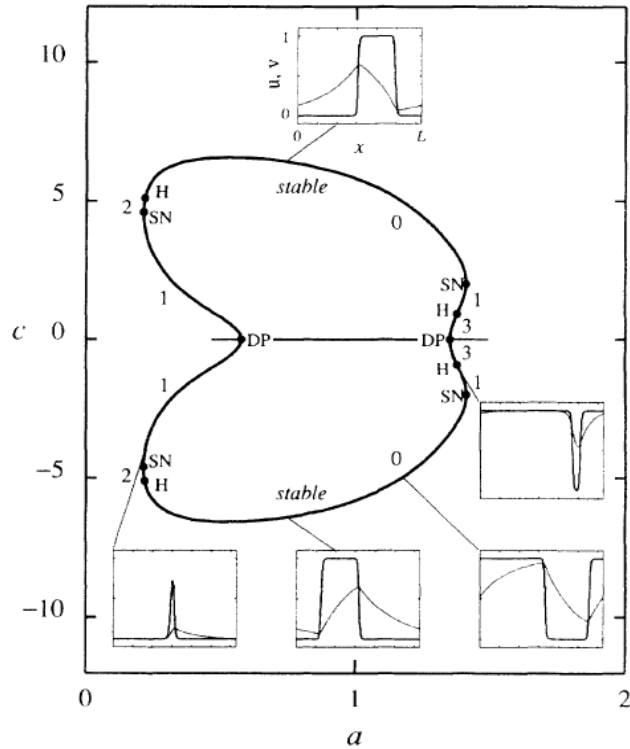
$$\begin{bmatrix} \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial \zeta} \\ \frac{\partial \dot{\zeta}}{\partial \theta} & \frac{\partial \dot{\zeta}}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Function $W(\theta)$ is even about some θ_0 .

Marginal and bifurcating eigenvectors are odd about θ_0

Drifting W is asymmetric in θ .

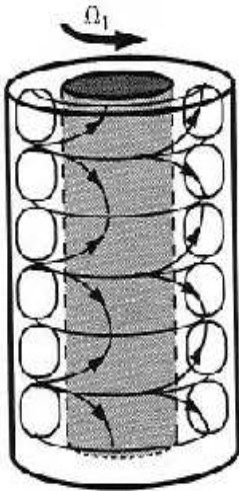
Drift pitchfork in reaction-diffusion equations



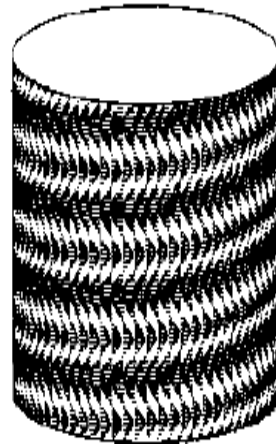
speed

From Kness, Tuckerman & Barkley, *Phys. Rev. A* 46, 5054 (1992).

$O(2)$ and $SO(2)$



Axisymmetric Taylor vortices
Tagg, Nonlinear Science Today 4,
1 (1994).



Spiral Taylor vortices
Antonijoan et al., Phys. Fluids 10,
829 (1995).

No requirement of $\kappa f = f\kappa \implies f$ complex

$$\dot{z} = (\mu + i\omega - (\alpha + i\beta)|z|^2)z$$

$$(\dot{r} + ri\dot{\theta})e^{i\theta} = ((\mu - \alpha r^2) + i(\omega + \beta r^2)) r e^{i\theta}$$

$$\dot{r} = (\mu - \alpha r^2) r$$

$$\dot{\theta} = \omega + \beta r^2$$

Breaking of $SO(2) \implies$ motion along θ direction

Hopf bifurcation and $O(2)$ symmetry

Need four-dimensional eigenspace:

$$u(\theta, t) = (z_+(t) + z_-(t))e^{i\theta} + (\bar{z}_+(t) + \bar{z}_-(t))e^{-i\theta}$$

where z_{\pm} are complex amplitudes (i.e. amplitude and phase) of left-going and right-going traveling waves.

At linear order:

$$\frac{d}{dt} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} i\omega z_+ \\ -i\omega z_- \end{pmatrix}$$

$$u(\theta, t) = z_+(0)e^{i(\theta+\omega t)} + z_-(0)e^{i(\theta-\omega t)} + \bar{z}_+(0)e^{-i(\theta+\omega t)} + \bar{z}_-(0)e^{-i(\theta-\omega t)}$$

$z_{\pm}(0)$ arbitrary initial amplitudes.

