Binary fluid convection
as a
2×2 matrix problem

Laurette Tuckerman

PMMH-ESPCI-CNRS

France

Convection due to two competing or cooperating forces

1960s: **Veronis, Sani, Nield, Baines, Gill, Chandrasekhar**

1970s: **Caldwell, Hurle, Jakeman, Schechter, Velarde, Platten, Chavepeyer, Huppert, Moore, Gershuni, Zhukhovitskii**

Codimension-two (Takens-Bogdanov) point

1980s: **Knobloch, Platten, Legros, Proctor, Da Costa, Weiss, Guckenheimer, Bretherton, Spiegel, Brand, Le Gal, Pocheau, Croquette, Coullet, Fauve, Tirapegui, Walden, Kolodner, Passner, Surko, Rehberg, Ahlers, Deane, Toomre, Moses, Steinberg, Cross, Lücke, Fineberg, Linz, Silber, Müller, Lhost, Bensimon, Pumir, Shraiman**

1990s: **Clune, Rucklidge, Riecke, Schöpf, Zimmermann, Predtechensky, Mc-Cormick, Swift, Rossberg, Swinney, Barten, Kamps, Schmitz, Dominguez-Lerma, Cannell, Hollinger, Büchel, Fütterer, Jung, Bergeon, Henry, Benhadid, Huke, Bestehorn**

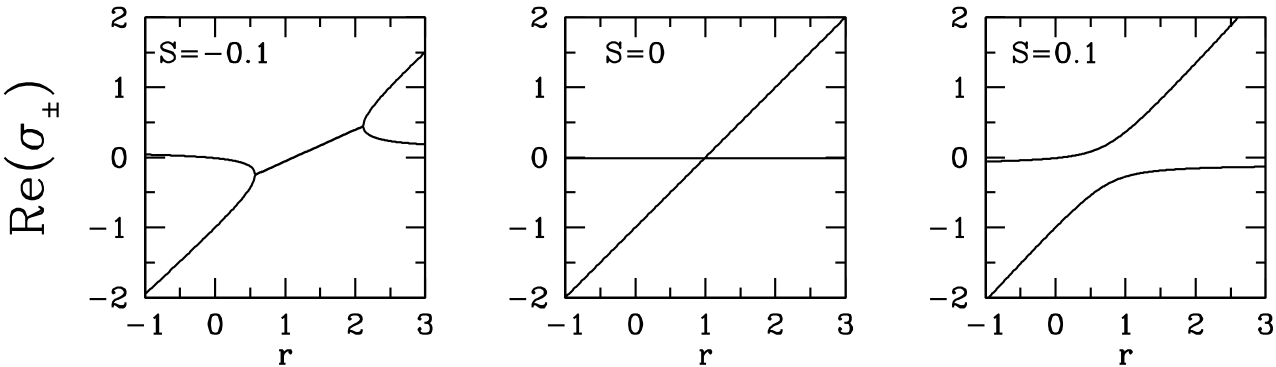
Bergeon, Henry, BenHadid & Tuckerman,
Marangoni convection in binary mixtures with Soret effect, JFM, 1998

Tuckerman, Thermosolutal and binary fluid convection as a $\mathbf{2 \times 2}$ *matrix problem, Physica D, 2001*

