Course: Nonlinear Dynamics

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Maps, Period Doubling and Floquet Theory

Discrete Dynamical Systems or Mappings

$$y_{n+1} = g(y_n) \qquad y, g \in \mathcal{R}^N$$

Fixed point: $\bar{y} = g(\bar{y})$

1D linear stability of $\bar{y}$: Set $y_n = \bar{y} + \epsilon_n$

$$\begin{aligned} \bar{y} + \epsilon_{n+1} &= g(\bar{y} + \epsilon_n) \\ &= g(\bar{y}) + g'(\bar{y})\epsilon_n + \frac{1}{2}g''(\bar{y})\epsilon_n^2 \cdots \\ \epsilon_{n+1} &\approx g'(\bar{y})\epsilon_n \end{aligned}$$

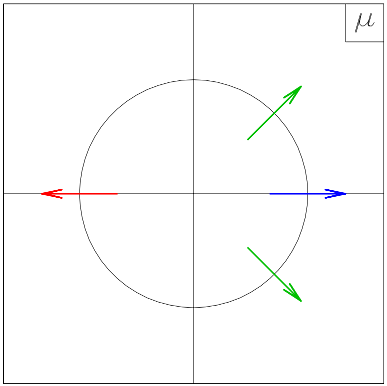
$|g'(\bar{y})| <> 1 \iff |\epsilon| \downarrow\uparrow \iff \bar{y}$ **stable / unstable**

Superstability: $g'(\bar{y}) = 0 \Longrightarrow \epsilon_{n+1} \approx \frac{1}{2}g''(\bar{y})\epsilon_n^2$

Multidimensional system

$g'(\bar{y}) \Longrightarrow Dg(\bar{y})$ **(Jacobian)**

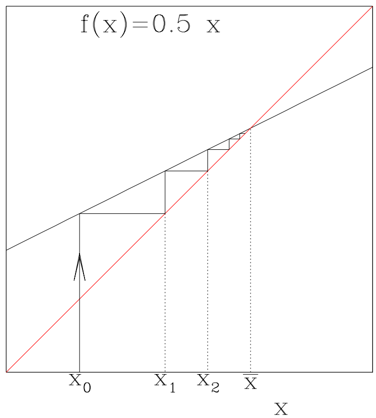
$\bar{y}$ **stable** $\Longleftrightarrow$ **all eigenvalues** μ **of** $Dg(\bar{y})$ **inside unit circle**

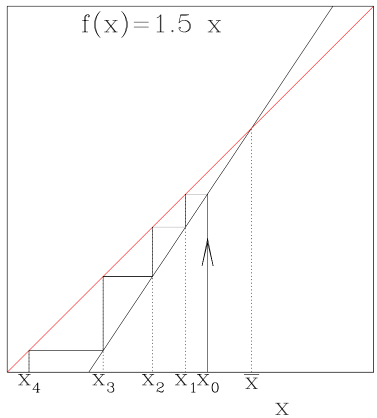


μ **exits at (–1,0)** μ **exits at** $e^{\pm i\theta}$ μ **exits at (1,0)**

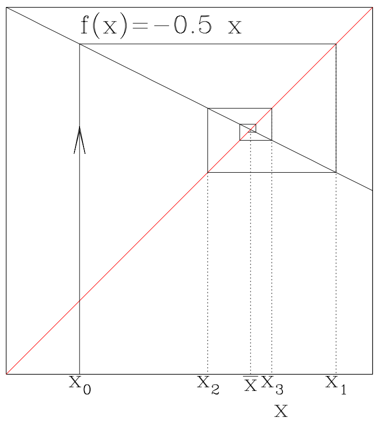
Illustrate exit at $(\pm 1, 0)$ via graphical construction

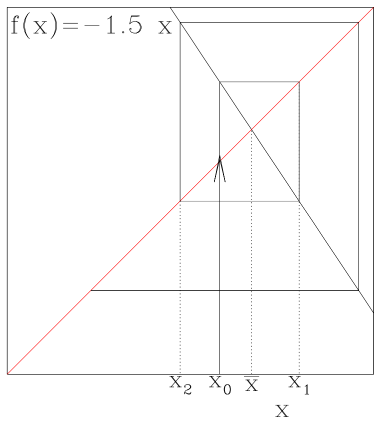
$(+1, 0)$





$(-1, 0)$



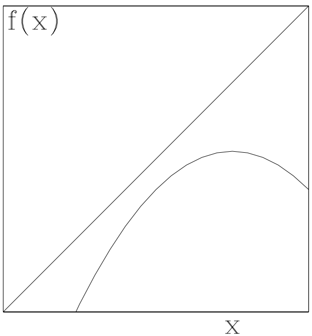


Change of stability $\Longleftrightarrow$ Bifurcations

Saddle-node bifurcation:

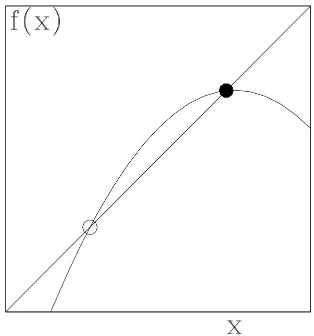
$$\begin{aligned} \dot{x} &= \mu - x^2 \\ x_{n+1} - x_n &= \mu - x_n^2 \\ x_{n+1} = f(x_n) &\equiv x_n + \mu - x_n^2 \\ f'(x_n) &= 1 - 2x_n \end{aligned}$$

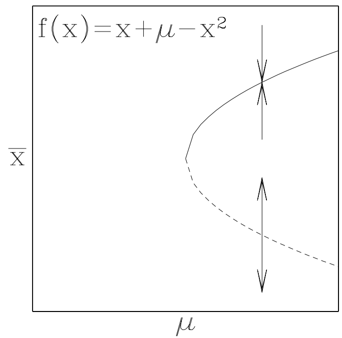
$\mu < 0$



no fixed point

$\mu > 0$





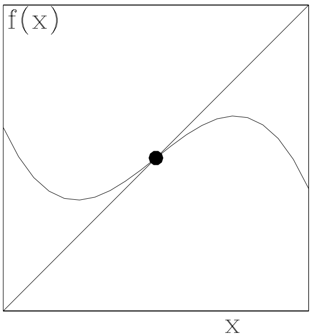
$\bar{x} = \pm\sqrt{\mu}$

$f'(\pm\sqrt{\mu}) = 1 \mp 2\sqrt{\mu} \lessgtr 1$

Supercritical pitchfork bifurcation:

$$
\begin{aligned}
\dot{x} &= \mu x - x^3 \\
x_{n+1} - x_n &= \mu x_n - x_n^3 \\
x_{n+1} = f(x_n) &\equiv x_n + \mu x_n - x_n^3 \\
f'(x_n) &= 1 + \mu - 3x_n^2
\end{aligned}
$$

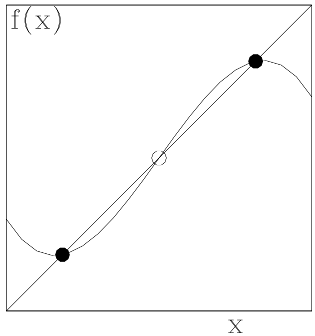
$\mu < 0$



$\bar{x} = 0$

$f'(0) = 1 + \mu < 1$

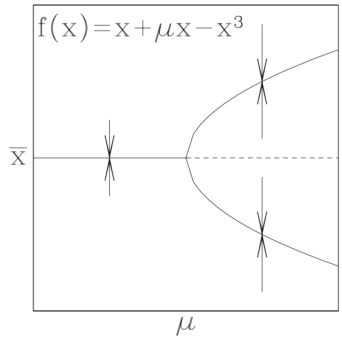
$\mu > 0$



$\bar{x} = 0, \pm\sqrt{\mu}$

$f'(0) = 1 + \mu > 1$

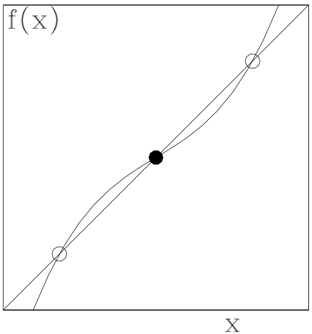
$f'(\pm\sqrt{\mu}) = 1 - 2\mu < 1$

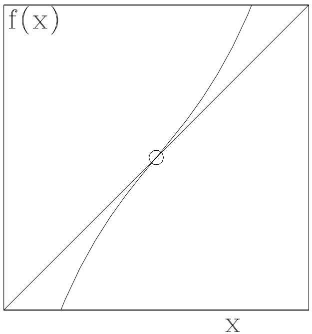


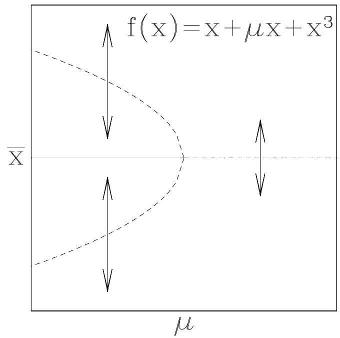
Subcritical pitchfork bifurcation:

$$\begin{aligned}
\dot{x} &= \mu x + x^3 \\
x_{n+1} - x_n &= \mu x_n + x_n^3 \\
x_{n+1} = f(x_n) &\equiv x_n + \mu x_n + x_n^3 \\
f'(x_n) &= 1 + \mu + 3x_n^2
\end{aligned}$$

