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Dynamical Systems

Dynamical Systems

$$\dot{x} = f(x), \qquad x, f \text{ vectors in } \mathcal{R}^N$$

Examples:

–Normal form of saddle-node bifurcation ($N = 1$)

$$\dot{x} = \mu - x^2$$

–Lorenz model ($N = 3$)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10(y-x) \\ -xz + rx - y \\ xy - 8z/3 \end{pmatrix}$$

–Navier-Stokes equations ($N \gg 1$)

$$\frac{d}{dt}\mathrm{u} = -(\mathrm{u} \cdot \nabla)\mathrm{u} - \nabla p + \frac{1}{Re}\Delta \mathrm{u}$$

Navier-Stokes:

$$\mathbf{u}(\mathbf{x}) = (u(x,y,z), v(x,y,z), w(x,y,z)) \qquad N = \infty$$

3D Numerical Discretization:

$$N = 3 \times N_x \times N_y \times N_z \approx 3 \times 100^3 = 3 \times 10^6$$

Non-autonomous system:

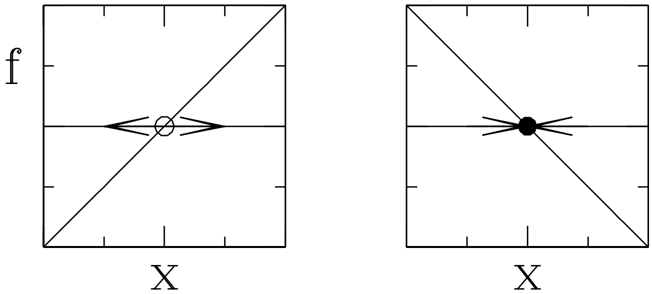
$$\dot{x} = f(x,t) \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} f(x,\theta) \\ 1 \end{pmatrix} \text{ with } \theta \equiv t$$

Higher-order system:

$$\ddot{x} = f(x,\dot{x}) \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f(x,y) \end{pmatrix} \text{ with } y \equiv \dot{x}$$

Fixed points and linear stability. $\dot{x} = f(x)$

unstable



stable

$0 = f(\bar{x})$ **Fixed point $\bar{x}$**

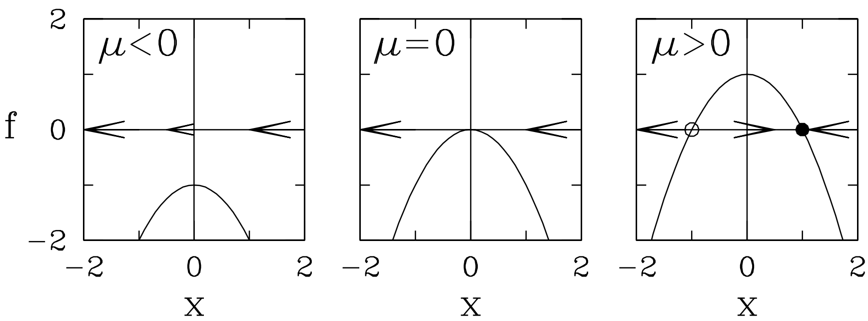
Linear stability of $\bar{x}$

$$\frac{d}{dt}(\bar{x} + \epsilon(t)) = f(\bar{x} + \epsilon)$$

$$\dot{\epsilon} = f(\bar{x}) + f'(\bar{x})\epsilon + \frac{1}{2}f''(\bar{x})\epsilon^2 + \cdots \approx f'(\bar{x})\epsilon$$

$$\epsilon(t) = e^{tf'(\bar{x})}\epsilon(0) \begin{cases} \text{increases if } f'(\bar{x}) > 0 \\ \text{decreases if } f'(\bar{x}) < 0 \end{cases}$$

Saddle-node Bifurcations



Normal form:

$$\dot{x} = f(x) = \mu - x^2$$

Fixed points:

$$\bar{x}_{\pm} = \pm\sqrt{\mu} \qquad \textbf{for } \mu > 0$$

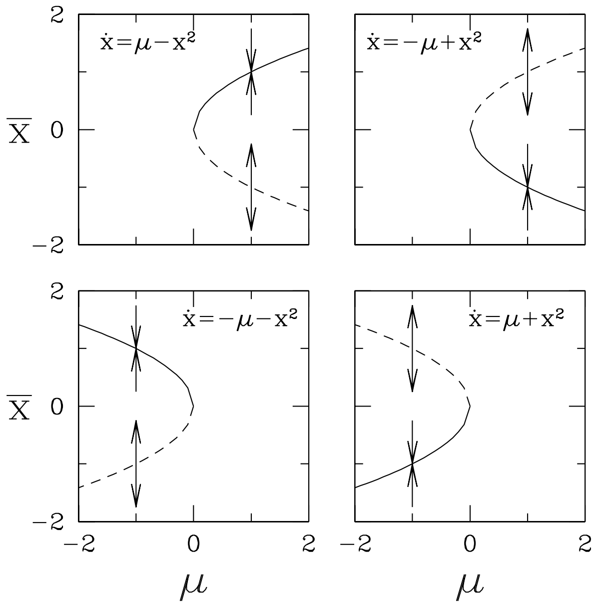
Stability:

$$f'(\bar{x}_{\pm}) = -2\bar{x}_{\pm} = -2(\pm\sqrt{\mu}) = \mp 2\sqrt{\mu}$$

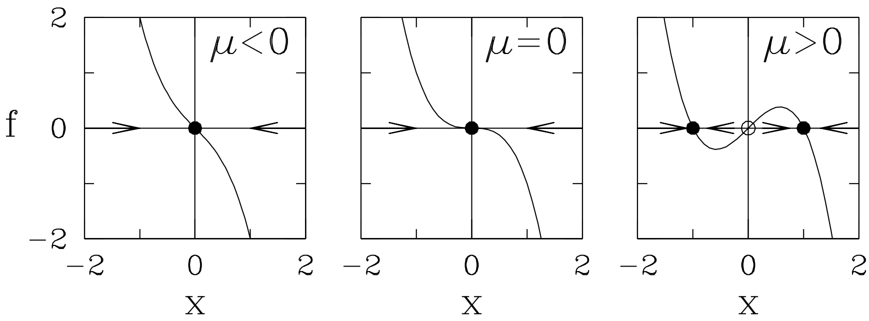
$$f'(\bar{x}_{+}) = f'(\sqrt{\mu}) = -2\sqrt{\mu} < 0 \implies \bar{x}_{+} \textbf{ stable}$$

$$f'(\bar{x}_{-}) = f'(-\sqrt{\mu}) = 2\sqrt{\mu} > 0 \implies \bar{x}_{-} \textbf{ unstable}$$

$$
\begin{aligned}
f(x,\mu) &= c_{00} + c_{10}x + c_{01}\mu + c_{20}x^2 + \ldots \qquad \text{general quadratic polynomial} \\
&= \left(c_{01}\mu + c_{00} - \frac{c_{10}^2}{4c_{20}}\right) + c_{20}\left(x + \frac{c_{10}}{2c_{20}}\right)^2 \\
&= \pm\tilde{\mu} \pm \tilde{x}^2 \qquad \text{four cases, depending on signs of } c\text{'s}
\end{aligned}
$$



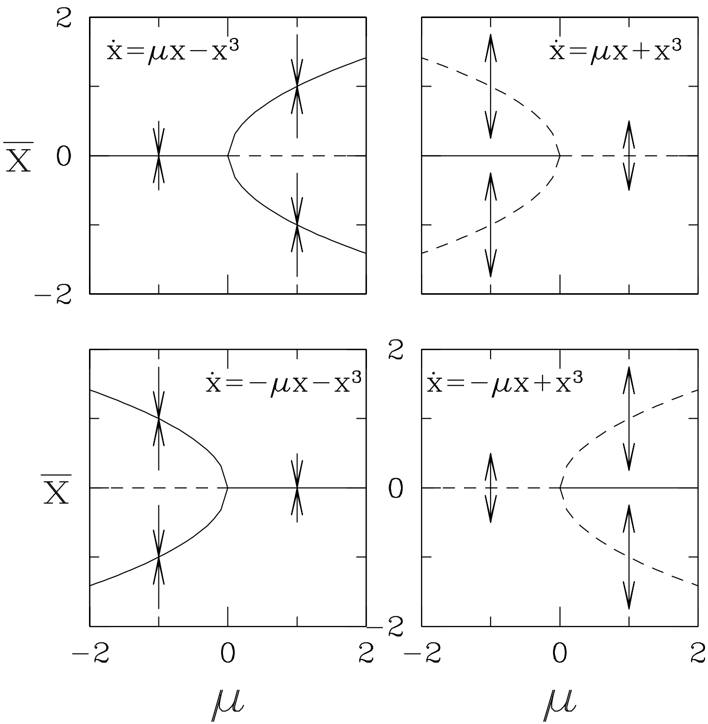
Pitchfork Bifurcations



one fixed point **three fixed points**

Normal form: $\dot{x} = f(x) = \mu x - x^3$

Symmetry $\Longrightarrow$ $f(x)$ **odd in** x**.**



Supercritical

$$f(x,\mu) = \mu x - x^3$$
$$f(x,\mu) = -\mu x - x^3$$

Subcritical

$$f(x,\mu) = \mu x + x^3$$
$$f(x,\mu) = -\mu x + x^3$$

Supercritical pitchfork bifurcations

Fixed points:

$$0 = \bar{x}(\mu - \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \textbf{for all } \mu \\ \bar{x} = \pm\sqrt{\mu} & \textbf{for } \mu > 0 \end{cases}$$

Stability:

$$f'(\bar{x}) = \mu - 3\bar{x}^2 = \begin{cases} \mu & \textbf{for } \bar{x} = 0 \\ \mu - 3\mu = -2\mu & \textbf{for } \bar{x} = \pm\sqrt{\mu} \quad \textbf{stable} \end{cases}$$

Subcritical pitchfork bifurcations

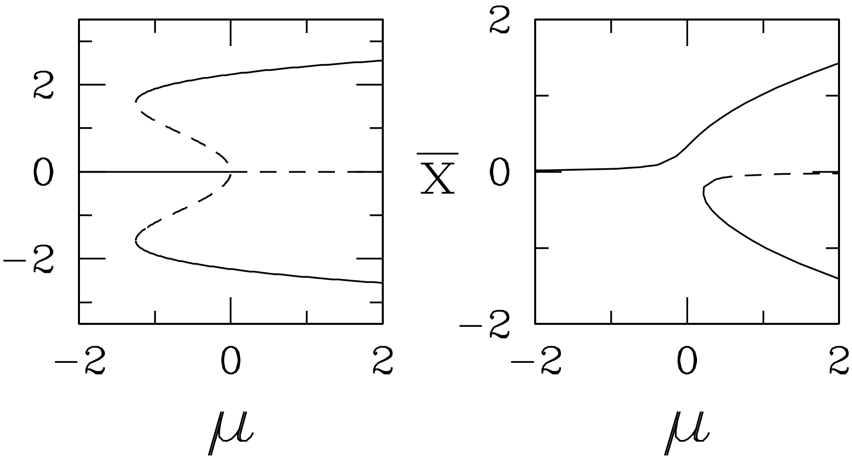
Fixed points:

$$0 = \bar{x}(\mu + \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \textbf{for all } \mu \\ \bar{x} = \pm\sqrt{-\mu} & \textbf{for } \mu < 0 \end{cases}$$