Addition of nonlinear terms compatible with $O(2)$ symmetry greatly restricts possible equilibria.

Appropriate representation of $O(2)$ on (z_+, z_-) :

$$S_{\theta_0}(z_+, z_-) = (e^{i\theta_0} z_+, e^{i\theta_0} z_-)$$

$$\kappa(z_+, z_-) = (\bar{z}_-, \bar{z}_+)$$

Simplest cubic order equivariant evolution equations:

$$\frac{d}{dt} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} [\mu + i\omega + a|z_-|^2 + b(|z_+|^2 + |z_-|^2)] z_+ \\ [\mu - i\omega + \bar{a}|z_+|^2 + \bar{b}(|z_+|^2 + |z_-|^2)] z_- \end{pmatrix}$$

Substitute $z_{\pm} = r_{\pm} e^{i\phi_{\pm}}$:

$$\dot{r}_{\pm} = (\mu + a_r r_{\mp}^2 + b_r (r_+^2 + r_-^2)) r_{\pm}$$

$$\dot{\phi}_{\pm} = \pm(\omega + a_i r_{\mp}^2 + b_i (r_+^2 + r_-^2))$$

Equations do not involve phases ϕ_{\pm} (all phases equivalent).

Solutions with $\dot{r}_{\pm} = 0$:

origin : $r_+ = 0, r_- = 0$

left traveling waves : $r_+ = \sqrt{\frac{-\mu}{b_r}}, r_- = 0,$

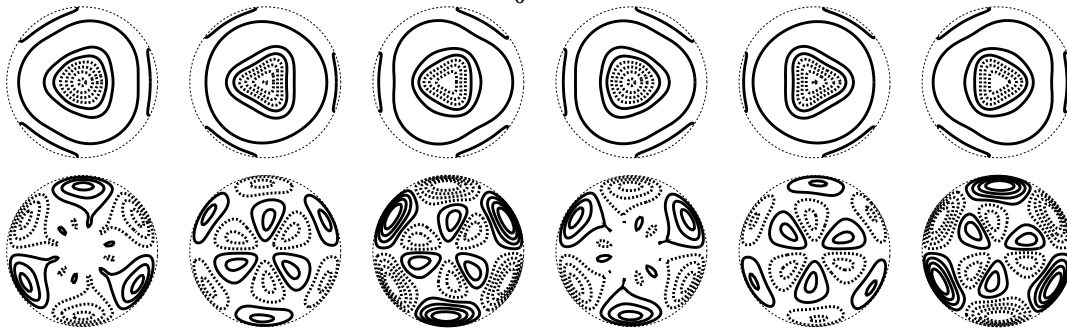
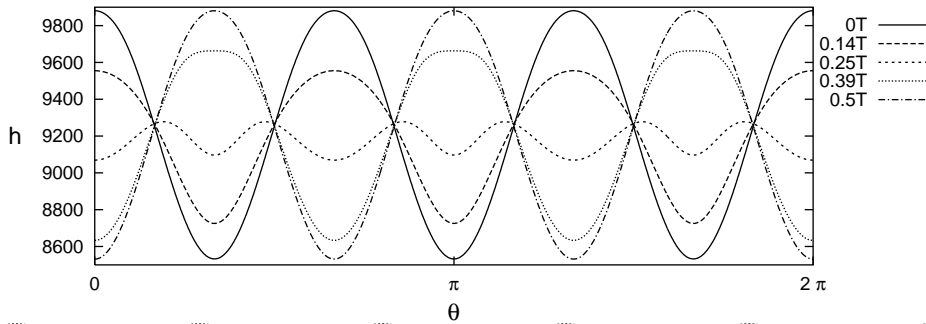
$$\dot{\phi}_+ = \omega - \mu \frac{b_i}{b_r}$$