$\mu < 0$ $\mu > 0$







$\bar{x} = 0, \pm\sqrt{\mu}$ $\bar{x} = 0$

$f'(0) = 1 + \mu < 1$ $f'(0) = 1 + \mu > 1$

$f'(\pm\sqrt{\mu}) = 1 - 2\mu < 1$

Eig at	Bifurcations	Leads to
$(+1, 0)$	saddle-node, pitchfork, transcritical	other steady states
$e^{\pm i\theta}$	secondary Hopf or Neimark-Sacker	torus (next chapter)
$(-1, 0)$	flip or period-doubling	two-cycle

Period-doubling: impossible for continuous dynamical systems

Illustrate via logistic map: $x_{n+1} = a x_n (1 - x_n)$

$$\text{Next value } x_{n+1} \text{ is } \begin{cases} \text{multiple of } x_n \text{ for } x_n \text{ small, but} \\ \text{reduced when } x_n \text{ too large} \end{cases}$$

Mentioned in 1800s, popularized in 1970s: models population

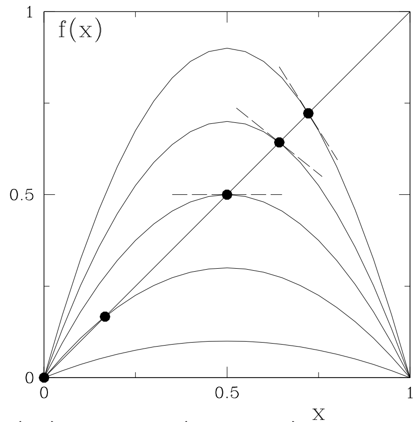
Famous period-doubling cascade discovered in 1970s
by Feigenbaum in Los Alamos, U.S., and
by Coullet and Tresser in Nice, France

Logistic Map

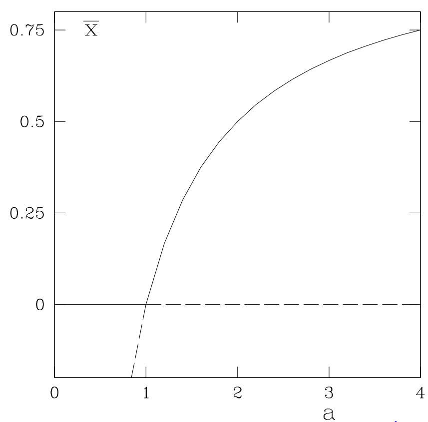
$$x_{n+1} = f(x_n) \equiv a x_n (1 - x_n) \quad \textbf{for } x_n \in [0,1], \quad 0 < a < 4$$

Fixed points: $\bar{x} = a\bar{x}(1 - \bar{x})$

$$\Longrightarrow \begin{cases} \bar{x} = 0 & \textbf{or} \\ 1 = a(1-\bar{x}) \Longrightarrow 1 - \bar{x} = 1/a \Longrightarrow \bar{x} = 1 - 1/a & \end{cases}$$



$f(x) = ax(1-x)$ **for**
$a = 0.4, 1.2, 2.0, 2.8, 3.6$

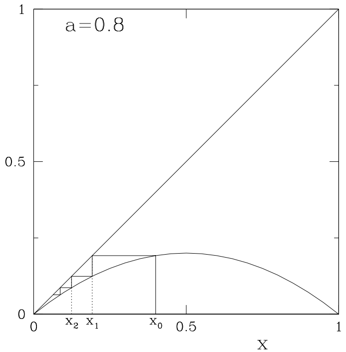


$\bar{x} = 0$ **and** $\bar{x} = 1 - 1/a$
as function of a

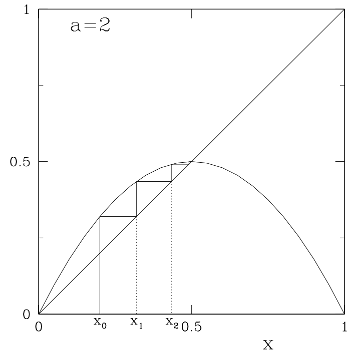
Stability

$$\begin{aligned}
f(x) &= ax(1-x) \\
f'(x) &= a(1-x) - ax = a(1-2x) \\
f'(0) &= a \Longrightarrow |f'(0)| < 1 \text{ for } a < 1 \\
f'\left(1-\frac{1}{a}\right) &= a\left(1-2\left(1-\frac{1}{a}\right)\right) = -a+2 \\
-1 &< f'\left(1-\frac{1}{a}\right) < 1 \\
-1 &< -a+2 < 1 \\
-1 &< a-2 < 1 \\
\textbf{transcrit:}\quad 1 &< a < 3 \qquad ???
\end{aligned}$$

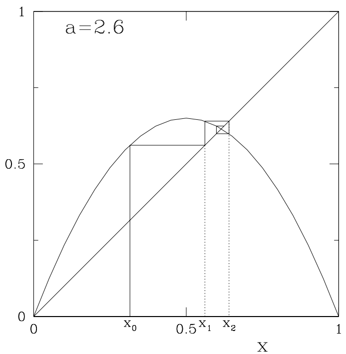
Graphical construction



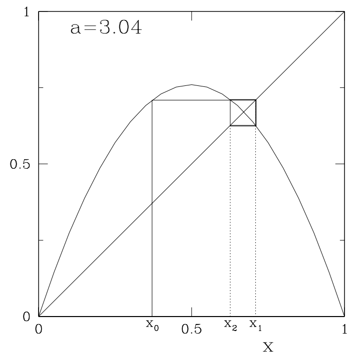
$$a = 0.8: \quad x_n \to 0$$



$$a = 2.0: \quad x_n \to 1 - 1/a$$



$$a = 2.6: \quad x_n \to 1 - 1/a$$



$a = 3.04:$ $\quad x_n \to$ 2−cycle

Seek two-cycle formed at $a = 3$, where $f'(\bar{x}) = -1$

$f(x_1) = x_2$ **and** $f(x_2) = x_1 \Longrightarrow f^2(x_1) \equiv f(f(x_1)) = x_1$

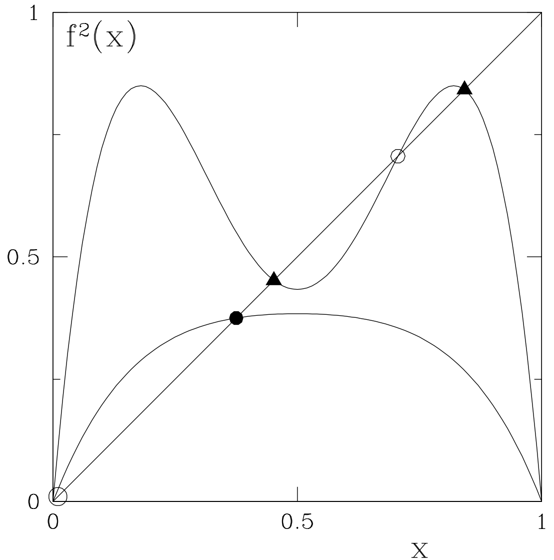
$$\begin{aligned} f^2(x) &= af(x)[1 - f(x)] = a\,(ax(1-x))\,[1 - ax(1-x)] \\ &= a^2x(1-x)\left[1 - ax + ax^2\right] \\ &= a^2x\left[1 - ax + ax^2 - x + ax^2 - ax^3\right] \\ &= a^2x\left[1 - (1+a)x + 2ax^2 - ax^3\right] \end{aligned}$$