Stability:

$$f'(\bar{x}) = \mu + 3\bar{x}^2 = \begin{cases} \mu & \textbf{for } \bar{x} = 0 \\ \mu + 3(-\mu) = -2\mu & \textbf{for } \bar{x} = \pm\sqrt{\mu} \textbf{ unstab} \end{cases}$$

Variations of pitchfork bifurcations



$$\dot{x} = \mu x + x^3 - \frac{x^5}{10}$$

stabilizing term
prevents trajectories
from evolving to ∞
pitchfork $+$ 2 saddle-nodes

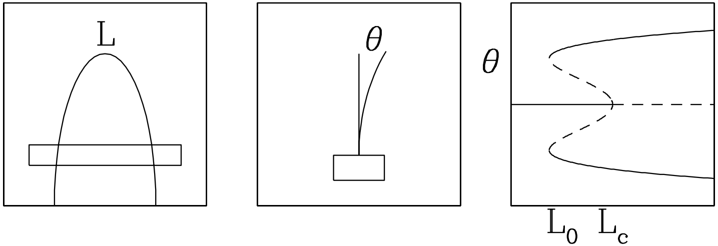
$$\dot{x} = \frac{1}{27} + \mu x - x^3$$

imperfect
symmetry broken

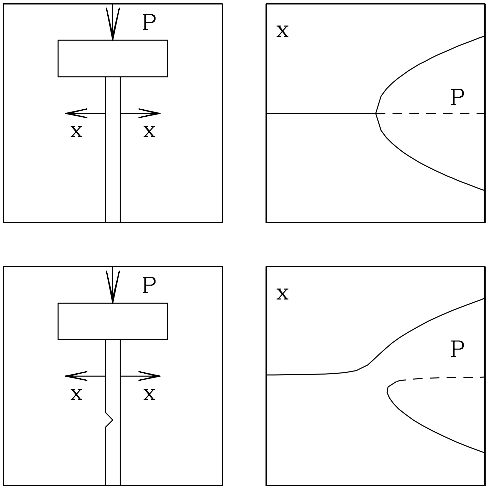
1 saddle-node

Simple mechanical examples

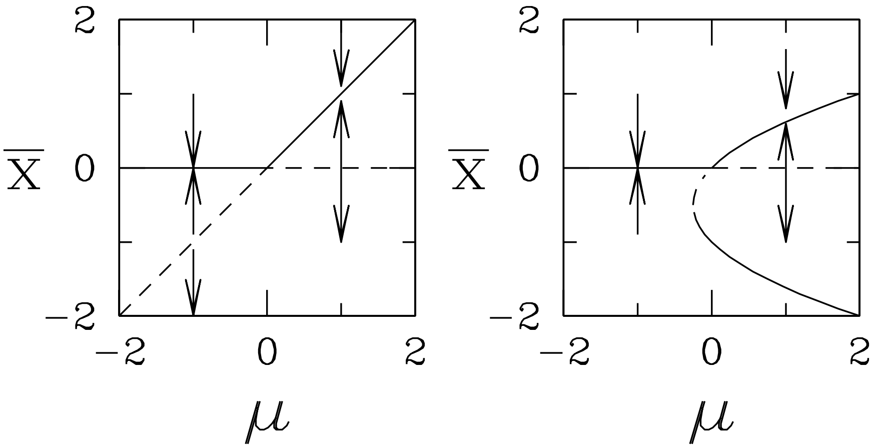
Wire loop: straight or leaning?



Weighted beam: weakness $\longrightarrow$ imperfect pitchfork



Transcritical Bifurcations



with stabilizing term

Normal form:

$$\dot{x} = \mu x - x^2$$

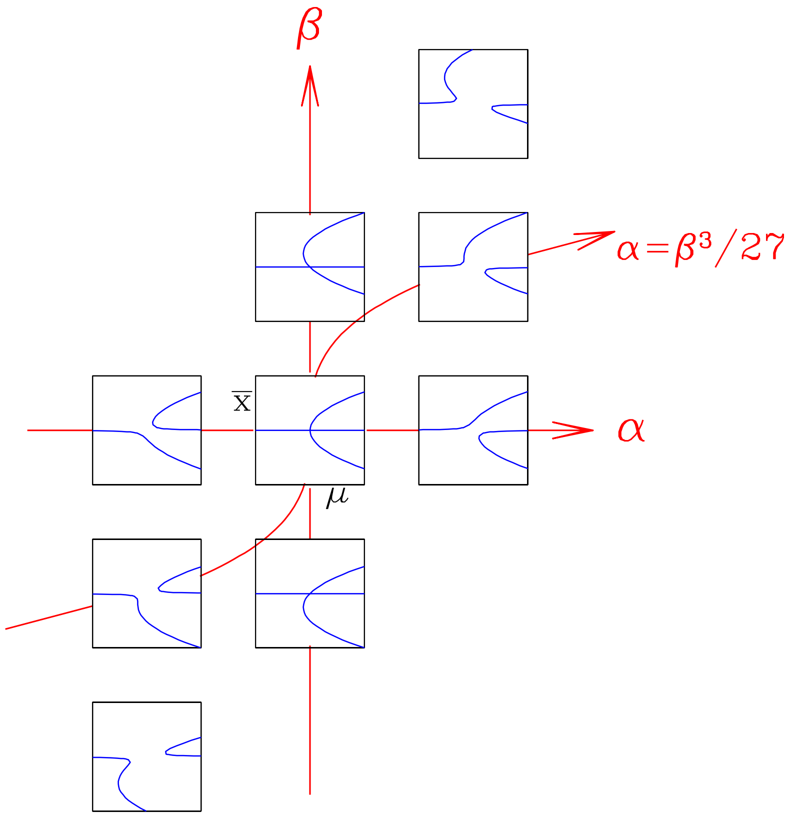
Fixed points:

$$0 = \bar{x}(\mu - \bar{x}) \Longrightarrow \begin{cases} \bar{x} = 0 \\ \bar{x} = \mu \end{cases}$$

