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Numerical Methods for Differential Equations in Physics

Temporal Discretization

$$\frac{du}{dt} = f(u) \qquad u, f \text{ in } \mathcal{R}^N$$

Goal: turn differential equation into difference equation

First order methods:

Forwards (Explicit) Euler:

$$u^{\mathrm{FE}}(t + \Delta t) = u^{\mathrm{FE}}(t) + \Delta t f(u^{\mathrm{FE}}(t))$$

explicit: “=” is an assignment statement

Backwards (Implicit) Euler:

$$u^{\mathrm{BE}}(t + \Delta t) = u^{\mathrm{BE}}(t) + \Delta t f(u^{\mathrm{BE}}(t + \Delta t))$$

implicit: “=” is an equation to be solved for $u^{\mathrm{BE}}(t + \Delta t)$

Taylor series:

$$
\begin{aligned}
u(t+\Delta t) &= u(t) + \Delta t \frac{du}{dt} + \frac{\Delta t^2}{2}\frac{d^2u}{dt^2} + \ldots \\
&= u(t) + \Delta t f(u(t)) + \frac{\Delta t^2}{2} f'(u(t)) f(u(t)) + \ldots
\end{aligned}
$$

Forwards (Explicit) Euler:

$$
u^{\mathrm{FE}}(t+\Delta t) = u^{\mathrm{FE}}(t) + \Delta t f(u^{\mathrm{FE}}(t))
$$

Backwards (Implicit) Euler:

$$
\begin{aligned}
&u^{\mathrm{BE}}(t+\Delta t) = u^{\mathrm{FE}}(t) + \Delta t f(u^{\mathrm{BE}}(t+\Delta t)) \\
&= u^{\mathrm{BE}}(t) + \Delta t \left(f(u^{\mathrm{BE}}(t)) + \Delta t f'(u^{\mathrm{BE}}(t)) f(u^{\mathrm{BE}}(t)) + \ldots \right) \\
&= u^{\mathrm{BE}}(t) + \Delta t f(u^{\mathrm{BE}}(t)) + \Delta t^2 f'(u^{\mathrm{BE}}(t)) f(u^{\mathrm{BE}}(t)) + \ldots
\end{aligned}
$$

First order: Δt terms match Taylor series but not Δt^2 terms

Second order methods:

Adams-Bashforth (explicit)

$$u^{\mathrm{AB}}(t+\Delta t) = u^{\mathrm{AB}}(t)+\Delta t\left(\frac{3}{2}f(u^{\mathrm{AB}}(t)) - \frac{1}{2}f(u^{\mathrm{AB}}(t-\Delta t))\right)$$

Crank-Nicolson (implicit) also called trapezoidal

$$u^{\mathrm{CN}}(t+\Delta t) = u^{\mathrm{CN}}(t)+\Delta t\left(\frac{1}{2}f(u^{\mathrm{CN}}(t)) + \frac{1}{2}f(u^{\mathrm{CN}}(t+\Delta t))\right)$$

Backwards Differentiation (implicit)

$$u^{\mathrm{BD}}(t+\Delta t) = \frac{4}{3}u^{\mathrm{BD}}(t)-\frac{1}{3}u^{\mathrm{BD}}(t-\Delta t)+\frac{2}{3}\Delta t f(u^{\mathrm{BD}}(t+\Delta t))$$

Second order: Δt, Δt^2 terms match Taylor series but not Δt^3

For fixed T, take $T/\Delta t$ steps

If one-step error is Δt^p,
total error at time T is $(T/\Delta t)\Delta t^p = T\Delta t^{p-1}$

First-order methods have one-step error Δt^2
and fixed-time error Δt

Second-order methods have one-step error Δt^3
and fixed-time error Δt^2

Have assumed $f(u(t))$, but can also have $f(u(t), t)$

Second or higher order differential equations:
Write $u_0 = u$, $u_1 = u_0'$, $u_2 = u_0''$, etc.

