# Math or Physics? Or Natural Philosophy?

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How many times have you asked or been asked, "Which do you prefer, math or physics?" You may have formulated an answer to this: probably "physics", since you are reading the Normale Physics Review. But what does this really mean? It is obvious that math and physics are closely related. Is there a clear-cut distinction between the two? If so, what is it? And how can mathematics and physics be defined? A common starting point is: Mathematics consists of posing and proving theorems. Physics consists of discovering new fundamental laws and testing them. Sociologically, neither is anywhere close to being true. The overwhelming majority of people called physicists never discover new fundamental laws, even those who publish many papers and receive awards. And many people called mathematicians do not prove theorems.

In studying electromagnetism or quantum mechanics or advanced classical mechanics, a great deal of what we learn is differential equations such as the Poisson or Schrodinger equations and techniques and special functions for solving them. Is this math or physics? Most will probably agree that this is math, or perhaps mathematical methods for physics, but this was not clear in my mind for a long time. Basically, this is the mathematics/physics that was developed in the 19th century, before the two diverged.

Mathematics has marched triumphantly across the landscape of the natural sciences, inventing and applying seemingly strange objects like complex numbers and quaternions. The alchemists were unable to explain the chemical elements, but the periodic table was finally ordered by the spherical harmonics. Biology is next on the agenda. In his famous article "The unreasonable effectiveness of mathematics in the natural sciences" [Communications on Pure and Applied Mathematics 13, 1, 1960], physicist Eugene Wigner states "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." Mathematician and philosopher Bertrand Russell believed that this was due to the nature of humans: "physics is mathematical ... because ... it is only its mathematical properties that we can discover" [An Outline of Philosophy, 1927]. Physicist Max Tegmark, on the other hand, believed that this was due to the nature of physics: "the physical world is a mathematical structure, and we are simply uncovering this bit by bit." [Found. Phys. 38, 101, 2008]. Another question about human nature: Is mathematics invented or discovered?

In the Great Hall of Ecole Polytechnique (sorry!) there is a stone tablet on which are engraved the names of the great professors of yore (a romantic expression for the past, often used semi-ironically, like "d'antan" or "jadis" in French): Ampere, Arago, Cauchy, Liouville, Hermite, Hadamard, Lagrange, Fourier, Poisson, Ampere, Mathieu, Navier, Painlevé, Poincaré. None of the above are designated as professors of physics. Instead, they are professors of mathematics or of mechanics. Mechanics? Where does that fit into our classification? Mechanics is the physics that precedes the early 20th century revolutions of quantum mechanics and relativity. There seems little reason to distinguish it logically from physics; clearly, if anything, mechanics is a particular and large branch of physics. In the last few decades, physics journals and departments have exploded with new types of mechanics problems: the behavior of granular media, new kinds of elastic shells. This brings up a new question (which we will not address): what is the difference, if any, between physics and engineering? Is it in the object of study? Clearly not, from the example of mechanics above. Is it in the degree of applicability? But don't physicists also aspire to be useful? Is it in the degree of generality? What about the construction of a new type of telescope or laser?

Nowadays, Physical Review contains articles on stock pricing, recessions, traffic jams, income inequality, the ranking of sports teams. One hears new terms such as econophysics and sociophysics. In what sense are these physics? Certainly not, if physics is defined to be an object of study. Perhaps, if physics is defined as a collection of methods, or a way of thinking. Is that what physics is? Mightn't we say just as easily -- and perhaps more appropriately -- that this research uses a collection of mathematical methods that are usually used or that originated in physics? Or is the rule that when people who call themselves physicists start to study something else, they call it a kind of physics? Perhaps the definition is sociological or tautological: physics is what is done by people who call themselves physicists, who work in physics departments and publish in what are called physics journals.

My own field is fluid dynamics, in which a (the?) major problem is turbulence. Fluid motion is described by equations derived by Navier in France in 1822 and Stokes in England in 1844. (Note that even when there were so few scientific journals and articles, scientists did not read them.) The Navier-Stokes equations are perfectly adequate descriptions of the flow of incompressible Newtonian fluids and computers are perfectly capable of solving them to increasingly high Reynolds numbers each year, reproducing the phenomenology of turbulence seen in the laboratory. What then is "the problem of turbulence"? I will propose an analogy. Outside of the relativistic and quantum-mechanical domains, Newton's laws are perfectly good descriptions of the motion of particles. If we know the positions and velocities of a million or more particles at some time, along with the forces acting on them and the shape and nature of the domain, we can enter all of this into a computer and Newton's laws will give the positions and velocities of the particles at a later time. Yet, this procedure seems rather short on elegance, generality, or insight.