Seek fixed points of f^2:

$$\begin{aligned} 0 &= f^2(x) - x = x\left[a^2(1 - (1+a)x + 2ax^2 - ax^3) - 1\right] \\ &\quad \text{contains factors } x \text{ and } (x - (1 - 1/a)) \\ &= -ax(a(x-1)+1)\left[ax^2 - (a+1)x + (a+1)/a\right] \end{aligned}$$

$$x_{1,2} = \frac{a + 1 \pm \sqrt{(a-3)(a+1)}}{2a} \text{ for } a > 3$$

f^2 undergoes pitchfork bifurcation



a	Fixed Points					
1.6	$x = 0$	○	unstable	$x = 1 - 1/a$	●	stable
2.3	$x = 0$	○	unstable	$x_{1,2}$	▲	stable
	$x = 1 - 1/a$	○	unstable	together comprise two-cycle for f		

Stability of two-cycle

$$\frac{d}{dx}f^2(x_1) = f'(f(x_1))f'(x_1) = f'(x_2)f'(x_1)$$

$$x_1 + x_2 = 1 + \frac{1}{a} = \frac{a+1}{a} \qquad x_1 x_2 = \frac{a+1}{a^2}$$

$$\begin{aligned} f'(x_1)f'(x_2) &= a(1-2x_1)\,a(1-2x_2) \\ &= a^2(1-2(x_1+x_2)+4x_1x_2) \\ &= a^2\left(1-2\left(\frac{a+1}{a}\right)+4\left(\frac{a+1}{a^2}\right)\right) \\ &= a^2-2a(a+1)+4(a+1) \\ &= -a^2+2a+4 \end{aligned}$$

$$0 = f'(x_1)f'(x_2) - 1 = -a^2+2a+4-1 = -a^2+2a+3$$

$$a = \frac{2 \pm \sqrt{4+12}}{2} = \frac{1 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 3$$ **pitchfork → 2-cycle**

$$0 = f'(x_1)f'(x_2) + 1 = -a^2+2a+4+1 = -a^2+2a+5$$

$$a = \frac{2 \pm \sqrt{4+20}}{2} = 1+\sqrt{6} = 3.44948\ldots$$ **flip bif → 4-cycle**

Period-doubling cascade

Successive period-doubling bifs occur at successively smaller intervals in a and *accumulate* at $a = 3.569945672\ldots$.

n	2^n	a_n	$\Delta_n \equiv a_n - a_{n-1}$	$\delta_n \equiv \Delta_{n-1}/\Delta_n$
0	1	1		
1	2	3	2	
2	4	3.44948	0.449	4.45
3	8	3.54408	0.0948	4.747
4	16	3.56872	0.0244	4.640
5	32	3.5698912	0.00116	4.662
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	3.569945672	0	4.669

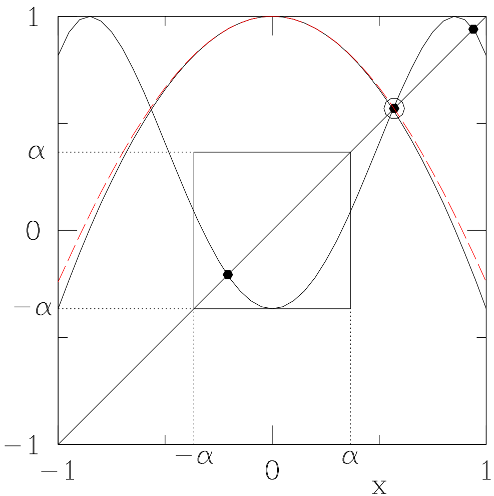
Renormalization

Feigenbaum (1979), Collet & Eckmann (1980), Lanford (1982)

$$f(x) = 1 - rx^2 \qquad \textbf{on} \qquad [-1, 1]$$

T **acts on mappings** f**:**

$$(Tf)(x) \equiv -\frac{1}{\alpha} f^2(-\alpha x)$$



Seek fixed point of $T: f(x) \approx (Tf)(x)$

$$
\begin{aligned}
1 - rx^2 &\approx -f^2(-\alpha x)/\alpha \\
&= -f(1 - r(\alpha x)^2)/\alpha \\
&= -(1 - r(1 - r(\alpha x)^2)^2)/\alpha \\
&= -(1 - r(1 - 2r(\alpha x)^2 + r^2(\alpha x)^4))/\alpha \\
&= -(1 - r + 2r^2(\alpha x)^2 - r^3(\alpha x)^4)/\alpha
\end{aligned}
$$

Matching constant and quadratic terms:

$$\frac{r-1}{\alpha} = 1 \qquad\qquad \frac{2r^2\alpha^2}{\alpha} = r$$

$$r - 1 = \alpha \qquad\qquad 2r\alpha = 1 \Longrightarrow 2r(r-1) = 1$$

$$\alpha = 0.366 \qquad\qquad r = (1 + \sqrt{3})/2 = 1.366$$

$$\begin{array}{ccccccccc} f(x) & \rightarrow & (Tf)(x) & \rightarrow & (T^2f)(x) & \rightarrow & \ldots & \rightarrow & \phi(x) \\ \text{2nd order} & \rightarrow & \text{4th order} & \rightarrow & \text{8th order} & \rightarrow & \ldots & \rightarrow & \end{array}$$

where limiting function

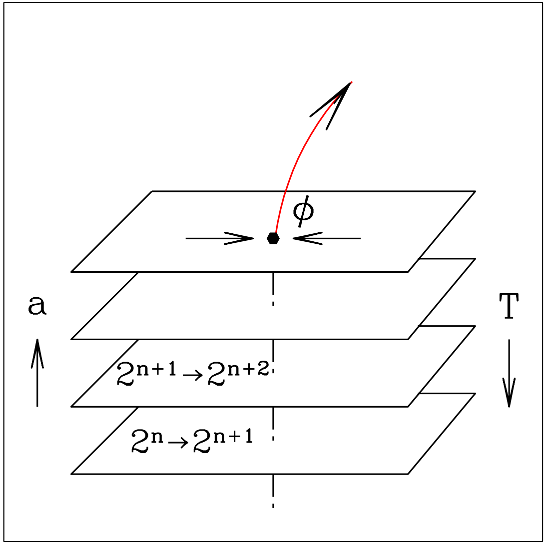
$$\phi(x) = 1 - 1.528x^2 + 0.105x^4 + 0.0267x^6 + \ldots$$

is fixed point of T (mapping-of-mappings), i.e.

$$T(\phi) = \phi$$

Single unstable direction with multiplier $\delta = 4.6692$ (table)

ϕ, δ, are universal for all map families with quadratic maxima, such as $a \sin \pi x$, $1 - rx^2$, $ax(1-x)$.



If f has a stable 2-cycle, then f^2 has a stable fixed point.

More generally, if f has a stable 2^n cycle, then f^2 has a stable 2^{n-1} cycle.

Since T involves taking f^2, then T takes maps with 2^n cycles to maps with 2^{n-1} cycles.

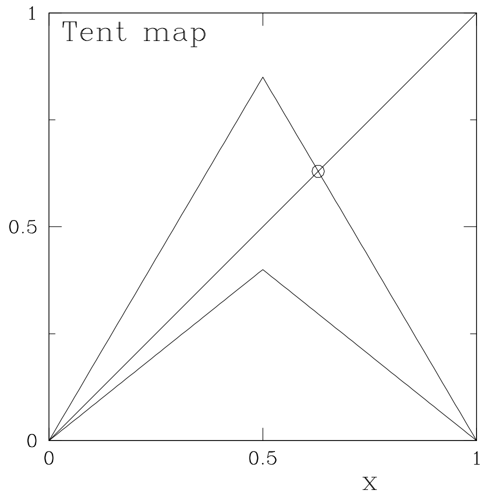
Other classes of unimodal maps, characterized by the nature of their extrema, undergo a period-doubling cascade. Each class has its own asymptotic value of δ and α. Examples are functions with a quartic maximum or the tent map

$$f(x) = \begin{cases} ax & \textbf{for } x < 1/2 \\ a(1-x) & \textbf{for } x > 1/2 \end{cases}$$