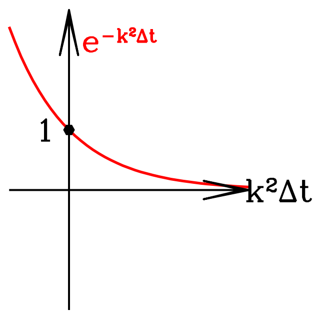
$$\frac{du}{dt} = f(u) \qquad u = (u_0, u_1, \ldots), f = (f_0, f_1, \ldots)$$

Stability: example of heat equation with periodic boundary conditions

$$\partial_t u = \partial_{xx} u$$
$$u(x,t) = \sum_{k=1}^{k_{max}} \hat{u}_k(t) \sin kx$$
$$\partial_t \hat{u}_k = -k^2 \hat{u}_k$$

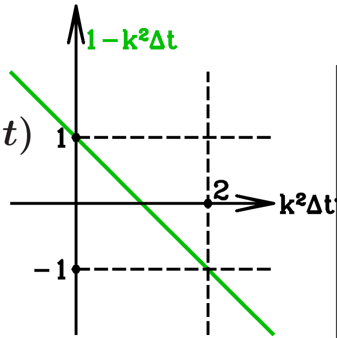
EXACT SOLUTION

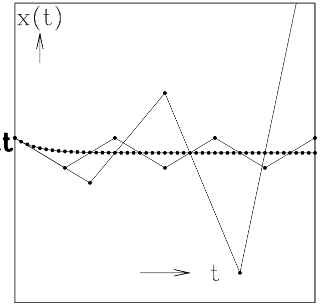
$$\hat{u}_k(t+\Delta t) = e^{-k^2 \Delta t} \hat{u}_k(t)$$



Explicit Euler

$$\hat{u}_k^{\mathrm{FE}}(t+\Delta t) = \hat{u}_k^{\mathrm{FE}}(t) - k^2 \Delta t \hat{u}_k^{\mathrm{FE}}(t)$$
$$= (1-k^2\Delta t)\hat{u}_k^{\mathrm{FE}}(t)$$





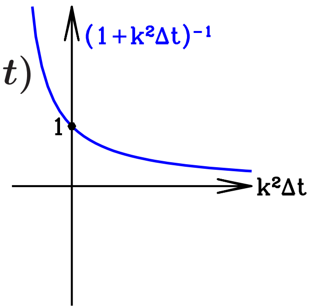
As $k_{max} \to \infty$, $\Delta t_{max} = \frac{2}{k_{max}^2} \to 0$

Implicit Euler

$$\hat{u}_k^{\mathrm{BE}}(t+\Delta t) = \hat{u}_k^{\mathrm{BE}}(t) - k^2 \Delta t \hat{u}_k^{\mathrm{BE}}(t+\Delta t)$$
$$(1+k^2\Delta t)\hat{u}_k^{\mathrm{BE}}(t+\Delta t) = \hat{u}_k^{\mathrm{BE}}(t)$$
$$\hat{u}_k^{\mathrm{BE}}(t+\Delta t) = (1+k^2\Delta t)^{-1}\hat{u}_k^{\mathrm{BE}}(t)$$

Matrix version:

$$u^{\mathrm{BE}}(t+\Delta t) = (I-\Delta t L)^{-1} u^{\mathrm{BE}}(t)$$



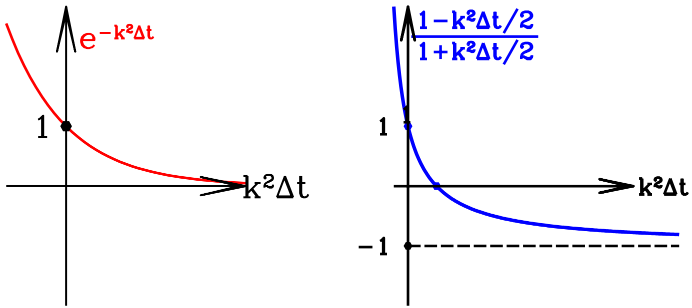
Crank-Nicolson:

$$u^{\mathrm{CN}}(t+\Delta t) = u^{\mathrm{CN}}(t) + \frac{\Delta t}{2}\left(f(u^{\mathrm{CN}}(t)) + f(u^{\mathrm{CN}}(t+\Delta t))\right)$$

$$u^{\mathrm{CN}}(t+\Delta t) - \frac{\Delta t}{2} f(u^{\mathrm{CN}}(t+\Delta t)) = u^{\mathrm{CN}}(t) + \frac{\Delta t}{2} f(u^{\mathrm{CN}}(t))$$

$$\left(1 + \frac{k^2\Delta t}{2}\right)\hat{u}_k^{\mathrm{CN}}(t+\Delta t) = \left(1 - \frac{k^2\Delta t}{2}\right)\hat{u}_k^{\mathrm{CN}}(t)$$

$$\hat{u}_k^{\mathrm{CN}}(t+\Delta t) = \frac{1 - \frac{k^2\Delta t}{2}}{1 + \frac{k^2\Delta t}{2}}\quad \hat{u}_k^{\mathrm{CN}}(t)$$



$|$**Amp. factor**$| < 1$ **but spurious large-**k **behavior: slow oscillatory decay**

A-stable methods:

$$\frac{du}{dt} = -k^2 u \Longrightarrow \qquad u_{\text{exact}}(t) = e^{-k^2 t}$$

$$\Longrightarrow \qquad \lim_{t\to\infty} u_{\text{exact}}(t) = 0$$

$$\textbf{A-stable method} \Longrightarrow \qquad \lim_{t\to\infty} u_{\text{numerical}}(t) = 0$$

Define $q \equiv -k^2 \Delta t$ and generalize to q complex

Define amplification factor $\Phi(q)$

Consider behavior for $\mathcal{R}e(q) < 0$

A-stable: $|\Phi(q)| < 1$

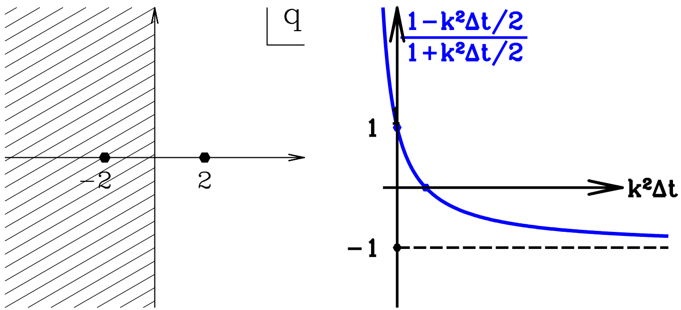
L-stable: $|\Phi(q)| \to 0$ as $q \to \infty$

Crank-Nicolson: seek stability region in complex plane, where

$$\left|\Phi^{\mathrm{CN}}(q)\right| \equiv \left|\frac{1+q/2}{1-q/2}\right| \leq 1$$

$$|1+q/2| \leq |1-q/2|$$

q **is closer to** -2 **than to 2:** q **is in left half plane**



A-stable $\Longleftrightarrow$ **stability region contains negative real axis**

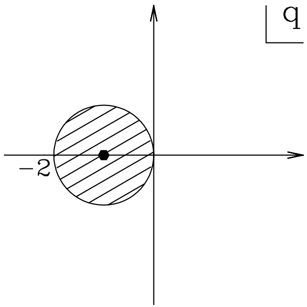
$\Longleftrightarrow$ **For** $\mathcal{R}e(q) < 0$**, have** $|\Phi(q)| < 1$ ✓ **YES**

L-stable $\Longleftrightarrow$ **For** $\mathcal{R}e(q) < 0$**, have** $|\Phi(q)| \to 0$ **as** $q \to \infty$