This is where statistical mechanics enters. Boltzmann did not discover any new force or law; what he did was to situate the problem on a higher level, defining macroscopic variables that condense the Newtonian description into a much more useful one. This is quite different from the revolutions of quantum mechanics or relativity, which addressed incorrect results produced by the previously accepted Newtonian laws of physics. I propose that the "problem of turbulence" is similar, that while the Navier-Stokes equations are not wrong, the description they provide is at too low a level to be useful despite their accuracy. We seek a formalism that will not invalidate or correct the Navier-Stokes equations, but will sit on top of them to provide the macroscopic predictions that we seek. Basically, we will have the "answer" to the turbulence problem once we know what the question is.

Returning to statistical mechanics, what is it? Is it physics, as you have been taught? Or might it actually be math? I can propose a different distinction between physics and math. The laws of gravitation or electromagnetism could logically be other than they are. But mathematics proceeds by pure logical reasoning and could not be other than it is. By this definition, much of statistical mechanics would be math; it is derived by pure thought.

An important development from the last quarter of the 20th century is what is sometimes called the chaos revolution, developed by researchers such as May, Lorenz, Feigenbaum and Swinney in the U.S. and Coullet, Tresser, Pomeau, Manneville, Ruelle, Libchaber, Berge, Dubois, Fauve in France (yes, France, and even ENS, played a crucial role in the chaos revolution!) These researchers discovered that small changes in initial conditions could grow exponentially, leading to completely different endpoints. Of course, this is true of any system in which there is exponential growth, but they showed that this could also be true of systems in which the trajectories occupy a bounded subspace, called strange attractors. This led to completely rethinking ideas in several fields, such as meteorology. You have probably heard of the butterfly effect, by which a butterfly flapping its wings in Brazil might lead to the formation of a distant tornado. The separation of trajectories (outcomes) implies that another digit of accuracy in initial meteorological conditions (i.e. a 10-fold reduction in uncertainty) leads to only an additional day (for example) of accurate prediction. In 1969, MIT meteorologist Edward Lorenz, proposed that the time horizon for weather prediction was limited to two weeks; 50 years later, atmospheric scientists reiterated this estimate [E. Lorenz, Tellus 21, 289, 1969; F. Zhang et al. J. Atmos. Sci. 76, 1077, 2019]. Hamiltonian chaos has come to play an important role in plasma physics, which in turn describes prospective nuclear fusion reactors such as the ITER tokamak at Cadarache in Provence. Chaos theory, more properly called dynamical systems theory, concerns the qualitative behavior of differential equations, and builds on the field developed in the 1880s by Henri Poincaré. (The Institut Henri Poincaré is a block away from ENS, on rue Pierre et Marie Curie). [Dynamical systems theory is the right way to understand the behavior of nonlinear differential or difference equations, just as linear algebra is the right way to understand the behavior of systems of linear equations. Linear algebra is viewed as math and, moreover, a fundamental and necessary part of every quantitative person's education. Yet dynamical systems theory, in contrast, is viewed as a somewhat specialized topic.] Is chaos theory math or physics?

To address this question, let us focus just on physics (or physicists) for the moment. The physicist Victor Weisskopf (1908-2002) divided physics into "intensive" and "extensive". Intensive physics is the formulation of new fundamental laws, as in high energy physics. Extensive physics is the explanation of phenomena in terms of known fundamental laws, as in condensed matter physics and plasma physics. Philip Anderson (1923-2020), who won the Nobel Prize in physics in 1977 for his work in magnetic and disordered systems, had strong opinions about this. He rejected the classification of extensive physics as less worthy than intensive physics, giving the word "emergence" to the complexity that connects the different levels. Presenting a hierarchy of elementary particle physics, many-body physics, chemistry, molecular biology, cell biology, ... going all the way up to physiology, psychology, social sciences, Anderson states: "At each level of complexity entirely new properties appear and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other. But this hierarchy does not imply that science X is just applied Y. At each stage entirely new laws, concepts and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one." [P.W. Anderson, Science 177, 393, 1972]. Both statistical mechanics and chaos theory would seem to be examples of the "emergence" or "complexity" described by Anderson.

I have asked a great many questions but proposed very few answers. To the extent that I have a conclusion, it is this. Between the time of Aristotle and the mid 19th century, the discipline that attempted to catalogue and explain the wonders of the world was called natural philosophy; the subdivisions of physics, chemistry and biology date from the mid $19^{th}$ century. But today, it should not be necessary to decide whether what you are doing is called math or physics or pure or applied physics or chemistry or engineering. Call yourself a natural philosopher and a seeker of truth.