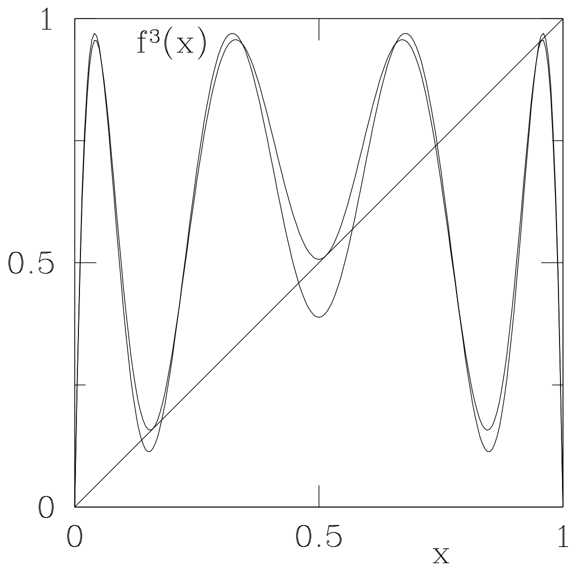
$a < 1$: 0 is stable fixed point

$a > 1$: 0 is unstable fixed point

$\bar{x} = a/(1+a)$ is stable



Periodic Windows

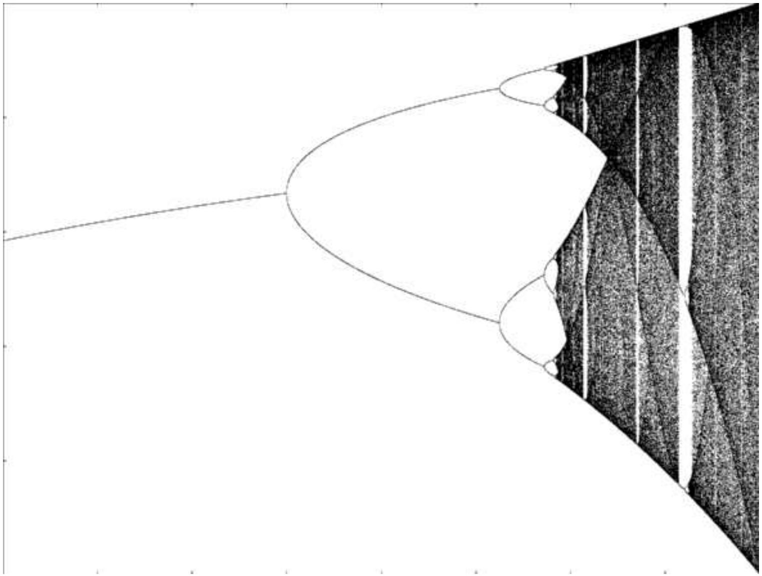


f^3 has 3 saddle-node bifurcations at $a_3 = 1 + \sqrt{8} = 3.828$

$\Longrightarrow$ **two 3-cycles**, one stable and one unstable ($|(f^3)'| \gtrless 1$)

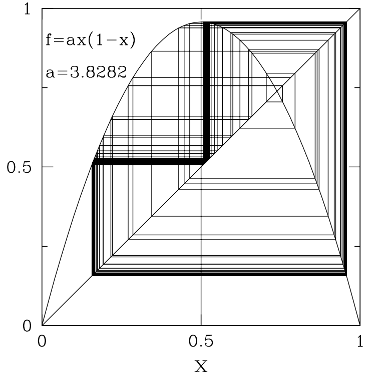
Stable and unstable n-cycles created when f^n crosses diagonal

Bifurcation Diagram for Logistic Map

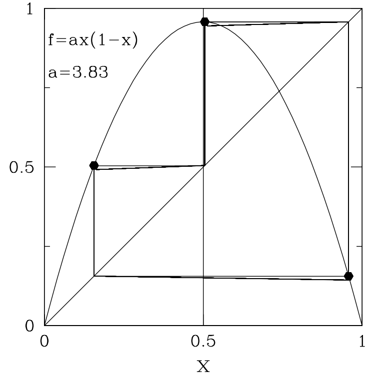


Each n-cycle undergoes period-doubling cascade

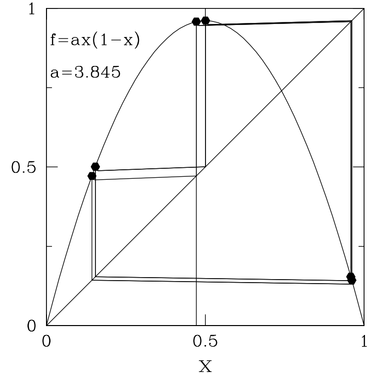
Three-cycle: before and after



$a = 3.8282$
intermittency



$a = 3.83$
3-cycle

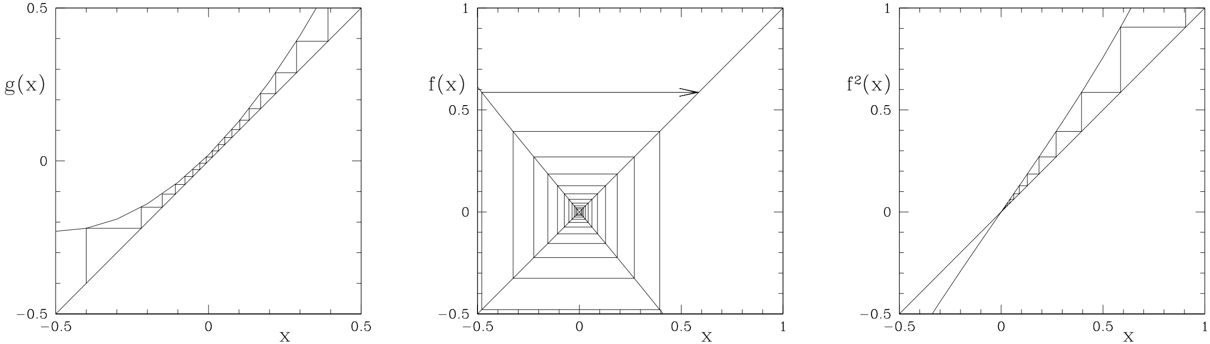


$a = 3.845$
6-cycle

Stable three-cycle for $1 + \sqrt{8} = 3.828427 < r < 3.857$

Intermittency

Slow dynamics near a bifurcation



Type I | **Type III**

near saddle-node bif | **near period-doubling bif** | **near pitchfork bif**

$g = x_n + 0.2 + x_n^2$ | $f = -1.2\, x_n + 0.1\, x_n^3$ | **of** f^2

Slow dynamics near *ghosts* of not-yet-created or *unstable* fixed points

Assume that another mechanism re-injects trajectories back to $x \approx 0$

Type II intermittency associated with a subcritical Hopf bifurcation

Sharkovskii Ordering

$$\begin{array}{ll}
3 \rhd 5 \rhd 7 \rhd 9 \ldots & \text{(odd numbers)} \\
2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd 2\cdot 9 \ldots & \text{(multiples of 2)} \\
2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd 2^2\cdot 9 \ldots & \text{(multiples of } 2^2\text{)} \\
\vdots & \vdots \\
\ldots 2^3 \rhd 2^2 \rhd 2 \rhd 1 & \text{(powers of 2)}
\end{array}$$

Sharkovskii's Theorem:

f has a k-cycle $\Longrightarrow$ f has an ℓ-cycle for any $\ell \rhd k$

f has a 3-cycle $\Longrightarrow$ f has an ℓ-cycle for any ℓ

(Most cycles not stable)

Logistic map for any $r > r_3$ has cycles of *all* lengths.

Period-doubling in Rayleigh-Bénard convection

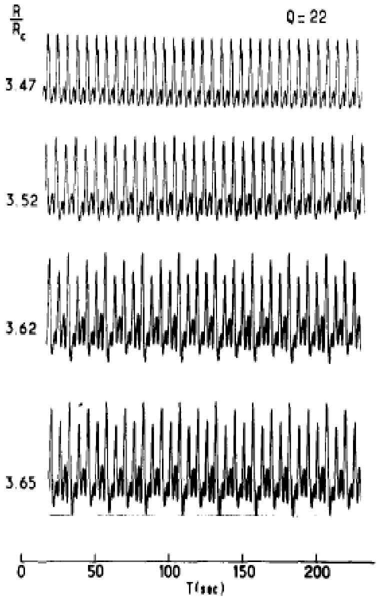


Fig. 3. The period doubling cascade ($\Gamma = 4$, $Q = 22$).