right traveling waves : $r_+ = 0, r_- = \sqrt{\frac{-\mu}{b_r}},$

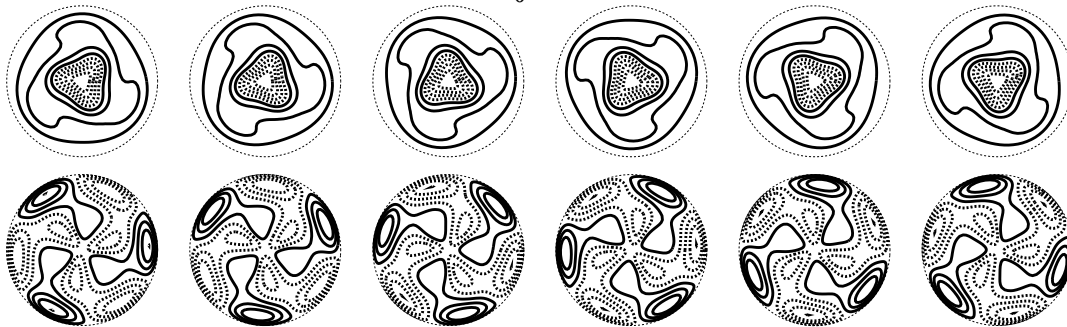
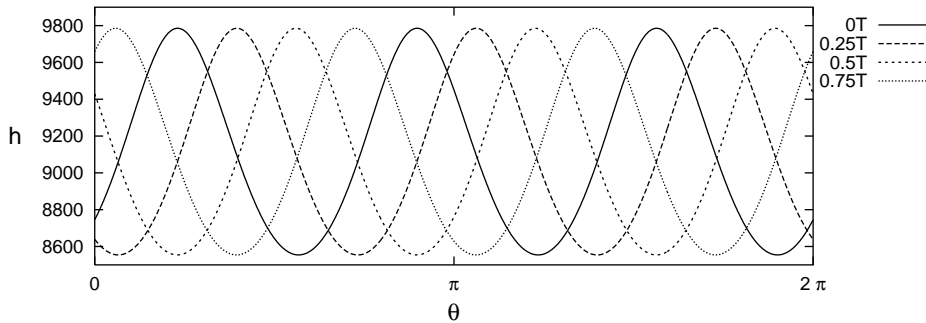
$$\dot{\phi}_- = - \left(\omega - \mu \frac{b_i}{b_r} \right)$$

standing waves : $r_+ = r_- = \sqrt{\frac{-2\mu}{a_r + 2b_r}},$

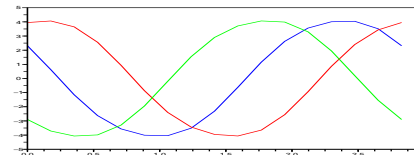
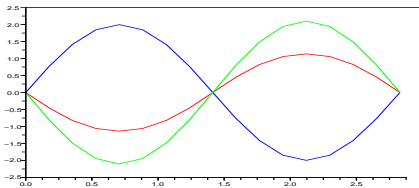
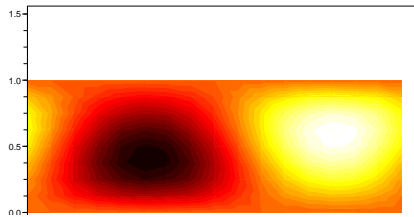
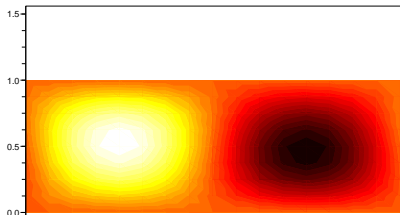
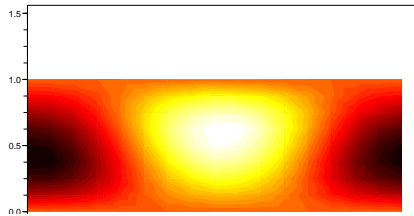
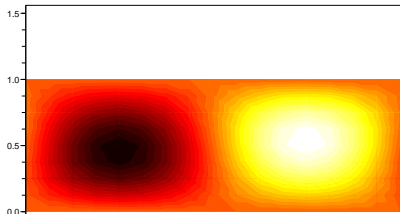
$$\dot{\phi}_{\pm} = \pm \left(\omega - 2\mu \frac{a_i + 2b_i}{a_r + 2b_r} \right)$$



Standing waves in Rayleigh-Bénard convection in a cylinder with $\Gamma = R/H = 1.47$ and $Pr = 1$ at $Ra = 26\,000$. From Borońska & Tuckerman, *J. Fluid Mech.* 559, 279 (2006).



Travelling wave in Rayleigh-Bénard convection in a cylinder with $\Gamma = R/H = 1.47$ and $Pr = 1$ at $Ra = 26\,000$. From Borońska & Tuckerman, *J. Fluid Mech.* 559, 279 (2006).