$\times$ **NO**

Forwards Euler

$$|1 + q| < 1$$

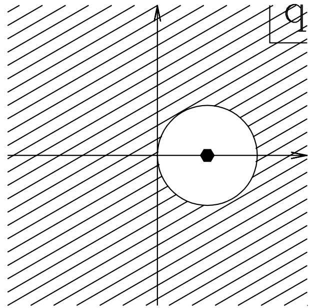


For q real, require $-2 < q < 0$

Backwards Euler

$$1/|1 - q| < 1$$

$$|1 - q| > 1$$



All $q < 0$ in stability region

Adams-Bashforth:

$$u^{\text{AB}}(t+\Delta t) = u^{\text{AB}}(t) + \frac{\Delta t}{2}\left(3f(u^{\text{AB}}(t)) - f(u^{\text{AB}}(t-\Delta t))\right)$$
$$= u^{\text{AB}}(t) + q\left(\frac{3}{2}u^{\text{AB}}(t) - \frac{1}{2}u^{\text{AB}}(t-\Delta t)\right)$$

$$\begin{pmatrix} u^{\text{AB}}(t+\Delta t) \\ u^{\text{AB}}(t) \end{pmatrix} = \begin{pmatrix} 1+\frac{3q}{2} & -\frac{q}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u^{\text{AB}}(t) \\ u^{\text{AB}}(t-\Delta t) \end{pmatrix}$$

Require that both eigenvalues obey $|\lambda| < 1$

Eigenvalues λ obey characteristic polynomial

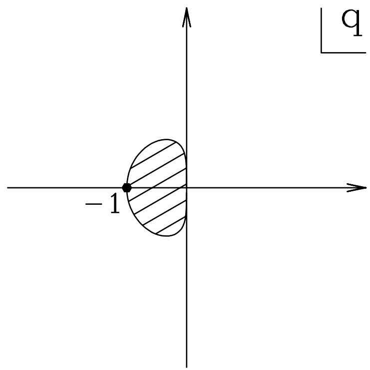
$$0 = \left(1+\frac{3q}{2}-\lambda\right)(-\lambda) + \frac{q}{2}$$
$$= \lambda^2 - \lambda\left(1+\frac{3q}{2}\right) + \frac{q}{2}$$
$$\frac{q}{2} = \frac{\lambda^2-\lambda}{3\lambda-1}$$

Set $\lambda = e^{i\theta}$:

$$\frac{q}{2} = \frac{\lambda^2 - \lambda}{3\lambda - 1} = \frac{e^{2i\theta} - e^{i\theta}}{3e^{i\theta} - 1} = \frac{\cos 2\theta - \cos\theta + i(\sin 2\theta - \sin\theta)}{(3\cos\theta - 1) + 3i\sin\theta}$$

$$= \frac{(\cos 2\theta - \cos\theta)\,(3\cos\theta - 1) + (\sin 2\theta - \sin\theta)3\sin\theta}{(3\cos\theta - 1)^2 + (3\sin\theta)^2}$$

$$+\, i\,\frac{(\cos 2\theta - \cos\theta)\,(-3\sin\theta) + (\sin 2\theta - \sin\theta)(3\cos\theta - 1)}{(3\cos\theta - 1)^2 + (3\sin\theta)^2}$$



not A-stable

Backwards Differentiation:

$$u^{\mathrm{BD}}(t+\Delta t) = \frac{4}{3}u^{\mathrm{BD}}(t) - \frac{1}{3}u^{\mathrm{BD}}(t-\Delta t) + \frac{2}{3}\Delta t f(u^{\mathrm{BD}}(t+\Delta t)$$

$$= \frac{4}{3}u^{\mathrm{BD}}(t) - \frac{1}{3}u^{\mathrm{BD}}(t-\Delta t) + \frac{2}{3}q u^{\mathrm{BD}}(t+\Delta t)$$

$$\left(1-\frac{2}{3}q\right)u^{\mathrm{BD}}(t+\Delta t) = \frac{4}{3}u^{\mathrm{BD}}(t) - \frac{1}{3}u^{\mathrm{BD}}(t-\Delta t)$$

$$u^{\mathrm{BD}}(t+\Delta t) = \frac{1}{1-\frac{2q}{3}}\left(\frac{4}{3}u^{\mathrm{BD}}(t) - \frac{1}{3}u^{\mathrm{BD}}(t-\Delta t)\right)$$

$$\begin{pmatrix} u^{\mathrm{BD}}(t+\Delta t) \\ u^{\mathrm{BD}}(t) \end{pmatrix} = \begin{pmatrix} \frac{4}{3-2q} & \frac{-1}{3-2q} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u^{\mathrm{BD}}(t) \\ u^{\mathrm{BD}}(t-\Delta t) \end{pmatrix}$$