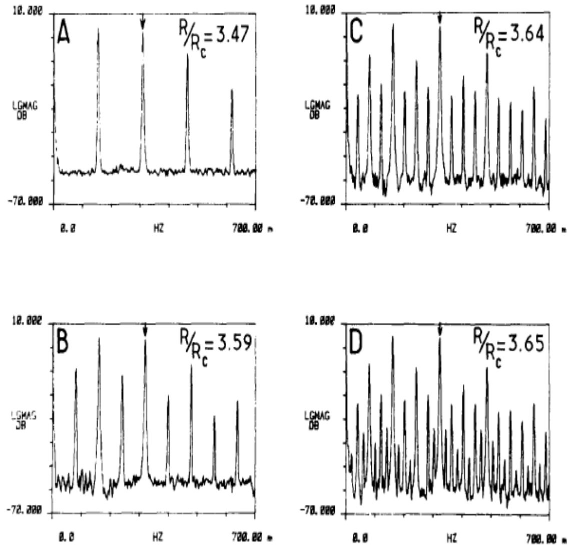


Fig. 4. Fourier spectrum for the period-doubling cascade.

From Libchaber, Fauve & Laroche, Physica D (1983).

A: freq f and $f/2 \Longrightarrow$ B: $f/4 \Longrightarrow$ C: $f/8 \Longrightarrow$ D: $f/16$

Rayleigh-Bénard convection in small-aspect-ratio container

Type III intermittency

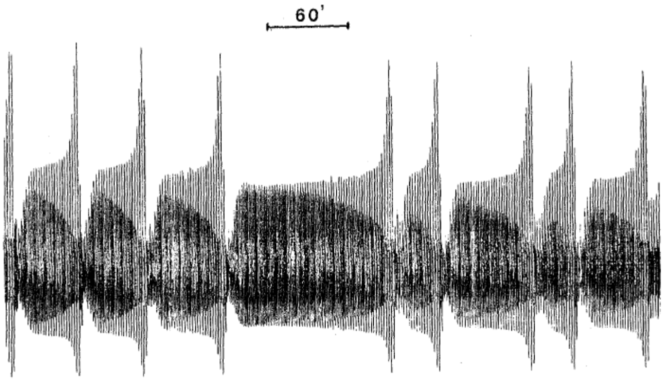


FIG. 1. Time dependence of light intensity, roughly proportional to the horizontal temperature gradient near the cold plume. $N_{\mathrm{Ra}}/N_{\mathrm{Ra},c} \approx 420$.

Timeseries

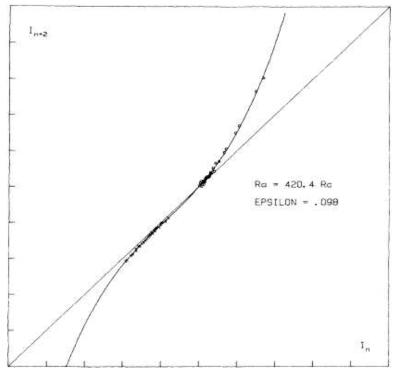


FIG. 2. Return map $I_{n+2} = f(I_n)$, constructed from the data shown in Fig. 1 superposing two laminar periods. The amplitudes of the light modulation in the turbulent bursts have not been drawn.

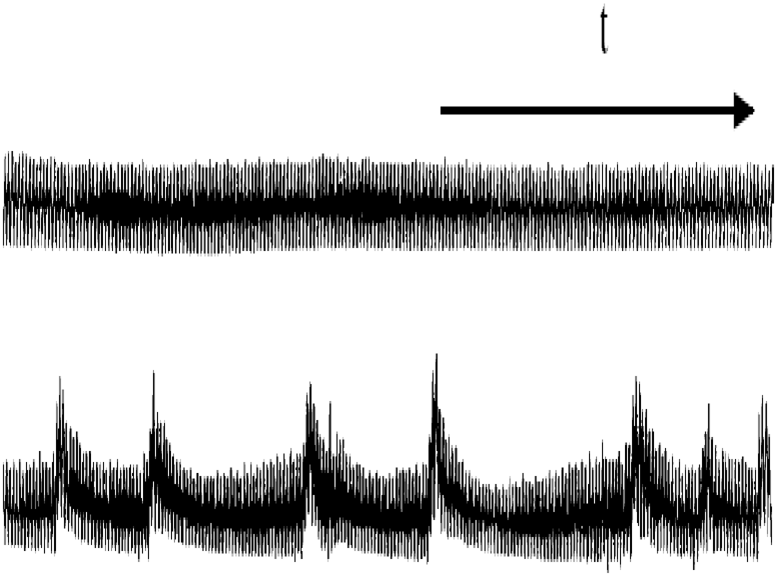
Poincaré map
maxima (I_k, I_{k+2})

Subharmonic (period-doubled signal) grows til burst, followed by laminar phase

From M. Dubois, M.A. Rubio & P. Bergé, Phys. Rev. Lett. **51**, *1446 (1983).*

Rayleigh-Bénard convection in small-aspect-ratio container

Type I intermittency

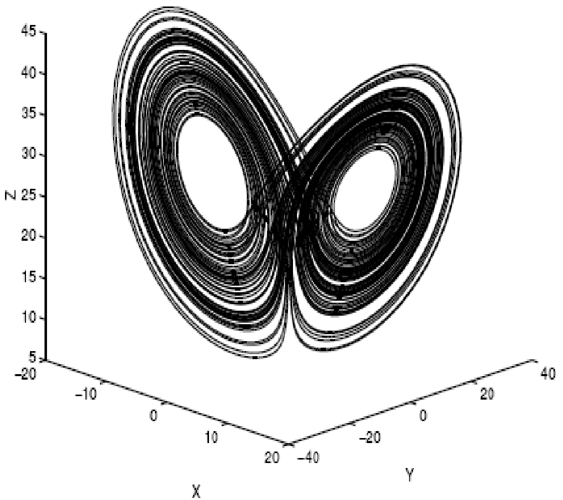


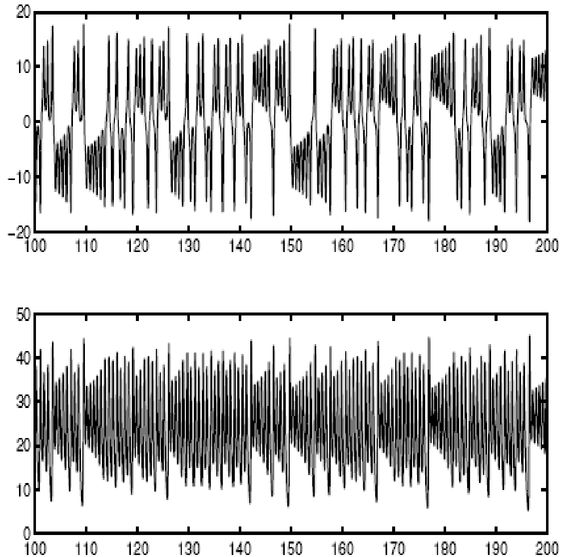
non-intermittent timeseries
$Ra = 280\ Ra_c$

intermittent timeseries
$Ra = 300\ Ra_c$

From P. Bergé, M. Dubois, P. Manneville & Y. Pomeau, J. Phys. (Paris) Lett. **41**, *341 (1980).*