Basic idea: 2×2 matrix depending on control parameter r

$$\begin{pmatrix} \sigma_T & \alpha \\ \beta & \sigma_C \end{pmatrix}$$

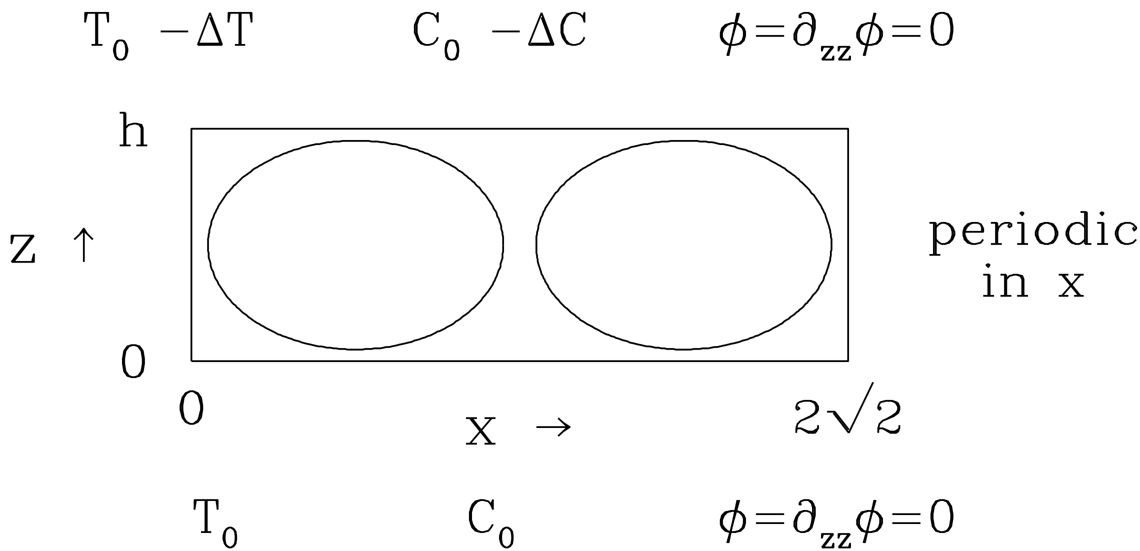
has eigenvalues $$\sigma_\pm = \frac{\sigma_T + \sigma_C}{2} \pm \sqrt{\left(\frac{\sigma_T - \sigma_C}{2}\right)^2 + \alpha\beta}$$



$\alpha\beta < 0$ **complex coalescence** | $\alpha\beta = 0$ **transverse crossing** | $\alpha\beta > 0$ **avoided crossing**

Sign of $\alpha\beta$ determines behavior of eigs where $\sigma_T(r)$ and $\sigma_C(r)$ intersect.

Simple Model: 2D Thermosolutal Problem



Vertical thermal and solutal gradients imposed at $z = 0, 1$

Boundary conditions: free-slip at $z = 0, 1$; periodic in x with length $2\sqrt{2}$

Streamfunction $\mathbf{U} = \nabla \times \psi(x, z)\mathbf{e}_y$

Density: $\rho(T,C) = \rho_0 + \rho_T(T - T_0) + \rho_C(C - C_0)$

Diffusivities: κ_T **(thermal),** κ_C **(solutal),** ν **(momentum)**

Conductive solution:

$T = T_0 - z\Delta T/h$, $C = C_0 - z\Delta C/h$, $\mathbf{U} = \nabla \times \psi \mathbf{e}_y = 0$

Four nondimensional parameters:

Fix: Lewis number $L \equiv \frac{\kappa_C}{\kappa_T} \ll 1$ Prandtl number $P \equiv \frac{\nu}{\kappa_T} \gg 1.$

Vary: Rayleigh number $R \equiv \frac{g\rho_T \Delta T h^3}{\nu\kappa_T}$ Separation ratio $S \equiv \frac{\rho_C \Delta C}{\rho_T \Delta T}$

Subtract conductive solution and nondimensionalize.

Governing Equations:

$$\begin{aligned}
\partial_t \tilde{T} &= \partial_x \tilde{\psi} + \mathbf{e}_y \cdot (\nabla\tilde{\psi} \times \nabla\tilde{T}) + \nabla^2 \tilde{T} \\
\partial_t \tilde{C} &= \partial_x \tilde{\psi} + \mathbf{e}_y \cdot (\nabla\tilde{\psi} \times \nabla\tilde{C}) + L\nabla^2 \tilde{C} \\
\partial_t \nabla^2\tilde{\psi} &= PR\partial_x(\tilde{T} + S\tilde{C}) + \mathbf{e}_y \cdot (\nabla\tilde{\psi} \times \nabla\nabla^2\tilde{\psi}) + P\nabla^4\tilde{\psi}
\end{aligned}$$

Linear Analysis:

$$\begin{Bmatrix} \tilde{T} \\ \tilde{C} \\ \tilde{\psi} \end{Bmatrix} (x, z, t) = \begin{Bmatrix} T\cos(kx) \\ C\cos(kx) \\ \psi\sin(kx) \end{Bmatrix} \sin(\pi z) e^{(k^2+\pi^2)\sigma t}$$

$$k = k_c = \pi/\sqrt{2}, \qquad \gamma^2 \equiv k^2 + \pi^2, \qquad r \equiv Rk^2/\gamma^6$$

$$P = \infty \quad \Longrightarrow \quad \psi = -r(T + SC)\gamma^2/k$$

2 × 2 system: $$\sigma \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r-1 & rS \\ r & rS - L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}$$