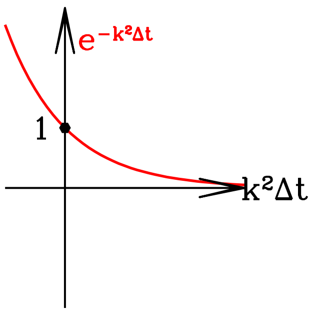
Eigenvalues λ obey characteristic polynomial

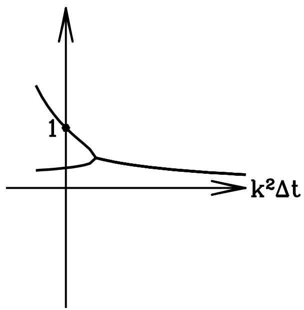
$$0 = \left(\frac{4}{3-2q} - \lambda\right)(-\lambda) + \frac{1}{3-2q}$$

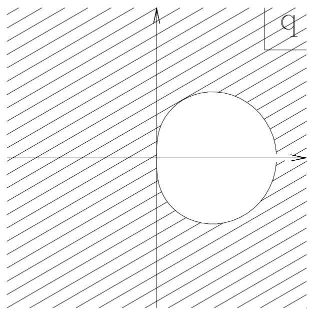
$$(2q-3) = -\frac{4}{\lambda} + \frac{1}{\lambda^2}$$

Require that both eigenvalues obey $|\lambda| < 1$ so set $\lambda = e^{i\theta}$

$$
\begin{aligned}
(2q - 3) &= -\frac{4}{e^{i\theta}} + \frac{1}{e^{2i\theta}} \\
&= -4(\cos\theta - i\sin\theta) + (\cos 2\theta - i\sin 2\theta) \\
q &= \frac{1}{2}(3 - 4\cos\theta + \cos 2\theta + i[4\sin\theta - \sin 2\theta])
\end{aligned}
$$







A-stable

General formalism:

$$\sum_{j=0}^{s} \alpha_j u^{n+1-j} = \Delta t \sum_{j=0}^{s} \beta_j f(u^{n+1-j})$$

Degrees of freedom $\{\alpha_j\}$, $\{\beta_j\}$ allow many routes to order-p accuracy.
Scale by setting $\alpha_0 = 1$. Explicit $\Longleftrightarrow$ $\beta_0 = 0$:

$$u^{n+1} = \sum_{j=1}^{s} \left(-\alpha_j u^{n+1-j} + \Delta t \beta_j f(u^{n+1-j})\right)$$

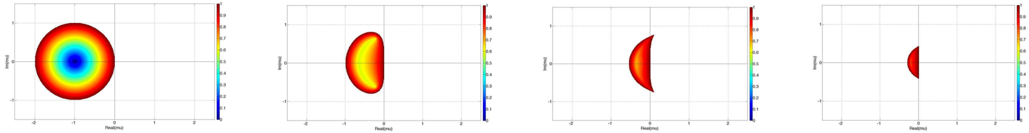
Adams-Bashforth (explicit): $\alpha_0 = 1, \alpha_1 = -1, \alpha_{j\geq 2} = 0, \beta_0 = 0$
Select $\beta_{1\leq j\leq p}$ to achieve p-order accuracy.

Adams-Moulton (implicit): $\alpha_0 = 1, \alpha_1 = -1, \alpha_{j\geq 2} = 0$
Select $\beta_{0\leq j\leq p-1}$ to achieve p-order accuracy.

