Master M1 (ICFP): Numerical Methods for Partial Differential Equations

Midterm 2019

Here are three timestepping schemes for solving the ordinary differential equation $\frac{du}{dt} = f(u)$

$$\text{Crank Nicolson} \qquad u_{n+1} = u_n + \frac{\Delta t}{2}\left(f(u_n) + f(u_{n+1})\right) \tag{1}$$

$$\text{Midpoint} \qquad u_{n+1} = u_n + \Delta t f\left(u_n + \frac{\Delta t}{2} f(u_n)\right) \tag{2}$$

$$\text{Leapfrog} \qquad u_{n+1} = u_{n-1} + 2\Delta t f(u_n) \tag{3}$$

1) Use Taylor series expansions of the exact and approximate solutions about u_n to show that all three methods are second-order accurate, i.e. that the single-step error is proportional to $(\Delta t)^3$. Find the constant of proportionality (the constant multiplying $(\Delta t)^3$) for each of the three methods. You may treat only the differential equation

$$\frac{du}{dt} = \lambda u \tag{4}$$

i.e. you do not need to consider the case of a general $f(u)$.
Exact solution:

$$u_{n+1} = e^{\lambda \Delta t} u_n = \left[1 + \lambda\Delta t + \tfrac{1}{2}(\lambda\Delta t)^2 + \tfrac{1}{6}(\lambda\Delta t)^3 \ldots\right] u_n$$

Crank-Nicolson:

$$\begin{aligned}
u_{n+1} &= \frac{1 + \frac{\lambda\Delta t}{2}}{1 - \frac{\lambda\Delta t}{2}} u_n \\
&= \left(1 + \frac{\lambda\Delta t}{2}\right)\left(1 + \frac{\lambda\Delta t}{2} + \left(\frac{\lambda\Delta t}{2}\right)^2 + \left(\frac{\lambda\Delta t}{2}\right)^3 + \ldots\right) \\
&= 1 + 2\left(\frac{\lambda\Delta t}{2}\right) + 2\left(\frac{\lambda\Delta t}{2}\right)^2 + 2\left(\frac{\lambda\Delta t}{2}\right)^3 + \ldots
\end{aligned}$$

The first non-matching term is that in $(\lambda\Delta t)^3$ and the error is

$$\left(\frac{1}{4} - \frac{1}{6}\right)(\lambda\Delta t)^3 = \frac{1}{12}(\lambda\Delta t)^3$$

Midpoint method:

$$\begin{aligned}
u_{n+1} &= u_n + \lambda\Delta t\left(u_n + \frac{\lambda\Delta t}{2} u_n\right) \\
&= 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2
\end{aligned}$$

This series has no term in $(\lambda\Delta t)^3$. The error is

$$\left(0 - \frac{1}{6}\right)(\lambda\Delta t)^3 = -\frac{1}{6}(\lambda\Delta t)^3$$

Leapfrog:

$$\begin{aligned}
u_{n+1} &= u_{n-1} + 2\lambda\Delta t u_n \\
&= u_n - \lambda\Delta t u_n + \frac{1}{2}(\lambda\Delta t)^2 u_n - \frac{1}{6}(\lambda\Delta t)^3 u_n + \ldots + 2\lambda\Delta t u_n \\
&= u_n + \lambda\Delta t u_n + \frac{1}{2}(\lambda\Delta t)^2 u_n - \frac{1}{6}(\lambda\Delta t)^3 u_n + \ldots
\end{aligned}$$

The first non-matching term is that in $(\lambda\Delta t)^3$ and the error is

$$\left(-\frac{1}{6} - \frac{1}{6}\right)(\lambda\Delta t)^3 = -\frac{1}{3}(\lambda\Delta t)^3$$

2) We now consider the stability of the three methods for the diffusion and the wave equation. You may use the result that the general solution to a linear constant-coefficient difference equation

$$0 = a_1 u_{n+1} + a_0 u_n + a_{-1} u_{n-1} + a_{-2} u_{n-2} + a_{-3} u_{n-3} \ldots \tag{5}$$

is a linear combination of solutions of the form

$$u_n = c r^n \tag{6}$$

2a) For the diffusion equation, we consider (4) for $\lambda = -k^2$ real and negative, motivated by

$$\begin{aligned}
\partial_t u &= \partial_{xx} u & u(x,t) &= \hat{u}(t)e^{ikx} \\
\frac{d\hat{u}}{dt} &= -k^2\hat{u} & \hat{u}(t) &= e^{-k^2 t}\hat{u}(0)
\end{aligned} \tag{7}$$

For the three methods, find the constraint on $\Delta t > 0$ such that the numerical sequence u_n decays when integrating (4) with λ real and negative.