Standing waves

Travelling waves

Thermosolutal convection with $S = -0.1$, $L = 0.1$, $Pr = 10$, $r \equiv Ra/Ra_c = 1.3$.

Standing vs. Travelling Waves

Jacobian in $(r_+, r_-, \phi_+, \phi_-)$ coordinates:

$$\begin{pmatrix} \mu + a_r r_-^2 + b_r (r_+^2 + r_-^2) + 2b_r r_+^2 & 2(a_r + b_r)r_- r_+ & 0 & 0 \\ 2(a_r + b_r)r_- r_+ & \mu + a_r r_+^2 + b_r (r_+^2 + r_-^2) + 2b_r r_-^2 & 0 & 0 \\ 2b_i r_+ & 2(a_i + b_i)r_- & 0 & 0 \\ -2(a_i + b_i)r_+ & -2b_i r_+ & 0 & 0 \end{pmatrix}$$

Block lower-triangular:

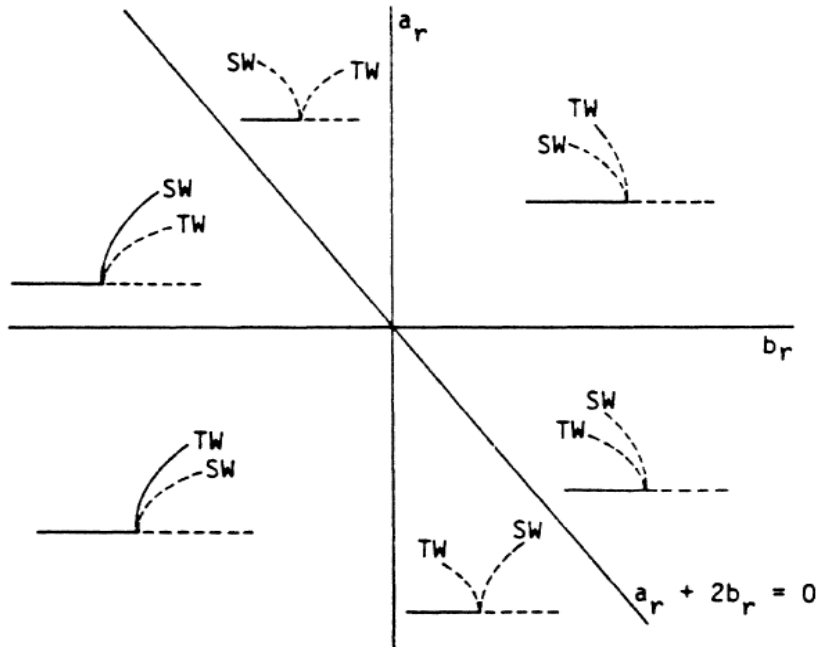
$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\left. \begin{array}{l} AX = \lambda X \\ CX + DY = \lambda Y \end{array} \right\} \implies \left\{ \begin{array}{l} X = 0 \\ (\lambda, Y) \text{ eig of } D \end{array} \right\} \text{ or } \left\{ \begin{array}{l} (\lambda, X) \text{ eig of } A \\ Y = (\lambda I - D)^{-1} CX \end{array} \right\}$$

Directions ϕ_{\pm} are neutral. For eigs in r_{\pm} directions, use

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \lambda_{\pm} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$$

Standing vs. Travelling Waves



Stability and branching direction of standing waves (SW) and travelling waves (TW) in (a_r, b_r) parameter plane. Either standing or traveling waves are stable, or neither are stable, depending on nonlinear coefficients a, b . (Knobloch, *Phys. Rev. A* **34**, 1538 (1986).)

Standing vs. Travelling Waves

origin : μ along r_+

μ along r_-

TW⁺ : -2μ along r_+

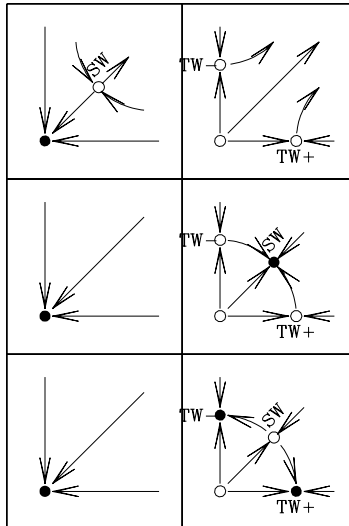
$-a_r\mu/b_r$ along r_-

TW⁻ : -2μ along r_-

$-a_r\mu/b_r$ along r_+

SW : -2μ along (r_+, r_-)

$2a_r\mu/(a_r + 2b_r)$ perp to (r_+, r_-)



Symmetries of Standing and Travelling Waves

Time translation: $T_{t_0} u(t) \equiv u(t + t_0)$

Group of all time translations: S^1

Symmetry group of homogeneous stationary state: $O(2) \times S^1$.