Backwards-Differentiation (implicit): $\alpha_0 = 1, \beta_{j\geq 1} = 0$

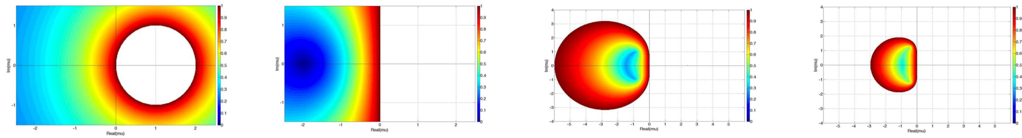
Select β_0, $\alpha_{1\leq j\leq p}$ to achieve p-order accuracy.

Stability regions of Adams-Bashforth formulas:



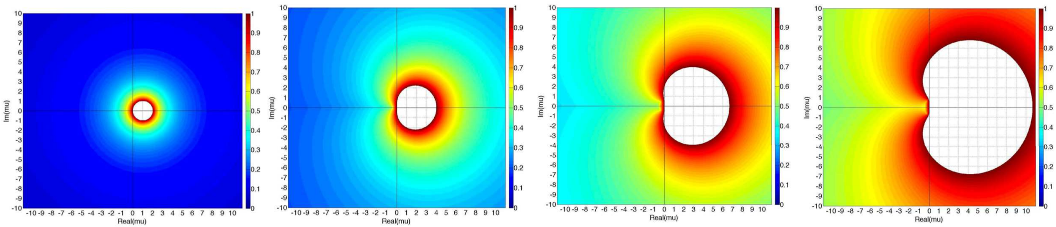
Forwards Euler

Stability regions of Adams-Moulton formulas:



Backwards Euler **Crank-Nicolson**

Stability regions of Backwards Differentiation formulas:



Backwards Euler

order 1 **order 2** **order 3** **order 4**

In general, increasing accuracy $\Longrightarrow$ decreasing stability

A-stable methods must be implicit and at most second order.

Explicit methods approximate exponential by polynomial:
Necessarily grow as $q \to \pm\infty$

Implicit methods approximate the exponential by a rational.

What about using the exponential itself?

This is sometimes possible if the operator:
–is linear
–can be cheaply diagonalized

Exponential of a matrix with real eigenvalues

$$
\begin{aligned}
e^{Lt} &= I + tL + \frac{t^2}{2}L^2 + \ldots \\
&= VV^{-1} + tV\Lambda V^{-1} + \frac{t^2}{2}V\Lambda V^{-1}\,V\Lambda V^{-1} + \\
&= V\left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \ldots\right]V^{-1} = Ve^{\Lambda t}V^{-1} \\
\Lambda^2 &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \\
e^{t\Lambda} &= \begin{pmatrix} 1 + t\lambda_1 + \frac{(t\lambda_1)^2}{2} + \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{(t\lambda_2)^2}{2} + \ldots \end{pmatrix} \\
&= \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}
\end{aligned}
$$

Imaginary Eigenvalues

$$\mathbf{L} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \qquad \lambda_{\pm} = \pm i\omega$$

$$\mathbf{L}^2 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}$$

$$\mathbf{L}^3 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = \begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}$$

$$e^{\mathbf{L}t} = \mathbf{I} + t\mathbf{L} + \frac{t^2}{2}\mathbf{L}^2 + \frac{t^3}{6}\mathbf{L}^3 + \ldots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} + \frac{t^3}{6}\begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}$$

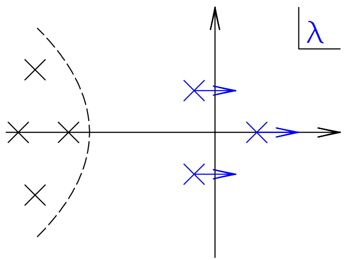
$$= \begin{pmatrix} 1 - \frac{1}{2}(t\omega)^2 + \ldots & -t\omega + \frac{1}{6}(t\omega)^3 + \ldots \\ t\omega - \frac{1}{6}(t\omega)^3 + \ldots & 1 - \frac{1}{2}(t\omega)^2 + \ldots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Complex Eigenvalues:

$$\exp\left[t\begin{pmatrix}\mu & -\omega\\ \omega & \mu\end{pmatrix}\right] = e^{\mu t}\begin{pmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{pmatrix}$$