Crank-Nicolson:

$$u_{n+1} = \frac{1 + \frac{\lambda\Delta t}{2}}{1 - \frac{\lambda\Delta t}{2}} u_n = \frac{1 - \frac{k^2\Delta t}{2}}{1 + \frac{k^2\Delta t}{2}} u_n$$

We require

$$|1 - \frac{k^2\Delta t}{2}| < |1 + \frac{k^2\Delta t}{2}|$$

If $-1 \leq k^2\Delta t \leq 1$, then we require

$$-k^2\Delta t < k^2\Delta t$$

If $k^2\Delta t > 1$, then we require

$$\begin{aligned} \frac{k^2\Delta t}{2} - 1 &< 1 + \frac{k^2\Delta t}{2} \\ -1 &< 1 \end{aligned}$$

which is satisfied for all $\Delta t > 0$. So Crank-Nicolson is always stable for the heat equation.

Midpoint method:

$$\begin{aligned} u_{n+1} &= \left(1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2\right) u_n \\ &= 1 - k^2\Delta t + \frac{1}{2}(-k^2\Delta t)^2 \end{aligned}$$

We require

$$|1 - k^2\Delta t + \frac{1}{2}(k^2\Delta t)^2| < 1$$

From the graph, we see that this inequality is satisfied as long as $0 < k^2\Delta t < 2$ and it is in this range of Δt that the midpoint method is stable for the heat equation.

Leapfrog method:

$$u_{n+1} = u_{n-1} + 2\lambda\Delta t u_n$$

Writing $u_n = cr^n$, we have

$$\begin{aligned} r^2 &= 1 + 2\lambda\Delta t\, r \\ 0 &= r^2 - 2\lambda\Delta t\, r - 1 \\ r_\pm &= \lambda\Delta t \pm \sqrt{(\lambda\Delta t)^2 + 1} \end{aligned}$$

For stability, we need $|r_\pm| < 1$ for both values of r when $\lambda = -k^2 < 0$. From the graph supplied, we see that $r_+ > 1$ for $\lambda > 0$ and $r_- < -1$ for $\lambda < 0$ so this method is never stable.

2b) For the wave equation, we consider (4) for $\lambda = ik$ imaginary. motivated by

$$\begin{aligned} \partial_t u &= \partial_x u & u(x,t) &= \hat{u}(t)e^{ikx} \\ \frac{d\hat{u}}{dt} &= ik\hat{u} & \hat{u}(t) &= e^{ikt}\hat{u}(0) \end{aligned} \tag{8}$$

For the three methods, find the constraint on $\Delta t > 0$ such that the numerical sequence u_n retains the same amplitude when integrating (4) with λ imaginary.

Crank-Nicolson:

$$u_{n+1} = \frac{1 + \frac{\lambda\Delta t}{2}}{1 - \frac{\lambda\Delta t}{2}} u_n = \frac{1 + \frac{ik\Delta t}{2}}{1 - \frac{ik\Delta t}{2}} u_n$$

We require

$$|1+\frac{ik\Delta t}{2}|=|1-\frac{ik\Delta t}{2}|$$

This is true for all values of Δt. So Crank-Nicolson is always stable for the wave equation.

Midpoint method:

$$\begin{aligned} u_{n+1} &= \left(1+\lambda\Delta t+\frac{1}{2}(\lambda\Delta t)^2\right)u_n \\ &= 1+ik\Delta t+\frac{1}{2}(ik\Delta t)^2 \end{aligned}$$

We require

$$\begin{aligned} 1 &= |1+ik\Delta t-\frac{1}{2}(k\Delta t)^2| \\ &= \left(1-\frac{1}{2}(k\Delta t)^2\right)^2+(k\Delta t)^2 \\ &= 1-(k\Delta t)^2+\frac{1}{4}(k\Delta t)^4+(k\Delta t)^2 \\ &= 1+\frac{1}{4}(k\Delta t)^4 \end{aligned}$$

This is only true when $\Delta t=0$. So the midpoint method is never stable for the wave equation.

Leapfrog method:
We have $u_n \sim r^n$ with

$$r_\pm=\lambda\Delta t\pm\sqrt{(\lambda\Delta t)^2+1}$$

For stability, we need $|r_\pm|=1$ for both values of r when $\lambda=ik$.

$$r_\pm=ik\Delta t\pm\sqrt{(ik\Delta t)^2+1}=ik\Delta t\pm\sqrt{1-(k\Delta t)^2}$$

For $|k\Delta t|<1$, the square root is real and we have

$$|r_\pm|^2=(k\Delta t)^2+\left(1-(k\Delta t)^2\right)=1$$

For $|k\Delta t|>1$, the square root is imaginary and we have

$$\begin{aligned} r_\pm &= ik\Delta t\pm i\sqrt{(k\Delta t)^2-1} \\ |r_\pm| &= |k\Delta t\pm\sqrt{(k\Delta t)^2-1}| \end{aligned}$$

From the graphs, we can see that $r_+>1$ for $|k\Delta t|>1$ and $r_-<1$ for $|k\Delta t|<-1$, leading to instability. Thus, we require $|k\Delta t|<1$ for stability.

To solve the diffusion equation, we can use the Crank-Nicolson method with any value of Δt or the midpoint method with $\Delta t<2/k^2$. To solve the wave equation, we can use the Crank-Nicolson method with any value of Δt or the leapfrog method with $\Delta t<1/|k|$.