"thermal eigenvalue" $\sigma_T = r - 1$ **has threshold** $r_T = 1$

"solutal eigenvalue" $\sigma_C = rS - L$ $r_C = L/S$

Eigenvalues intersect at $r_{\rm int} = \frac{1-S}{1-L}$ **with coupling** Sr^2

Eigenvalues:

$$\sigma_\pm = \frac{(r-1)+(rS-L)}{2} \pm \sqrt{\left(\frac{(r-1)-(rS-L)}{2}\right)^2 + Sr^2}$$

$$\equiv f_L \pm \sqrt{g_L}$$

complex coalescence $(S < 0)$, cross transversely $(S = 0)$, or avoided crossing $(S > 0)$

Inverting:

$$r = \frac{\sigma_\pm^2 + \sigma_\pm(1+L) + L}{\sigma_\pm(1+S) + (S+L)}$$

Each value of σ corresponds to a unique value of r.
In particular, unless $S = -L$, $\exists$ exactly one pitchfork bifurcation $\sigma = 0$ at

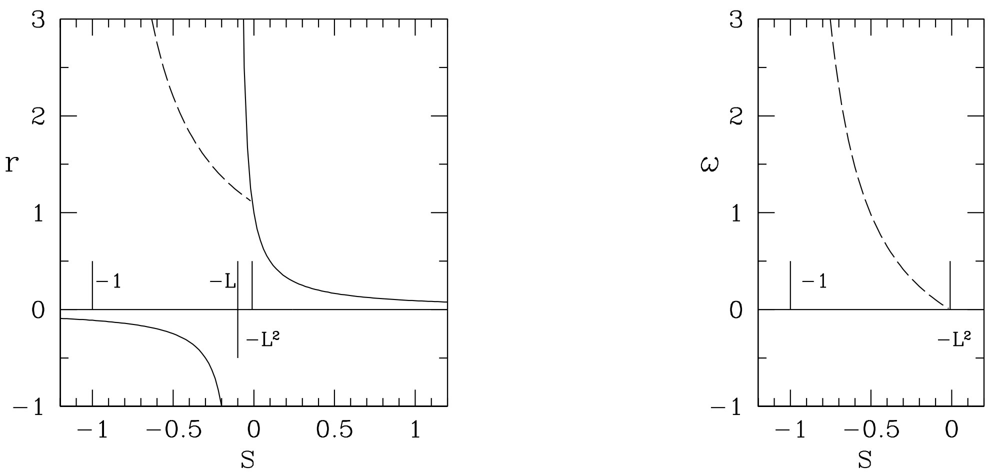
$$r = r_{PF} = \frac{L}{S+L}$$

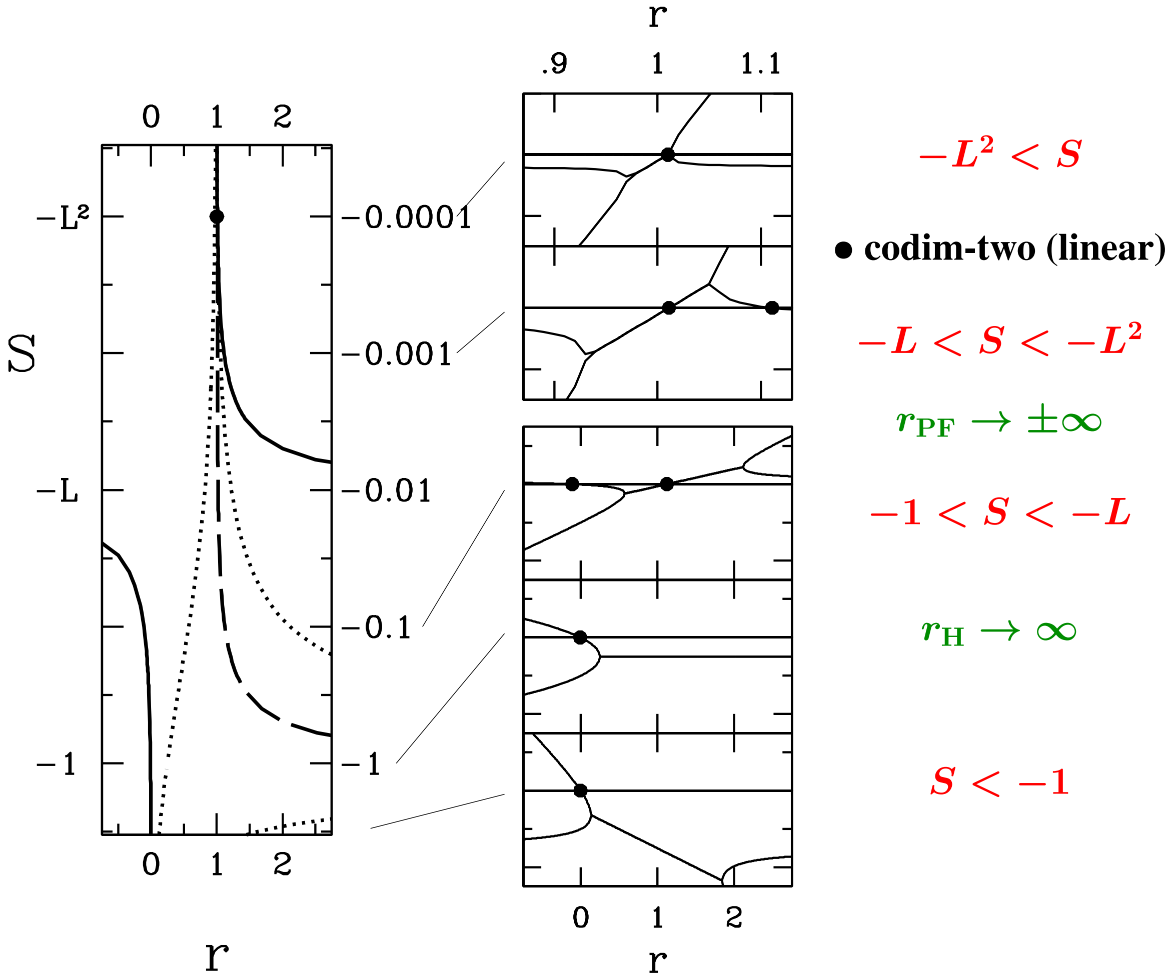
If $S < 0$, eigenvalues are complex over the range where $g_L < 0$:

$$\frac{1-L}{(1+\sqrt{-S})^2} \equiv r_{c-} < r < r_{c+} \equiv \frac{1-L}{(1-\sqrt{-S})^2}$$

There is a Hopf bifurcation if $f_L = 0$ in this range, i.e. if $-1 < S < -L^2$