Poincaré map of Lorenz system



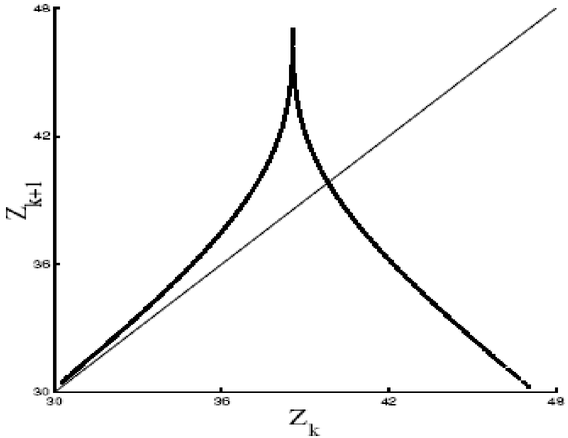


3D trajectory of the Lorenz system for standard chaotic value of $r = 28$

Timeseries of $X(t)$
Timeseries of $Z(t)$

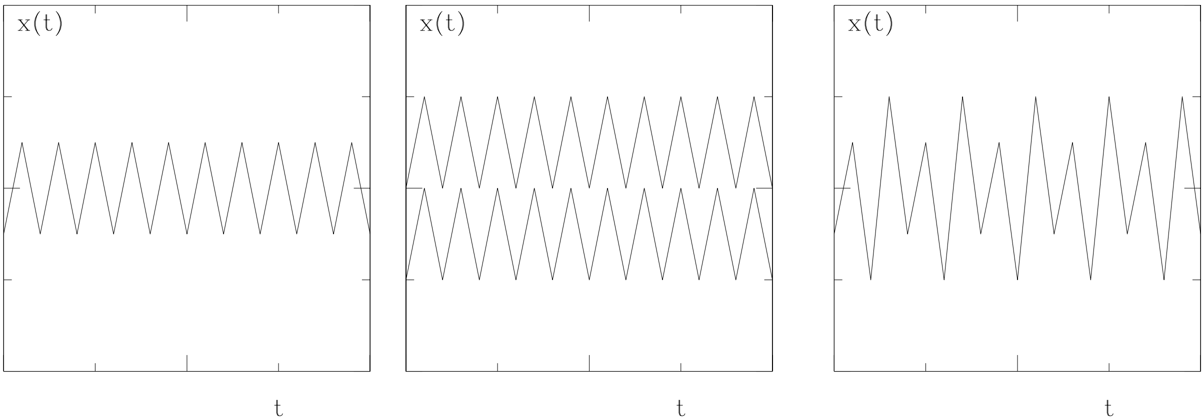
From P. Manneville, Course notes

Successive pairs of maxima of Z resemble tent map:

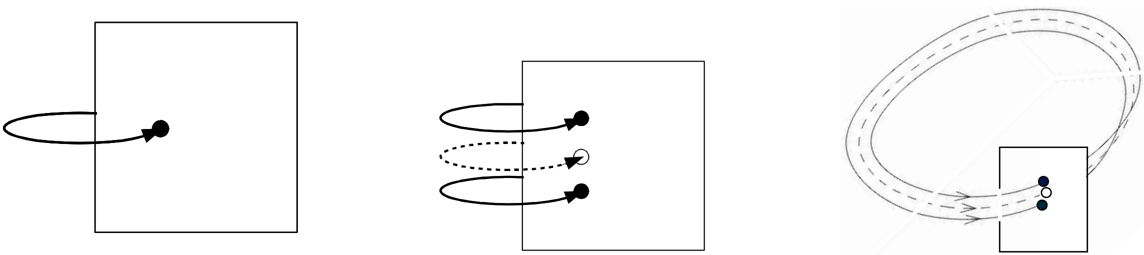


From P. Manneville, Course notes

From continuous flows to discrete maps



limit cycle **after pitchfork** **after period-doubling**



Where do discrete dynamical systems come from?

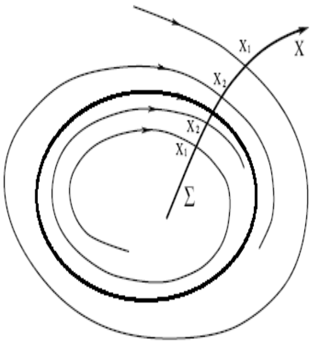
Hopf or global bifurcations $\Longrightarrow$ limit cycles

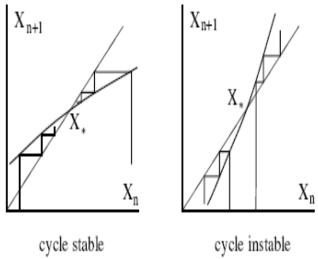
Study these using Poincaré or first return map:

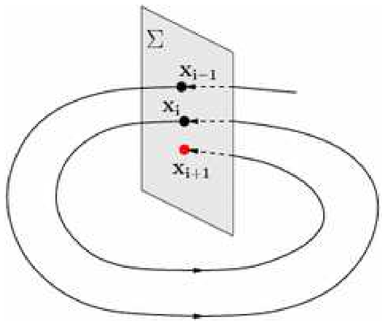
Continuous dynamical system $x(t) \in \mathcal{R}^d$,

$x^d(t)$ goes through β for some set of trajectories

$$y_{n+1} = g(y_n) \iff \begin{cases} x(t) = (y_n, \beta) & , \; \dot{x}^d(t) > 0 \\ x(t') = (y_{n+1}, \beta) & , \; \dot{x}^d(t') > 0 \\ \text{above not satisfied for any } t'' \in (t, t') & \end{cases}$$







From P. Manneville class notes, DEA Physique des Liquides

From Moehlis, Josic, & Shea-Brown, Scholarpedia

Floquet theory

Linear equations with constant coefficients:

$$a\ddot{x} + b\dot{x} + cx = 0 \implies x(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}$$

$$\text{where} \quad a\lambda^2 + b\lambda + c = 0$$

$$\dot{x} = cx \implies x(t) = e^{ct} x(0)$$

$$\sum_{n=0}^{N} c_n x^{(n)} = 0 \implies x(t) = \sum_{n=1}^{N} \alpha_n e^{\lambda_n t}$$

$$\text{where} \quad \sum_{n=0}^{N} c_n \lambda^n = 0$$

Generalize to linear equations with periodic coefficients:

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = 0 \implies x(t) = \alpha_1(t) e^{\lambda_1 t} + \alpha_2(t) e^{\lambda_2 t}$$

$a(t), b(t), c(t)$ **have period** $T \implies \alpha_1(t), \alpha_2(t)$ **have period** T

Floquet theory continued

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = 0 \Longrightarrow x(t) = \alpha_1(t)e^{\lambda_1 t} + \alpha_2(t)e^{\lambda_2 t}$$

$\alpha_1(t)$, $\alpha_2(t)$	**Floquet functions**
λ_1, λ_2	**Floquet exponents**
$e^{\lambda_1 T}$, $e^{\lambda_2 T}$	**Floquet multipliers**

λ_1, λ_2 not roots of polynomial $\Longrightarrow$ must calculate numerically or asymptotically

$$\dot{x} = c(t)x \;\Longrightarrow\; x(t) = e^{\lambda t}\alpha(t)$$

$$\sum_{n=0}^{N} c_n(t)x^{(n)} = 0 \;\Longrightarrow\; x(t) = \sum_{n=1}^{N} e^{\lambda_n t}\alpha_n(t)$$

Floquet theory and linear stability analysis

Dynamical system: $\dot{x} = f(x)$

Limit cycle solution: $\bar{x}(t+T) = \bar{x}(t)$ **with** $\dot{\bar{x}}(t) = f(\bar{x}(t))$

Stability of $\bar{x}(t)$: $x(t) = \bar{x}(t) + \epsilon(t)$

$$
\begin{aligned}
\dot{\bar{x}} + \dot{\epsilon} &= f(\bar{x}(t)) + f'(\bar{x}(t))\epsilon(t) + \ldots \\
\dot{\epsilon} &= f'(\bar{x}(t))\epsilon(t)
\end{aligned}
$$

Floquet form! $\epsilon(t) = e^{\lambda t}\alpha(t)$ **with $\alpha(t)$ of period T**

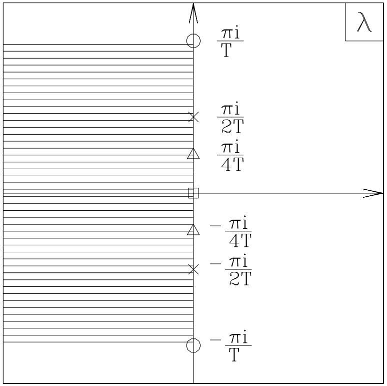
$\mathrm{Re}(\lambda) > 0 \implies \bar{x}(t)$ **unstable**

$\mathrm{Re}(\lambda) < 0 \implies \bar{x}(t)$ **stable**

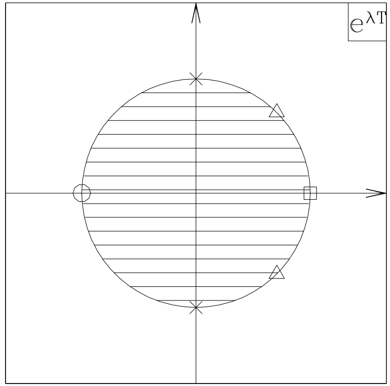
λ **complex** $\implies$ **most unstable or least stable pert has period different from $\bar{x}(t)$**