Stability:

$$f'(\bar{x}) = \mu - 2\bar{x} = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ -\mu & \text{for } \bar{x} = \mu \end{cases}$$

Unfolding of the pitchfork: $\dot{x} = \alpha + \mu x + \beta x^2 - x^3$



General conditions

$$\dot{x} = f(x, \mu) \text{ with } f(\bar{x}) = 0$$

	f	f_x	f_μ	f_{xx}	$f_{x\mu}$	f_{xxx}
steady state	0					
bifurcation	0	0	$\neq 0$			
saddle-node	0	0	$\neq 0$	$\neq 0$		
transcritical	0	0	0	$\neq 0$	$\neq 0$	
pitchfork	0	0	0	0	$\neq 0$	$\neq 0$

Systems with two or more dimensions

$$\dot{x} = f(x), \quad x, f \in \mathcal{R}^N$$

Fixed points: $0 = f(\bar{x})$

Stability:

$$\begin{aligned}
\frac{d}{dt}(\bar{x} + \epsilon) &= f(\bar{x} + \epsilon) \\
\dot{\epsilon} &= f(\bar{x}) + Df(\bar{x})\,\epsilon + \epsilon\, D^2 f(\bar{x})\,\epsilon + \ldots \\
\dot{\epsilon}_i &= f_i(\bar{x}) + Df(\bar{x})_{ij}\,\epsilon_j + \epsilon_j\,[D^2 f(\bar{x})]_{ijk}\,\epsilon_k + \ldots \\
&= f_i(\bar{x}) + \frac{\partial f_i}{\partial x_j}(\bar{x})\,\epsilon_j + \epsilon_j\,\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\bar{x})\,\epsilon_k + \ldots
\end{aligned}$$

$$\dot{\epsilon} = Df(\bar{x})\,\epsilon$$

$Df(\bar{x})$: **Jacobian matrix of f at $\bar{x}$**

$$\epsilon(t) = \exp(Df(\bar{x})t)\epsilon(0)$$

Exponential of a matrix: real eigenvalues

$$
\begin{aligned}
e^{At} &= I + tA + \frac{t^2}{2}A^2 + \ldots \\
&= VV^{-1} + tV\Lambda V^{-1} + \frac{t^2}{2}V\Lambda V^{-1}\,V\Lambda V^{-1} + \\
&= V\left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \ldots\right]V^{-1} = V e^{\Lambda t} V^{-1} \\
\Lambda^2 &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \\
e^{t\Lambda} &= \begin{pmatrix} 1 + t\lambda_1 + \frac{(t\lambda_1)^2}{2} + \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{(t\lambda_2)^2}{2} + \ldots \end{pmatrix} \\
&= \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}
\end{aligned}
$$

Stability in several dimensions

$\bar{x}$ **stable** $\Leftrightarrow Re(\lambda_i) < 0$ **for ALL** λ_i

$\bar{x}$ **unstable** $\Leftrightarrow Re(\lambda_1) > 0$ **even for ONE value of** λ_1

since perturbation in v_1 **direction will increase**

Scalar product with adjoint eigenvector v_1^T**:**

$$v_1^T Df(\bar{x}) = v_1^T \lambda_1$$

$\Longrightarrow$ **1D (or 2D) equation**

Directions $(Re(\lambda_i) < 0)$ **are uninteresting: contraction only**

Physical system $\Longrightarrow$ Polynomial equation:

- **Complicated equations in $N \gg 1$ variables.**

Calculate fixed points $\bar{x}$, Jacobians $Df(\bar{x})$ and spectra $\{\lambda_1, \lambda_2, \ldots\}$.

$\Longrightarrow$ Bifurcation if $Re(\lambda_i)$ changes sign.

- **Project onto corresponding adjoint eigenvector**

$\Longrightarrow$ Function of a single variable.

- **Taylor expansion about fixed point.**

Minimal truncation giving observed behavior

$\Longrightarrow$ Normal form of the bifurcation.

Imaginary Eigenvalues

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \qquad \lambda_\pm = \pm i\omega$$

$$A^2 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = \begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}$$

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \ldots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} + \frac{t^3}{6}\begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}$$

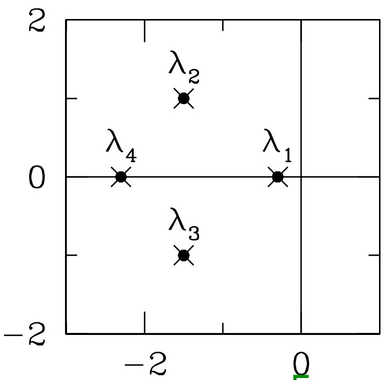
$$= \begin{pmatrix} 1 - \frac{1}{2}(t\omega)^2 + \ldots & -t\omega + \frac{1}{6}(t\omega)^3 + \ldots \\ t\omega - \frac{1}{6}(t\omega)^3 + \ldots & 1 - \frac{1}{2}(t\omega)^2 + \ldots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Complex Eigenvalues:

$$\exp\left[t\begin{pmatrix}\mu & -\omega\\ \omega & \mu\end{pmatrix}\right] = e^{\mu t}\begin{pmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{pmatrix}$$