Mixed spectrum:



$$\boldsymbol{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \mu & -\omega & 0\\ 0 & \omega & \mu & 0\\ 0 & 0 & 0 & \lambda_4\end{bmatrix}$$

$$Re(\lambda_1) \geq Re(\lambda_2) = Re(\lambda_3) \geq Re(\lambda_4)$$

$$\exp(\Lambda t) = \begin{bmatrix}e^{\lambda_1 t} & 0 & 0 & 0\\ 0 & e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) & 0\\ 0 & e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) & 0\\ 0 & 0 & 0 & e^{\lambda_4 t}\end{bmatrix}$$

Holds for any analytic function $f(L)$

$$
\begin{aligned}
f(L) &= \sum_j \frac{1}{j!} f^{(j)}(0) L^j \\
&= \sum_j \frac{1}{j!} f^{(j)}(0) \left(V \Lambda V^{-1}\right)^j \\
&= V \left[\sum_j \frac{1}{j!} f^{(j)}(0) \Lambda^j \right] V^{-1} = V f(\Lambda) V^{-1}
\end{aligned}
$$

where

$$
f(\Lambda) = f \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} = \begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_N) \end{pmatrix}
$$

$$\frac{du}{dt} = Lu \Longrightarrow u(t) = e^{Lt}u(0)$$

$$\begin{array}{c} u(0) \\ \downarrow \\ V^{-1}u(0) \\ \downarrow \\ e^{t\Lambda}V^{-1}u(0) \\ \downarrow \\ Ve^{t\Lambda}V^{-1}u(0) = e^{Lt}u(0) \end{array}$$

multiply by matrix V^{-1}

multiply by diagonal matrix $e^{\Lambda t}$

multiply by matrix V

Example: Heat Equation

$$\partial_t u = \partial^2_{xx} u$$

$$u(x,t) = \sum_{k=1}^{k_{max}} \hat{u}_k(t) \sin kx$$

$$\partial_t \hat{u}_k = -k^2 \hat{u}_k$$

Begin with $u(x_j, t=0)$ on a grid of values $\{x_0, x_1, \ldots x_n\}$

Take Fourier transform: $\{u(x_j, t=0)\} \Longrightarrow \{\hat{u}_k(t=0)\}$

$$\hat{u}_k(t) = e^{-k^2 t} \hat{u}_k(t=0)$$

Take inverse Fourier transform: $\{\hat{u}_k(t)\} \Longrightarrow \{u(x_j, t)\}$

Complete answer valid for all time.

Not usually possible, since most problems are:

–not linear

–not easily diagonalized

but exponential integration can be combined with other ideas

Time-splitting for $\frac{du}{dt} = Lu + N(u)$

Backwards Euler / Forwards Euler:

$$\begin{aligned} u(t+\Delta t) &= u(t) + \Delta t L u(t+\Delta t) + \Delta t N(u(t)) \\ (I - \Delta t L) u(t+\Delta t) &= u(t) + \Delta t N(u(t)) \\ u(t+\Delta t) &= (I - \Delta t L)^{-1} \left[u(t) + \Delta t N(u(t) \right] \end{aligned}$$

Exponential / Forwards Euler:

$$u(t+\Delta t) = e^{L\Delta t} u(t) + e^{L\Delta t/2} \Delta t N(u(t))$$

Exponential / Adams-Bashforth:

$$u(t+\Delta t) = e^{L\Delta t} u(t) + e^{L\Delta t/2} \frac{\Delta t}{2} \left(3N(u(t)) - N(u(t-\Delta t))\right)$$

$$\frac{du}{dt} = Lu + N(u)$$

Equivalent integral equation:

$$u(t + \Delta t) = e^{L\Delta t} \left[u(t) + \int_t^{t+\Delta t} d\tau \; e^{-L(\tau - t)} N(u(\tau)) \right]$$