$$r_{\rm H} = \frac{1+L}{1+S} \to \begin{cases} \infty \\ r_{\rm PF} \end{cases} \qquad \omega_{\rm H}^2 = -\frac{S+L^2}{1+S} \to \begin{cases} \infty \\ 0 \end{cases} \quad \text{as } S \to \begin{cases} -1 \\ -L^2 \end{cases}$$





r
.9
1
1.1
0
1
2
$-L^2$
$-L$
-1
S
-0.0001
-0.001
-0.01
-0.1
-1
r
$-L^2 < S$
● codim-two (linear)
$-L < S < -L^2$
$r_{PF} \to \pm\infty$
$-1 < S < -L$
$r_H \to \infty$
$S < -1$

Lorenz Model:
including nonlinear interactions

$$
\begin{aligned}
\boldsymbol{J}\left[\boldsymbol{\psi}, \boldsymbol{\nabla}^2\boldsymbol{\psi}\right] &= J[\psi, -\gamma^2\psi] \\
&= \partial_x\psi\ \partial_z(-\gamma^2\psi) - \partial_x(-\gamma^2\psi)\partial_z\psi = \mathbf{0}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{J}[\boldsymbol{\psi}, \boldsymbol{T}] &= \hat{\psi}\hat{T}\ [\partial_x(\sin qx \sin \pi z)\partial_z(\cos qx \sin \pi z) \\
&\quad -\partial_x(\cos qx\ \ \sin \pi z)\partial_z(\sin qx\ \ \sin \pi z)] \\
&= \hat{\psi}\hat{T}\ q\pi\ [\cos qx\ \ \sin \pi z\ \ \cos qx\ \ \cos \pi z \\
&\quad + \sin qx\ \ \sin \pi z\ \ \sin qx\ \ \cos \pi z] \\
&\quad + \hat{\psi}\hat{T}\ q\pi\ \left(\cos^2 qx + \sin^2 qx\right)\ \sin \pi z \cos \pi z \\
&= \hat{\psi}\hat{T}\ \frac{q\pi}{2}\ \sin 2\pi z
\end{aligned}
$$