Mixed spectrum:



$$\boldsymbol{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \mu & -\omega & 0\\ 0 & \omega & \mu & 0\\ 0 & 0 & 0 & \lambda_4\end{bmatrix}$$

$$Re(\lambda_1) \geq Re(\lambda_2) = Re(\lambda_3) \geq Re(\lambda_4)$$

$$\exp(\Lambda t) = \begin{bmatrix}e^{\lambda_1 t} & 0 & 0 & 0\\ 0 & e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) & 0\\ 0 & e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) & 0\\ 0 & 0 & 0 & e^{\lambda_4 t}\end{bmatrix}$$

Jordan blocks and transient growth

Star node: multiple of the identity $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ **with eigenvalues** $\lambda = \mu$

Eigenvectors:

$$\mu x_1 + 0x_2 = \mu x_1 \implies x_1 \text{ arbitrary}$$
$$0x_1 + \mu x_2 = \mu x_2 \implies x_2 \text{ arbitrary}$$

Two-dimensional eigenspace: $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Degenerate node: Jordan block $\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$ **with eigenvalues** $\lambda = \mu$

Eigenvectors:

$$\mu x_1 + x_2 = \mu x_1 \implies x_2 = 0$$
$$0x_1 + \mu x_2 = \mu x_2 \implies x_1 \text{ arbitrary}$$

One-dimensional eigenspace: $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Jordan blocks and transient growth

Eigenvector: $(A - \lambda I)x = 0$

Generalized eigenvector for Jordan block:

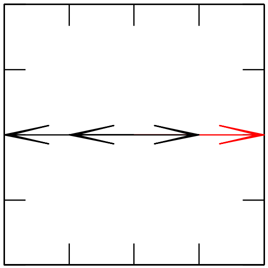
$$(A - \lambda I)v = x, \qquad (\lambda, x) \text{ eigen (value,vector)}, \qquad v \nparallel x$$

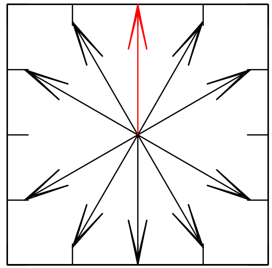
$$\left[\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \qquad \begin{matrix} c \neq 0 \\ v_2 \neq 0 \end{matrix}$$

$$(\mu - \lambda)v_2 = 0 \implies \lambda = \mu$$

$$(\mu - \lambda)v_1 + 1v_2 = c \implies v_2 = c \neq 0, \; v_1 \text{ arbitrary}$$

$$||x|| = 1 \implies x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left.\begin{matrix} \langle v, x \rangle = 0 \\ ||v|| = 1 \end{matrix}\right\} \implies v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

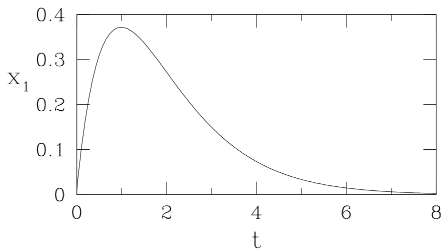




eigenvector x　　**generalized eigenvector v**

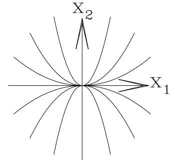
$$
\begin{aligned}
x_1 &= e^{\lambda t}(x_1(0) + x_2(0)t) \\
x_2 &= e^{\lambda t}x_2(0)
\end{aligned}
$$

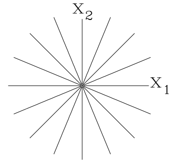
$$\lambda = -1 \Longrightarrow$$

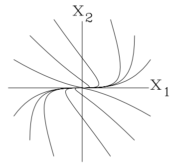


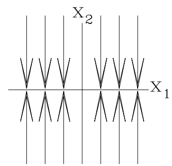
Linear behavior near a 2D fixed point

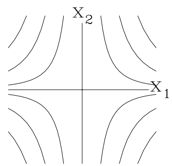
Name	Matrix	Behavior
Node: stable ($\lambda_2 < \lambda_1 < 0$) unstable ($\lambda_2 > \lambda_1 > 0$)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$x_1 = e^{\lambda_1 t} x_1(0)$ $x_2 = e^{\lambda_2 t} x_2(0)$
Star node: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$x_1 = e^{\lambda t} x_1(0)$ $x_2 = e^{\lambda t} x_2(0)$
Degenerate node: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$x_1 = e^{\lambda t}(x_1(0) + t x_2(0))$ $x_2 = e^{\lambda t} x_2(0)$
Non-isolated fixed points: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$	$x_1 = x_1(0)$ $x_2 = e^{\lambda t} x_2(0)$
Saddle: $\lambda_2 < 0 < \lambda_1$	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$x_1 = e^{\lambda_1 t} x_1(0)$ $x_2 = e^{\lambda_2 t} x_2(0)$