Region of stability

for exponent λ



for multiplier $e^{\lambda T}$



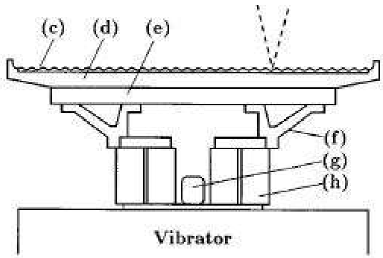
Imaginary part non-unique $\Longrightarrow$ choose Im$(\lambda) \in (-\pi i/T, \pi i/T]$

In $\mathcal{R}^N$, $\bar{x}(t)$ stable iff real parts of all λ_j are negative

Monodromy matrix: $\dot{M} = Df(\bar{x}(t))M$ with $M(t=0) = I$

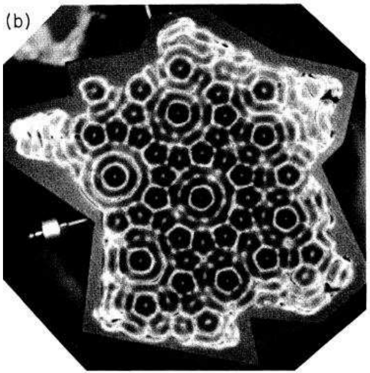
Floquet multipliers/functions = eigenvalues/vectors of $M(T)$

Faraday instability

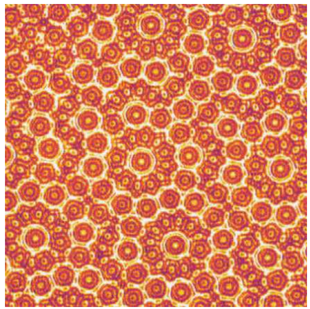


Faraday (1831): Vertical vibration of fluid layer $\Longrightarrow$ stripes, squares, hexagons

In 1990s: first fluid-dynamical quasicrystals:



Edwards & Fauve
J. Fluid Mech. (1994)



Kudrolli, Pier & Gollub
Physica D (1998)

Oscillating frame of reference $\Longrightarrow$ "oscillating gravity"

$$
\begin{aligned}
G(t) &= g\,(1 - a\cos(\omega t)) \\
G(t) &= g\,(1 - a\left[\cos(m\omega t) + \delta\cos(n\omega t + \phi_0)\right])
\end{aligned}
$$

Flat surface becomes linearly unstable for sufficiently high a

Consider domain to be horizontally infinite (homogeneous) $\Longrightarrow$ solutions exponential/trigonometric in $\mathrm{x} = (x, y)$

Seek bounded solutions $\Longrightarrow$ trigonometric: $\exp(i\mathrm{k}\cdot\mathrm{x})$

$$\textbf{Height}\ \ \zeta(x, y, t) = \sum_{\mathrm{k}} e^{i\mathrm{k}\cdot\mathrm{x}}\hat{\zeta}_k(t)$$

Oscillating gravity $\Longrightarrow$ temporal Floquet problem, $T = 2\pi/\omega$

$$\hat{\zeta}_k(t) = \sum_j e^{\lambda_k^j t} f_k^j(t)$$

$$\textbf{Height}\ \ \zeta(x,y,t) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\zeta}_k(t)$$

Ideal fluids (no viscosity), sinusoidal forcing $\Longrightarrow$ Mathieu eq.

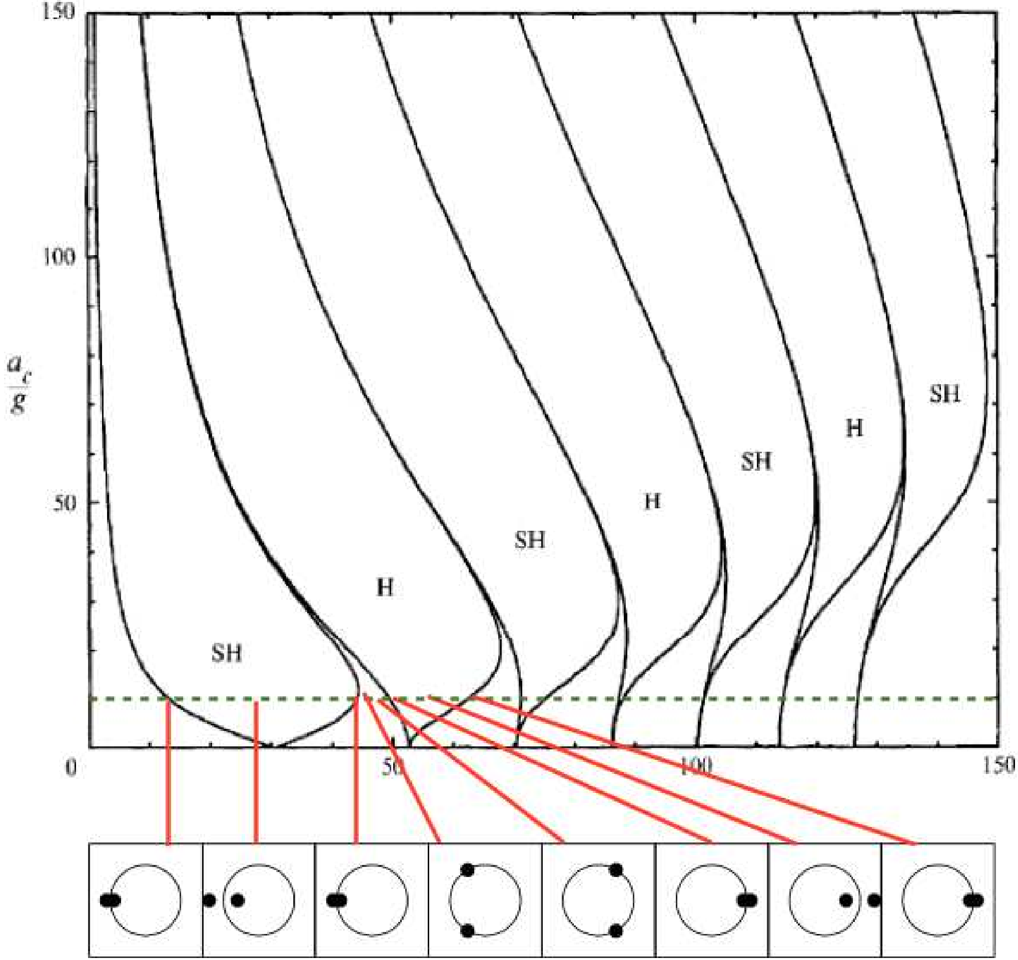
$$\partial_t^2\hat{\zeta}_k + \omega_0^2\left[1 - a\cos(\omega t)\right]\hat{\zeta}_k = 0$$

ω_0^2 **combines g, densities, surface tension, wavenumber k**

$$\hat{\zeta}_k(t) = \sum_{j=1}^{2} e^{\lambda_k^j t} f_k^j(t)$$

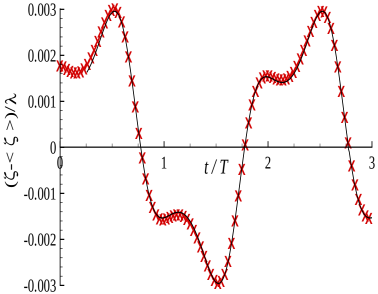
Re$(\lambda_k^j) > 0$ for some $j, k \Longrightarrow \hat{\zeta}_k \nearrow \Longrightarrow$ flat surface unstable
$\Longrightarrow$ Faraday waves with wavelength $2\pi/k$

Im(λ_k^j)	$e^{\lambda T}$	**waves**	**period**
0	1	**harmonic**	**same as forcing**
$\omega/2$	-1	**subharmonic**	**twice forcing period**