Travelling wave symmetries:

$$(T_{t_0} S_{\omega t_0} u)(\theta, t) \equiv u(\theta + \omega t_0, t + t_0) = u(\theta, t)$$

Group $\widetilde{SO}(2)$ (isomorphic to $SO(2)$)

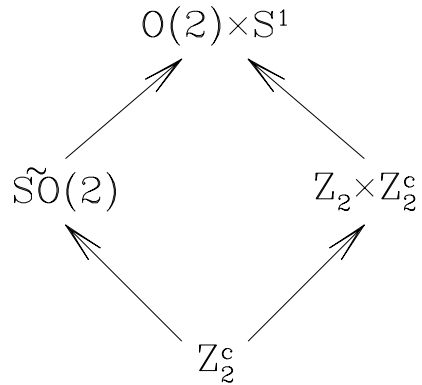
Standing wave symmetries (arbitrary symmetry axis at $\theta = 0$)

$$(\kappa u)(\theta, t) \equiv u(-\theta, t) = u(\theta, t) \quad Z_2$$

$$(T_{\pi/\omega} S_{\pi} u)(\theta, t) \equiv u(\theta + \pi, t + \pi/\omega) = u(\theta, t) \quad Z_2^c$$

Group $Z_2 \times Z_2^c$

Lattice of Isotropy Subgroups



Steady-state mode interactions

Bifurcations to two wavenumbers, m and n .

$$w(\rho, \theta) = \frac{1}{2} (z_m(\rho)e^{im\theta} + z_n(\rho)e^{in\theta} + \bar{z}_m(\rho)e^{-im\theta} + \bar{z}_n(\rho)e^{-in\theta})$$

$$S_{\theta_0}(z_m, z_n) = (e^{im\theta_0} z_m, e^{in\theta_0} z_n)$$

$$\kappa(z_m, z_n) = (\bar{z}_m, \bar{z}_n)$$

$$f(z_m, z_n) = \begin{pmatrix} f_m \\ f_n \end{pmatrix} (z_m, z_n) = \begin{pmatrix} f_{mpqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \\ f_{npqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \end{pmatrix}$$

$\kappa f = f \kappa \implies f$ real.

$$S_{\theta} f(z_m, z_n) = \begin{pmatrix} e^{im\theta} f_{mpqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \\ e^{in\theta} f_{npqrs} z_m^p z_n^q \bar{z}_m^r \bar{z}_n^s \end{pmatrix}$$

$$f S_{\theta}(z_m, z_n) = \begin{pmatrix} f_{mpqrs} (e^{im\theta} z_m)^p (e^{in\theta} z_n)^q (e^{-im\theta} \bar{z}_m)^r (e^{-in\theta} \bar{z}_n)^s \\ f_{npqrs} (e^{im\theta} z_m)^p (e^{in\theta} z_n)^q (e^{-im\theta} \bar{z}_m)^r (e^{-in\theta} \bar{z}_n)^s \end{pmatrix}$$

$$S_\theta f = f S_\theta \implies$$

$$f_{mpqrs} = 0 \text{ or } m = mp + nq - mr - ns$$

$$f_{npqrs} = 0 \text{ or } n = mp + nq - mr - ns$$

All invariants are products and sums of:

$$|z_m|^2, |z_n|^2, \text{ and } \Delta \equiv z_m^n \bar{z}_n^m + \bar{z}_m^n z_n^m$$

All equivariants are sums of:

$$\begin{pmatrix} z_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_n \end{pmatrix}, \begin{pmatrix} \bar{z}_m^{n-1} z_n^m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_m^n \bar{z}_n^{m-1} \end{pmatrix}$$

with coefficients which are invariants.