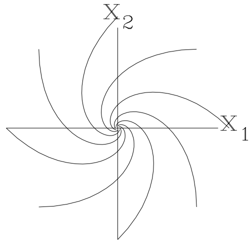




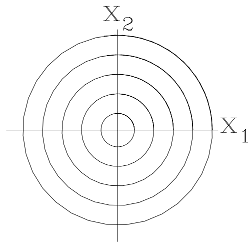


Linear behavior near a 2D fixed point

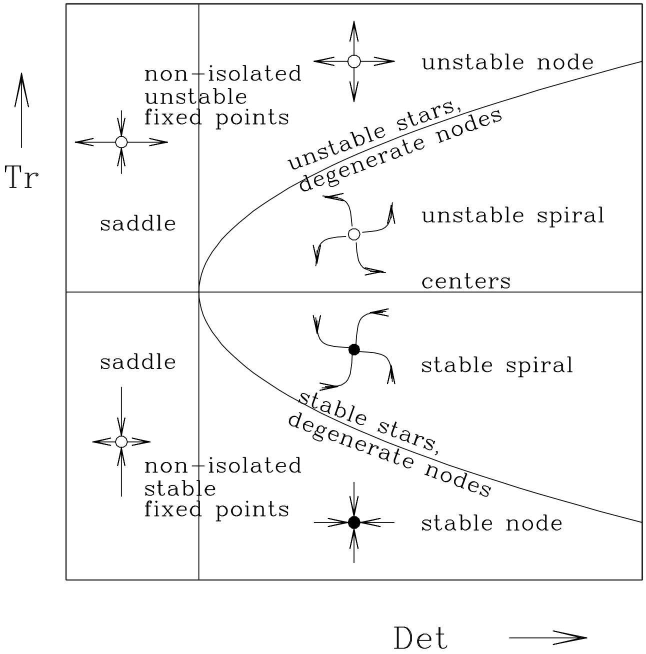
Name	Matrix	Behavior
Spiral: stable ($\mu < 0$) unstable ($\mu > 0$)	$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$	$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$



Name	Matrix	Behavior
Center:	$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$	$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$



Linear behavior near a 2D fixed point



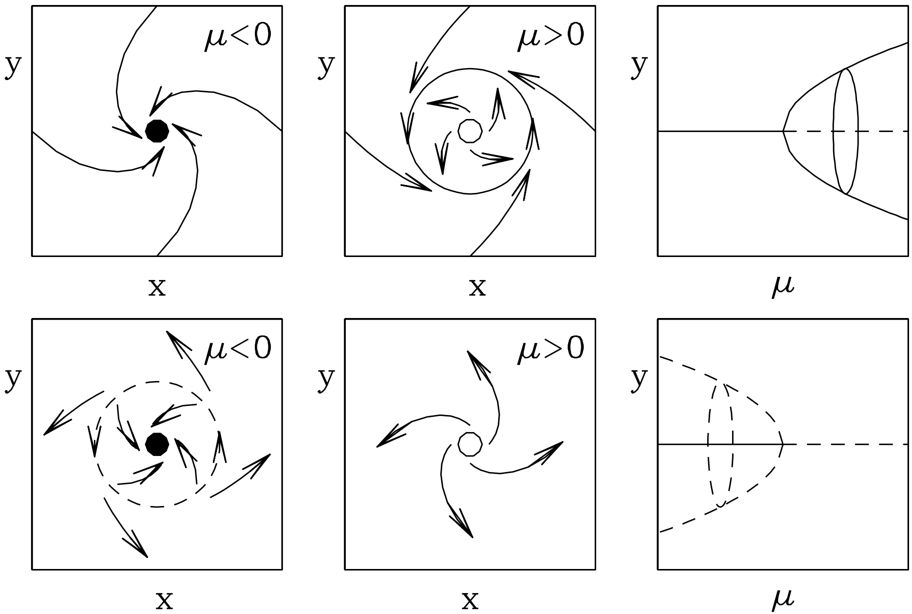
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{Tr} \pm \sqrt{\mathrm{Tr}^2 - 4\mathrm{Det}}\right)$$

$$\mathrm{Tr} \equiv a + d = \lambda_1 + \lambda_2$$

$$\mathrm{Det} \equiv ad - bc = \lambda_1 \lambda_2$$

Hopf Bifurcation



Normal Form:

$$\dot{z} = (\mu + i\omega)z - \alpha|z|^2 z$$

Cartesian Form:

$$\begin{aligned}
\dot{x} + i\dot{y} &= (\mu + i\omega)(x + iy) - (\alpha_r + i\alpha_i)(x^2 + y^2)(x + iy) \\
\dot{x} &= \mu x - \omega y - (x^2 + y^2)(\alpha_r x - \alpha_i y) \\
\dot{y} &= \omega x + \mu y - (x^2 + y^2)(\alpha_i x + \alpha_r y)
\end{aligned}$$

Polar Form:

$$\begin{aligned}
(\dot{r} + ir\dot{\theta})e^{i\theta} &= (\mu + i\omega)re^{i\theta} - (\alpha_r + i\alpha_i)r^2 re^{i\theta} \\
\dot{r} &= \mu r - \alpha_r r^3 \\
\dot{\theta} &= \omega - \alpha_i r^2
\end{aligned}$$

Trajectory:

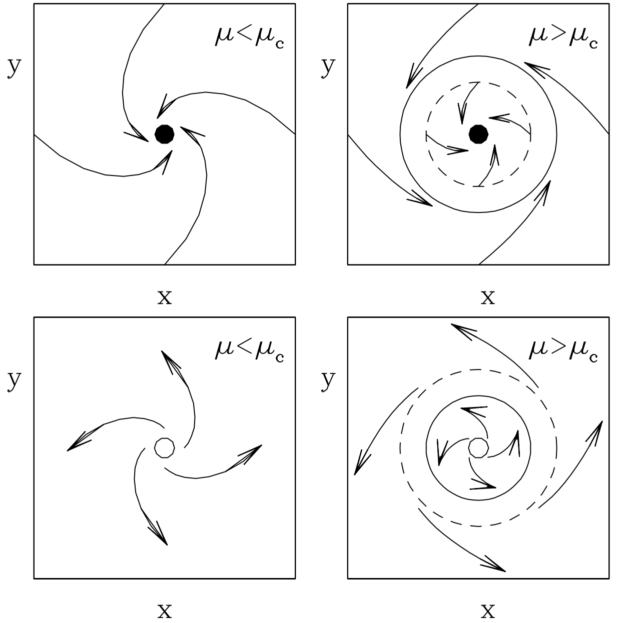
$$z(t) = \sqrt{\mu/\alpha_r}\, e^{i\omega(t - t_0)}$$