$\uparrow\quad\uparrow\qquad\qquad\uparrow\uparrow$

functions $\qquad$ **scalars**

$$\psi(x,z,t) = \hat{\psi}(t)\sin qx \ \sin \pi z$$
$$T(x,z,t) = \hat{T}_1(t)\cos qx \ \sin \pi z + \hat{T}_2(t)\sin 2\pi z$$

$$\begin{aligned} J[\psi, T_2] &= \hat{\psi}\hat{T}_2 \ [\partial_x(\sin qx \sin \pi z)\partial_z(\sin 2\pi z) \\ &\quad -\partial_x(\sin 2\pi z)\partial_z(\sin qx \ \sin \pi z)] \\ &= \hat{\psi}\hat{T}_2 \ q \ 2\pi \ \cos qx \ \sin \pi z \ \cos 2\pi z \\ &= \hat{\psi}\hat{T}_2 \ q \ \pi \cos qx \ (\sin \pi z + \sin 3\pi z) \end{aligned}$$

Including $\hat{T}_3(t)\cos qx \sin 3\pi z \Longrightarrow$ new terms $\Longrightarrow$ Closure problem for nonlinear equations

Lorenz (1963) proposed stopping at T_2.

$$\partial_t \hat{\psi} = Pr(q\hat{T}_1/\gamma^2 - \gamma^2\hat{\psi}) \qquad \sin qx \sin \pi z$$

$$\partial_t \hat{T}_1 + q\pi\hat{\psi}\hat{T}_2 = Ra\; q\; \hat{\psi} - \gamma^2\hat{T}_1 \qquad \cos qx \sin \pi z$$

$$\partial_t \hat{T}_2 + \frac{q\pi}{2}\hat{\psi}\hat{T}_1 = -(2\pi)^2\hat{T}_2 \qquad \sin 2\pi z$$

Define:

$$X \equiv \frac{\pi q}{\sqrt{2}\gamma^2}\hat{\psi}, \quad Y \equiv \frac{\pi q^2}{\sqrt{2}\gamma^6}\hat{T}_1, \quad Z \equiv \frac{\pi q^2}{\sqrt{2}\gamma^6}\hat{T}_2,$$

$$\tau \equiv \gamma^2 t, \quad r \equiv \frac{q^2}{\gamma^6}Ra, \quad b \equiv \frac{4\pi^2}{\gamma^2} = \frac{8}{3}, \quad \sigma \equiv Pr$$

Famous Lorenz Model:

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = -XZ + rX - Y$$

$$\dot{Z} = XY - bZ$$

Do the same for C

Nonlinear interactions generate terms of form $\left\{ \begin{matrix} T_2 \\ C_2 \end{matrix} \right\} \sin(2\pi z)$

Truncating at this order yields:

$$\partial_t \begin{pmatrix} T \\ C \\ \psi \\ T_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} -\gamma^2 & 0 & -k & 0 & 0 \\ 0 & -L\gamma^2 & -k & 0 & 0 \\ -P\gamma^4 r/k & -PS\gamma^4 r/k & -P\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & -4\pi^2 & 0 \\ 0 & 0 & 0 & 0 & -L4\pi^2 \end{pmatrix} \begin{pmatrix} T \\ C \\ \psi \\ T_2 \\ C_2 \end{pmatrix}$$

$$+ \pi k \psi \begin{pmatrix} -T_2 \\ -C_2 \\ 0 \\ \frac{1}{2}T \\ \frac{1}{2}C \end{pmatrix}$$

Veronis (1965)