Most general equivariant evolution equation is:

$$\frac{d}{dt} \begin{pmatrix} z_m \\ z_n \end{pmatrix} = a \begin{pmatrix} z_m \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ z_n \end{pmatrix} + c \begin{pmatrix} \bar{z}_m^{n-1} z_n^m \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ z_m^n \bar{z}_n^{m-1} \end{pmatrix}$$

where a, b, c, d are functions of $(|z_m|^2, |z_n|^2, \Delta)$.

Assume $m + n - 1 > 3$ and truncate to cubic order.

The most general set of equivariant equations is independent of m, n :

$$\dot{z}_m = (a_0 + a_m |z_m|^2 + a_n |z_n|^2) z_m$$

$$\dot{z}_n = (b_0 + b_m |z_m|^2 + b_n |z_n|^2) z_n$$

a, b real \implies phases play no role \implies replace (z_m, z_n) by (x_m, x_n) .

Steady states:

$$x_m = 0 \quad \text{or} \quad a_0 + a_m x_m^2 + a_n x_n^2 = 0 \quad \text{and}$$

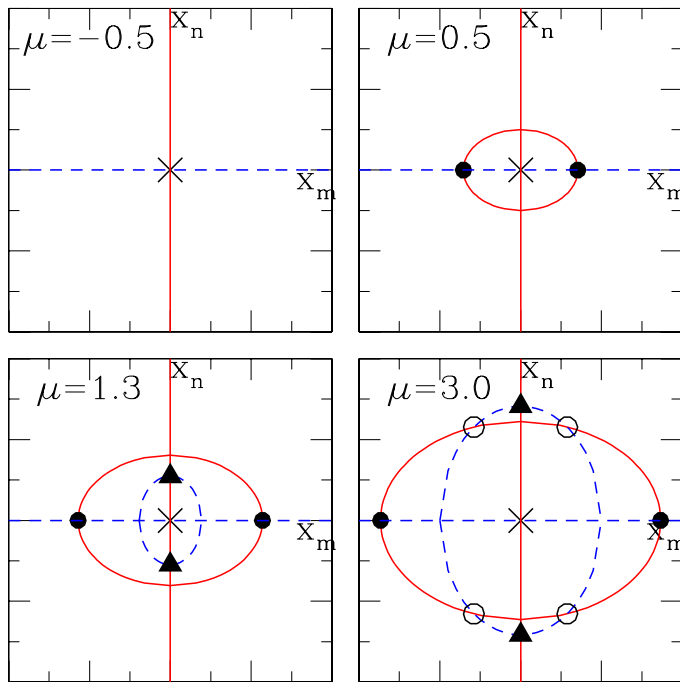
$$x_n = 0 \quad \text{or} \quad b_0 + b_m x_m^2 + b_n x_n^2 = 0$$

$$\text{origin :} \quad x_m = 0, x_n = 0$$

$$\text{pure } m \text{ modes :} \quad x_m^2 = -a_0/a_m, x_n = 0$$

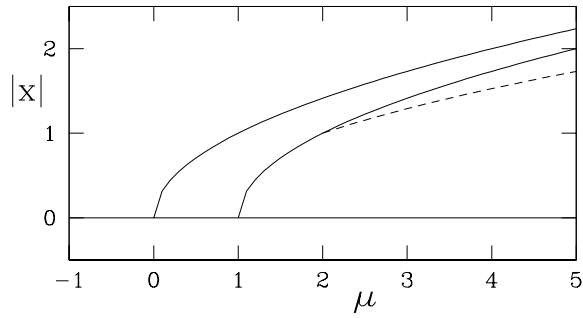
$$\text{pure } n \text{ modes :} \quad x_m = 0, x_n^2 = -b_0/b_n$$

$$\text{mixed modes :} \quad x_m^2 = \frac{a_0 b_n - b_0 a_n}{b_m a_n - a_m b_n},$$
$$x_n^2 = \frac{a_0 b_m - b_0 a_m}{-(b_m a_n - a_m b_n)}$$

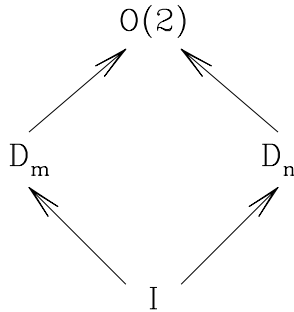


$$a_0 = \mu \quad b_0 = \mu - 1 \quad a_m = b_n = -1 \quad a_n = b_m = -2$$

$$\begin{aligned}
 x_m = 0 & \quad \text{or} \quad \mu - x_m^2 - 2x_n^2 = 0 \quad \text{and} \\
 x_n = 0 & \quad \text{or} \quad (\mu - 1) - 2x_m^2 - x_n^2 = 0
 \end{aligned}$$



**Bifurcation diagram for (x_m, x_n) . $|x| \equiv \sqrt{(x_m^2 + x_n^2)}$ as a function of μ .
Solid curves: pure modes. Dashed curve: mixed mode.**



Lattice of isotropy subgroups for (m, n) mode interaction.