Subcritical Form:

$$\dot{z} = (\mu + i\omega)z + \alpha|z|^2 z \qquad (\alpha_r > 0)$$

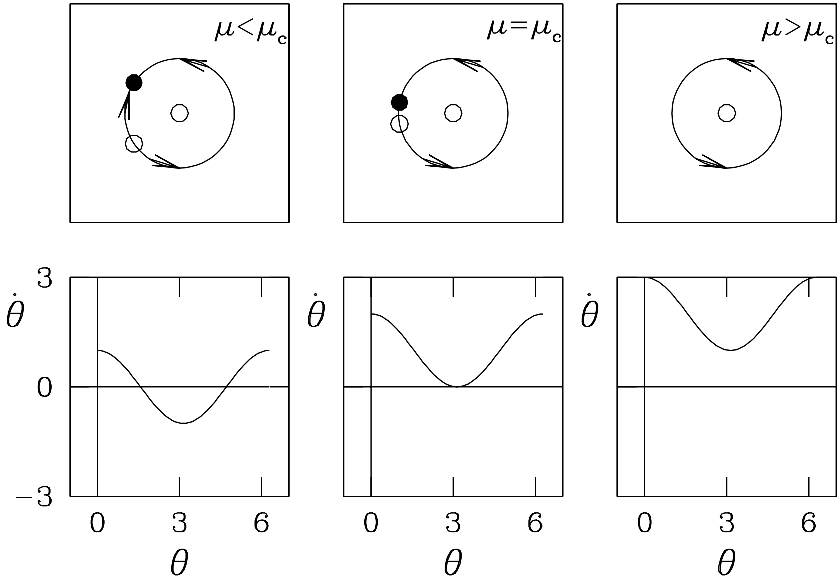
Global bifurcations $\Longrightarrow$ limit cycles

Saddle-node bifurcation OF two limit cycles



$$\dot{r} = \alpha r(\mu - \mu_c - (r^2 - r_c^2)^2)$$
$$\dot{\theta} = \omega$$

Saddle-node bifurcation IN a limit cycle

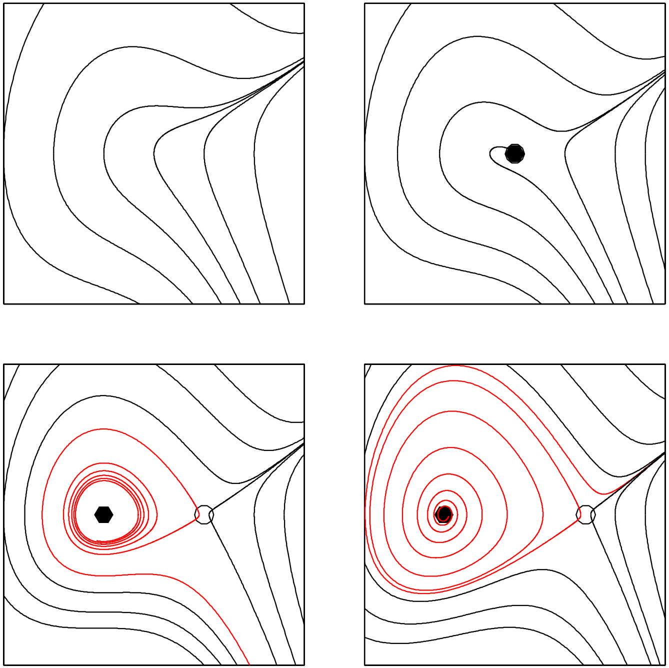


$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = \mu + 1 + \cos(\theta)$$

Homoclinic bifurcation

$\mu = -\frac{1}{2}$ $\qquad$ $\mu = -\frac{1}{4}$

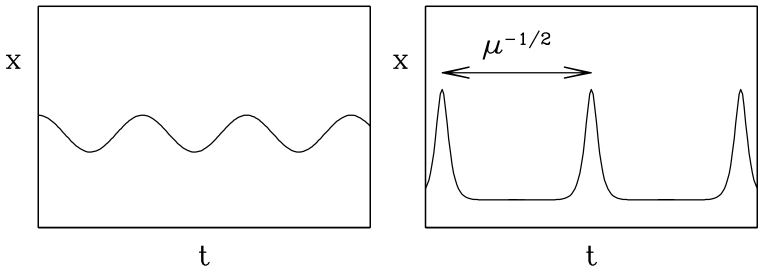
$\mu = 0$ $\qquad$ $\mu = \frac{1}{4}$



$$\dot{x} = y$$
$$\dot{y} = -\mu - x + x^2 - xy$$

Global bifurcations

Non-uniform trajectory:
slow phases near saddles or near “ghosts”of fixed points

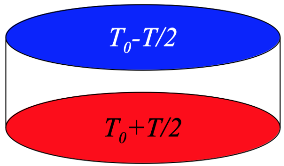


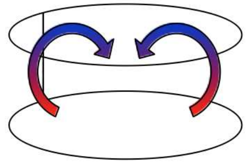
uniform trajectory **non-uniform trajectory**

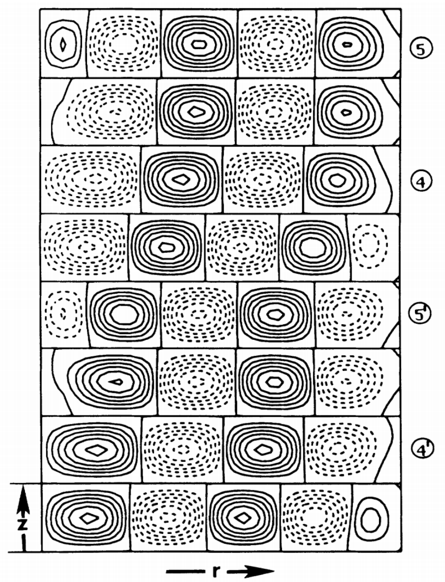
	Amplitude	Period
Supercritical Hopf	$O(\mu^{\frac{1}{2}})$	$O(1)$
Saddle-node *of* limit cycles	$O(1)$	$O(1)$
Saddle-node *in* limit cycle	$O(1)$	$O(\mu^{-\frac{1}{2}})$
Homoclinic	$O(1)$	$O(\log(\mu))$