Steady states obey strange eigenproblem

After eliminating ψ, T_2, C_2, steady states obey:

$$\begin{pmatrix} r-1 & rS \\ Lr & L(rS-L) \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} = \frac{1}{2}\left(\frac{r\gamma}{2}\right)^2 (T+SC)^2 \begin{pmatrix} T \\ C \end{pmatrix}$$

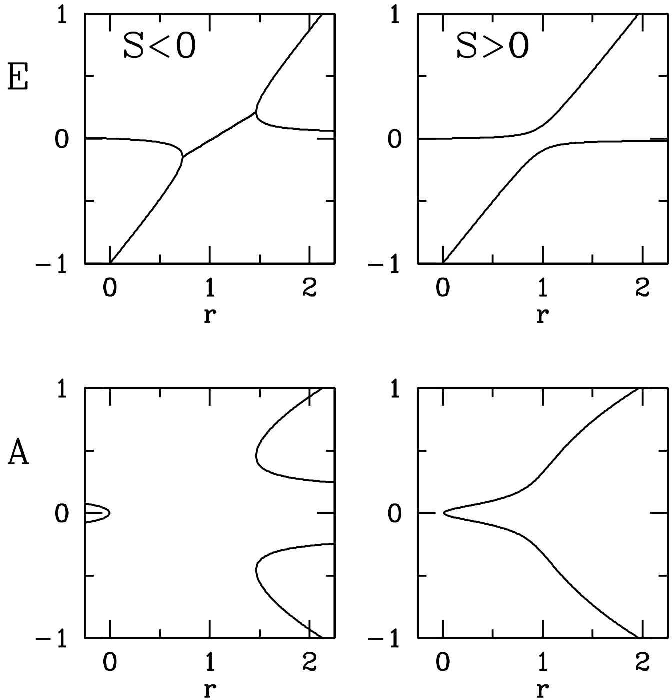
i.e. an eigenvalue problem for the square amplitude or energy:

$$E \equiv \frac{1}{2}\left(\frac{r\gamma}{2}\right)^2 (T+SC)^2$$

Solutions are:

$$E = \frac{(r-1)+L(rS-L)}{2} \pm \sqrt{\left(\frac{(r-1)-L(rS-L)}{2}\right)^2 + SLr^2}$$

$$\equiv f_{NL}(r) \pm \sqrt{g_{NL}(r)}$$



$S < 0$

Creation of two pairs of real solutions
$\pm\sqrt{E_\pm}$ via saddle-node bifurcations

$S > 0$

change of curvature between
weak solutal (“Soret”) and strong
thermal (“Rayleigh”) convective regimes

Exact analogy between growth rate and energy eigenproblems

- Translation from growth rate to energy via:

$$\begin{aligned} \sigma &\rightarrow E \\ L &\rightarrow L^2 \\ S &\rightarrow LS \end{aligned}$$

- Complex or negative σ indicate oscillating or decaying transients

$\longrightarrow$ Only real and positive values of E are meaningful

- Velocity streamfunction obeys $\psi = -r(T + SC)\gamma^2/k$ for:

$P = \infty$ in growth rate problem $\longrightarrow$ all P in energy problem

- Coupling has sign of S:

Sr^2 for growth-rate matrix $\longrightarrow$ LSr^2 for energy matrix

- **Each value of σ and of E is achieved exactly once:**

$$r = \frac{\sigma^2 + \sigma(1+L) + L}{\sigma(1+S) + (S+L)} \longrightarrow r = \frac{E^2 + E(1+L^2) + L^2}{E(1+LS) + L(S+L)}$$

- **At the pitchfork bifurcation, both σ and E necessarily vanish:**

$$r_{\mathrm{PF}} = \frac{L}{S+L} \longrightarrow r_{\mathrm{PF}} = \frac{L^2}{LS+L^2} = \frac{L}{S+L}$$

- **Transition from complex to positive real E is a saddle-node:**

$$r_{c+} = \frac{1-L}{(1-\sqrt{-S})^2} \longrightarrow r_{\mathrm{SN}} = \frac{1-L^2}{(1-\sqrt{-LS})^2}$$

- **Codimension-two bifurcations at $f = g = 0$:**

$$\left\{ \begin{array}{l} S_* = -L^2 \\ r_* = \frac{1}{1-L} \end{array} \right\} \text{ Takens-Bogdanov (PF/Hopf)} \longrightarrow \left\{ \begin{array}{l} S_* = -L^3 \\ r_* = \frac{1}{1-L^2} \end{array} \right\} \text{ degenerate PF (PF/SN)}$$