Case $(m, n) = (1, 2)$:

$$\dot{z}_1 = c_0 \bar{z}_1 z_2 + (a_0 + a_1 |z_1|^2 + a_2 |z_2|^2) z_1$$

$$\dot{z}_2 = d_0 z_1^2 + (b_0 + b_1 |z_1|^2 + b_2 |z_2|^2) z_2$$

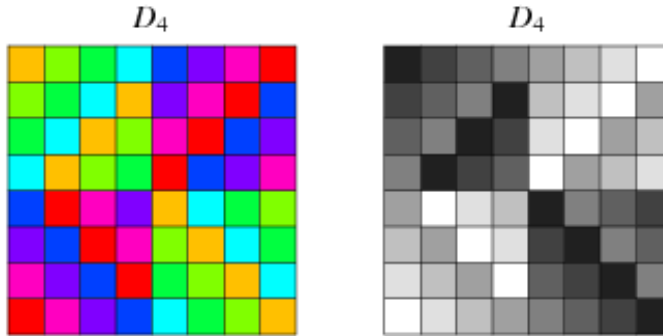
Case $(m, n) = (1, 3)$:

$$\dot{z}_1 = c_0 \bar{z}_1^2 z_3 + (a_0 + a_1 |z_1|^2 + a_3 |z_3|^2) z_1$$

$$\dot{z}_3 = d_0 z_1^3 + (b_0 + b_1 |z_1|^2 + b_3 |z_3|^2) z_3$$

\implies Interesting dynamics like heteroclinic orbits!

Group Table for D_4 , symmetries of a square ($\rho \equiv S_{\pi/2}$)



From Mathworld, by Eric Weisstein, Wolfram Research.

e	e	ρ	ρ^2	ρ^3	κ	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$
e	e	ρ	ρ^2	ρ^3	κ	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$
ρ	ρ	ρ^2	ρ^3	e	$\kappa\rho$	$\kappa\rho^2$	$\kappa\rho^3$	κ
ρ^2	ρ^2	ρ^3	e	ρ	$\kappa\rho^2$	$\kappa\rho^3$	κ	$\kappa\rho$
ρ^3	ρ^3	e	ρ	ρ^2	$\kappa\rho^3$	κ	$\kappa\rho$	$\kappa\rho^2$
κ	κ	$\kappa\rho^3$	$\kappa\rho^2$	$\kappa\rho$	e	ρ^3	ρ^2	ρ
$\kappa\rho$	$\kappa\rho$	κ	$\kappa\rho^3$	$\kappa\rho^2$	ρ	e	ρ^3	ρ^2
$\kappa\rho^2$	$\kappa\rho^2$	$\kappa\rho$	κ	$\kappa\rho^3$	ρ^2	ρ	e	ρ^3
$\kappa\rho^3$	$\kappa\rho^3$	$\kappa\rho^2$	$\kappa\rho$	κ	ρ^3	ρ^2	ρ	e

Quotient Groups

One one-element subgroup: $\{e\}$

Five two-element subgroups: $\{e, \rho^2\}$, $\{e, \kappa\}$, $\{e, \kappa\rho\}$, $\{e, \kappa\rho^2\}$, $\{e, \kappa\rho^3\}$

Two four-element subgroups: $\{e, \rho, \rho^2, \rho^3\}$, $\{e, \rho^2, \kappa, \kappa\rho^2\}$

$\{e, \rho, \rho^2, \rho^3\}$ is isomorphic to Z_4

$\{e, \rho^2, \kappa, \kappa\rho^2\}$ is isomorphic to $Z_2 \times Z_2$

Normal subgroup: $gng^{-1} \in N$ for all elements $g \in \Gamma$, $n \in N$

$N \equiv \{e, \rho, \rho^2, \rho^3\}$ is normal subgroup of $\Gamma \equiv D_4$

Can form quotient group Γ/N isomorphic to Z_2