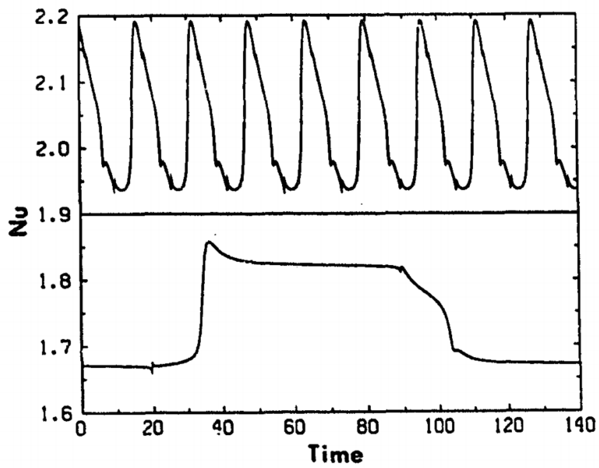
Limit cycles in fluid dynamics

Axisymmetric convection:

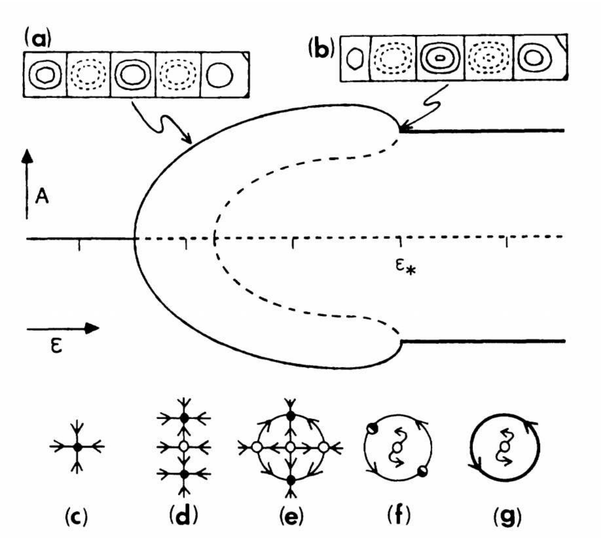




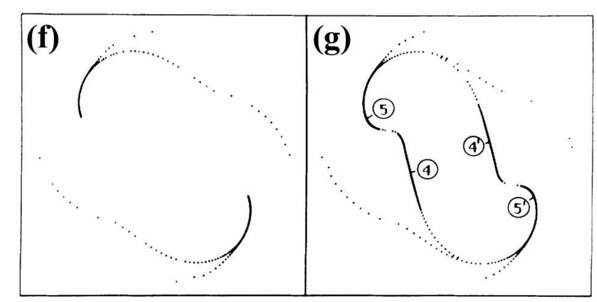




Bifurcation diagram:



Phase portraits: project trajectories onto two eigendirections

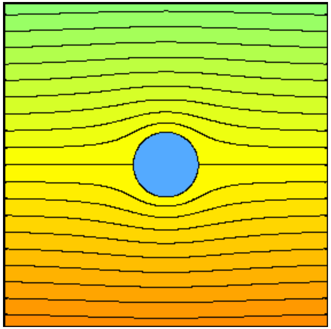


SNIPER global bifurcation: Saddle-Node In PERiodic Orbit

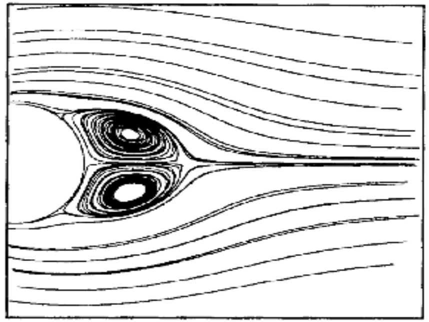
or Saddle-Node Infinite PERiod

Cylinder Wake: Hopf bifurcation

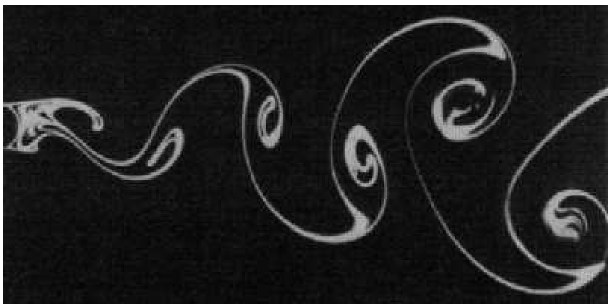
Ideal flow



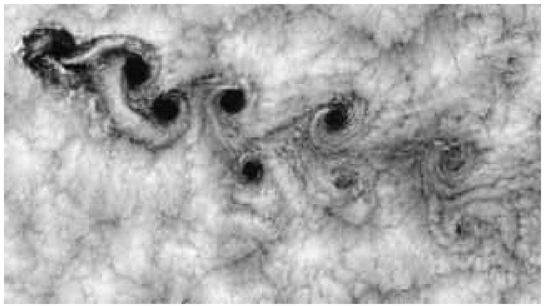
with downstream recirculation zone



von Kármán vortex street ($\mathrm{Re} \geq 46$)



Laboratory experiment (Taneda, 1982)



Off Chilean coast past Juan Fernandez islands