Approximate $e^{-L(\tau-t)}$ by a constant:

$$\textbf{value at } \tau = t : \int_t^{t+\Delta t} d\tau \; N(u(\tau))$$

$$\textbf{value at } \tau = t + \Delta t : e^{-L\Delta t} \int_t^{t+\Delta t} d\tau \; N(u(\tau))$$

$$\textbf{value at } \tau = t + \Delta t/2 : e^{-L\Delta t/2} \int_t^{t+\Delta t} d\tau \; N(u(\tau))$$

$$\textbf{average value : } \frac{1 - e^{-L\Delta t} - 1}{L\Delta t} \int_t^{t+\Delta t} d\tau \; N(u(\tau))$$

Integration scheme for N:

$$\frac{du_N}{dt} = N(u_N) \Leftrightarrow u_N(t+\Delta t) - u(t) \approx \int_t^{t+\Delta t} d\tau\; N(u_N(\tau))$$

Resulting "slaved exponential" integration scheme

$$u(t+\Delta t) = e^{L\Delta t}u(t) + \frac{e^{L\Delta t}-1}{L\Delta t}(u_N(t+\Delta t) - u(t))$$

If $N(u) = N$ is constant, scheme gives exact soln. for all t:

$$u(t) = e^{Lt}u(0) + \frac{e^{Lt}-1}{L}N$$

As $t \to \infty$, if L has negative eigs, then $u(t) \to -L^{-1}N$
If N is not constant but has a slower timescale than L, then numerical scheme is accurate for all Δt on timescale of N.

Runge-Kutta Methods

RK2 (two evaluations of f per timestep, order 2)

$$U_1 = u(t)$$
$$U_2 = u(t) + \Delta t f(U_1)$$
$$u(t+\Delta t) = u(t) + \frac{\Delta t}{2}\left(f(U_1) + f(U_2)\right)$$

RK4 (four evaluations of f per timestep, order 4)

$$F_1 = f(u(t))$$
$$F_2 = f\left(u(t) + \frac{\Delta t}{2} F_1\right)$$
$$F_3 = f\left(u(t) + \frac{\Delta t}{2} F_2\right)$$
$$F_4 = f\left(u(t) + \Delta t F_3\right)$$
$$u(t+\Delta t) = u(t) + \frac{\Delta t}{6}\left(F_1 + 2F_2 + 2F_3 + F_4\right)$$

Many others

For conservative (Hamiltonian, area-preserving) systems, numerical method should also preserve area.

Oscillator:

$$\frac{d^2u}{dt^2} = -\omega^2 u \Longleftrightarrow \begin{cases} \frac{du}{dt} = -\omega v \\ \frac{dv}{dt} = \omega u \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

or

$$\frac{d}{dt}(u + iv) = i\omega(u + iv)$$

Set $q = i\omega\Delta t$ **and consider** $|\Phi(q)|$

Forwards Euler:

$$\Phi^{\mathrm{FE}}(i\omega\Delta t) = 1 + i\omega\Delta t$$

$$|\Phi^{\mathrm{FE}}(i\omega\Delta t)|^2 = 1 + (\omega\Delta t)^2 > 1$$

Increases area

Backwards Euler:

$$\Phi^{\mathrm{BE}}(i\omega\Delta t) = \frac{1}{1 - i\omega\Delta t}$$

$$|\Phi^{\mathrm{BE}}(i\omega\Delta t)|^2 = \frac{1}{1 + (\omega\Delta t)^2} < 1$$

Decreases area

Crank-Nicolson:

$$\Phi^{\mathrm{CN}}(i\omega\Delta t) = \frac{1 + i\omega\Delta t/2}{1 - i\omega\Delta t/2}$$

$$|\Phi^{\mathrm{CN}}(i\omega\Delta t)|^2 = 1$$ **Preserves area**