Bifurcations corresponding to $\left\{ \begin{array}{l} \textbf{growth rate } \sigma \\ \textbf{energy } E \end{array} \right\} = f \pm \sqrt{g}$

S	f	g	σ	E
	$\pm\sqrt{g}$	$+$	**pitchfork**	**pitchfork**
$-$	0	$-$	**Hopf**	
$-$	$+$	0		**saddle-node**
0	0	0	**Takens-Bogdanov (Hopf/pitchfork)**	**degenerate pitchfork (super/subcritical)**

Lorenz-like Hybrid Model:

$$\frac{1}{\gamma^2}\frac{d}{dt}\begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r-1 & rS \\ r & rS-L \end{pmatrix}\begin{pmatrix} T \\ C \end{pmatrix}$$
$$- \frac{1}{2}\left(\frac{r\gamma}{2}\right)^2 (T+SC)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1/L \end{pmatrix}\begin{pmatrix} T \\ C \end{pmatrix}$$

$S = 0$ is an *organizing center*.

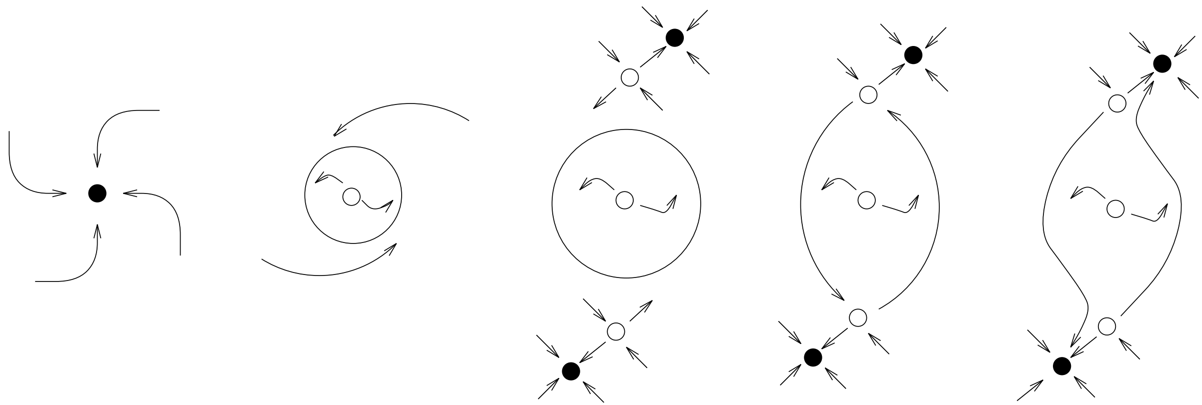
Reproduces:

- **linear stability diagram**
- **nonlinear steady states**
- **limit cycle created by Hopf, annihilated by global bifurcation**

Assumptions:

- $P = \infty$
- **truncate C_3, T_3, C_4, T_4, etc. despite slow damping of C_3, etc.**

Lorenz-like Hybrid Model for $S < 0$:



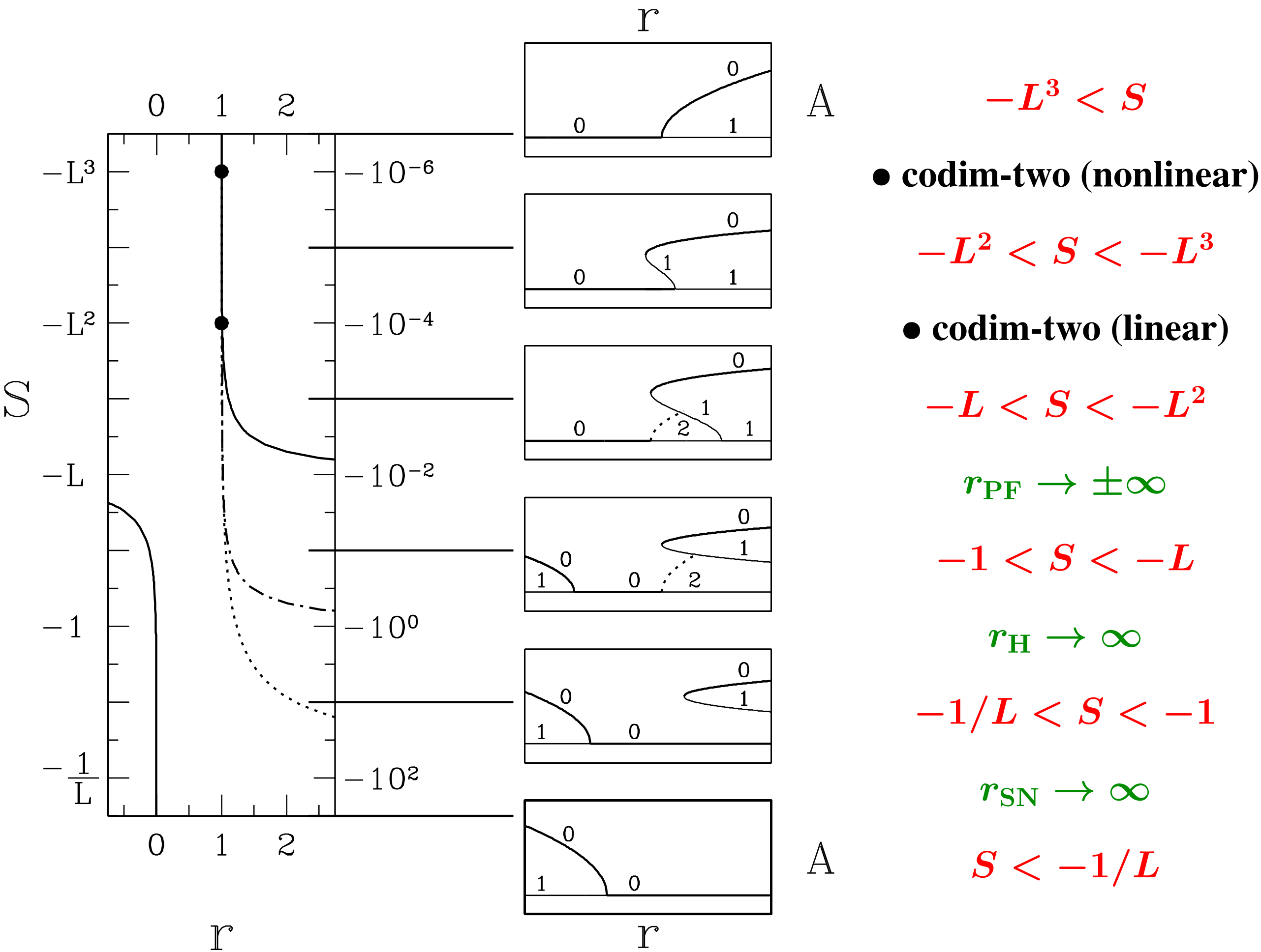
Hopf

Saddle-Node

Heteroclinic (Global)

r $\longrightarrow$

$$r_{\mathrm{H}} = \frac{1+L}{1+S} \qquad r_{\mathrm{SN}} = \frac{1-L^2}{(1-\sqrt{-LS})^2} \qquad r_{\mathrm{G}}$$



r
A
$-L^3 < S$
● codim-two (nonlinear)
$-L^2 < S < -L^3$
● codim-two (linear)
$-L < S < -L^2$
$r_{\rm PF} \to \pm\infty$
$-1 < S < -L$
$r_{\rm H} \to \infty$
$-1/L < S < -1$
$r_{\rm SN} \to \infty$
A
$S < -1/L$
S
$-L^3$
$-L^2$
$-L$
-1
$-\frac{1}{L}$
-10^{-6}
-10^{-4}
-10^{-2}
-10^{0}
-10^{2}
0
1
2
r

Conclusions

- **Organizing center $S = 0$: complex coalesce / avoided crossing.**

Complementary to Takens-Bogdanov point S_*.

- **Simple 2×2 matrix model captures all bifurcations.**

Yields analytic formulas.

- **Steady-state problem is *also* eigenproblem.**

Nonlinear problem reduced to eigenproblem $+$ one scalar equation.

- **Analogy between growth-rate and steady-state problems.**

Exact, quantitative correspondence.

- **Hybrid model also captures limit cycle and global bifurcation.**