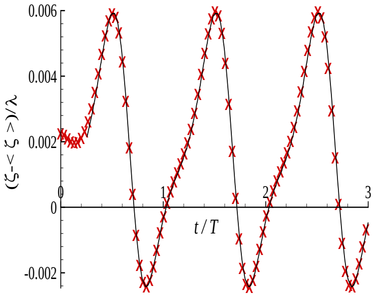


150
100
50
0
$\frac{a_c}{g}$
50
100
150
SH
H
SH
H
SH
H
SH

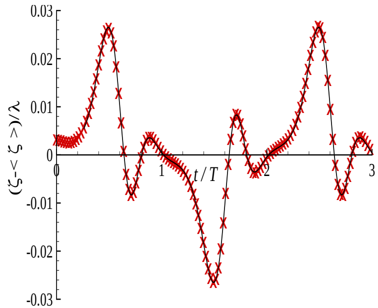
Floquet functions



within tongue 1 /2
subharmonic
$\mu = -1$



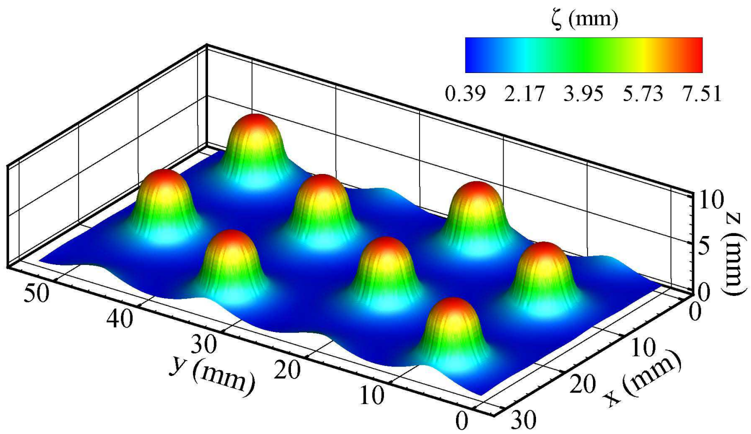
within tongue 2 /2
harmonic
$\mu = +1$



within tongue 3 /2
subharmonic
$\mu = -1$

From Périnet, Juric & Tuckerman, J. Fluid Mech. (2009)

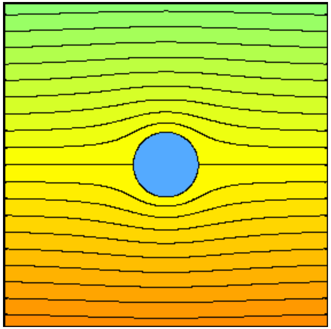
Hexagonal patterns in Faraday instability



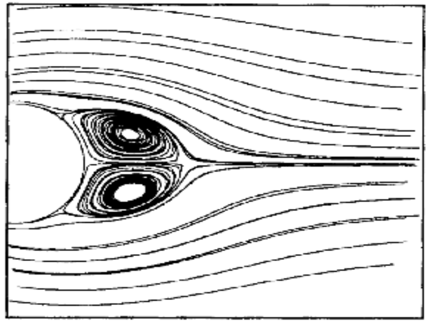
From Périnet, Juric & Tuckerman, J. Fluid Mech. (2009)

Cylinder wake

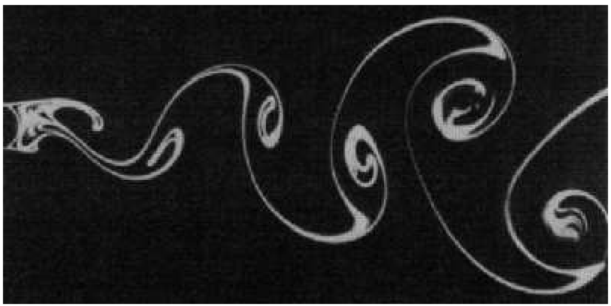
Ideal flow



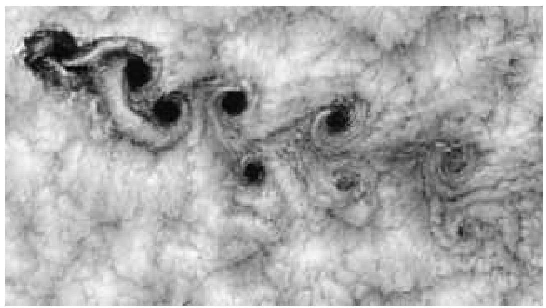
with downstream recirculation zone



Von Kármán vortex street ($\mathrm{Re} \geq 46$)

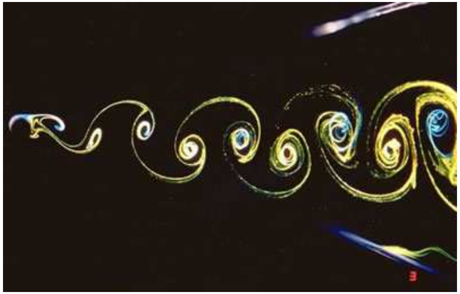


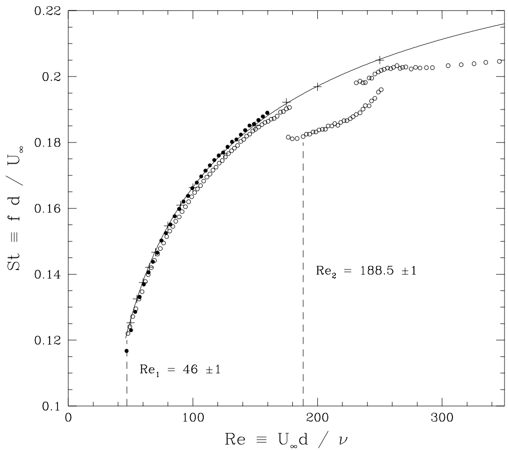
Laboratory experiment (Taneda, 1982)



Off Chilean coast past Juan Fernandez islands

von Kármán vortex street: $Re = U_\infty d/\nu \geq 46$





spatially:
two-dimensional (x, y)
(homogeneous in z)

temporally:
periodic, $St = fd/U_\infty$
appears spontaneously

$$\mathrm{U}_{2D}(x, y, t \bmod T)$$

Stability analysis of von Kármán vortex street

2D limit cycle $\mathrm{U}_{2D}(x, y, t \bmod T)$ obeys:

$$\partial_t \mathrm{U}_{2D} = -(\mathrm{U}_{2D} \cdot \nabla)\mathrm{U}_{2D} - \nabla P_{2D} + \frac{1}{Re}\Delta \mathrm{U}_{2D}$$

Perturbation obeys:

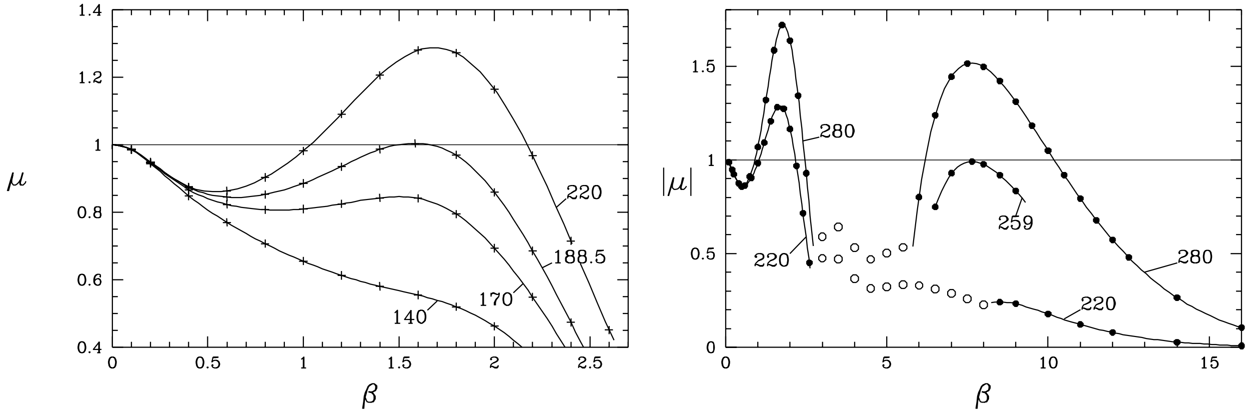
$$\partial_t \mathbf{u}_{3D} = -(\mathrm{U}_{2D}(t) \cdot \nabla)\mathbf{u}_{3D} - (\mathbf{u}_{3D} \cdot \nabla)\mathrm{U}_{2D}(t) - \nabla \boldsymbol{p}_{3D} + \frac{1}{Re}\Delta \mathbf{u}_{3D}$$

Equation homogeneous in z, periodic in t $\Longrightarrow$

$$\mathbf{u}_{3D}(x, y, z, t) \sim e^{i\beta z} e^{\lambda_\beta t} \mathrm{f}_\beta(x, y, t \bmod T)$$

Fix β, calculate largest $\mu = e^{\lambda_\beta T}$ via linearized Navier-Stokes

From Barkley & Henderson, J. Fluid Mech. (1996)



mode A: $\mathrm{Re_c} = 188.5$

$\beta_c = 1.585 \Longrightarrow \lambda_c \approx 4$

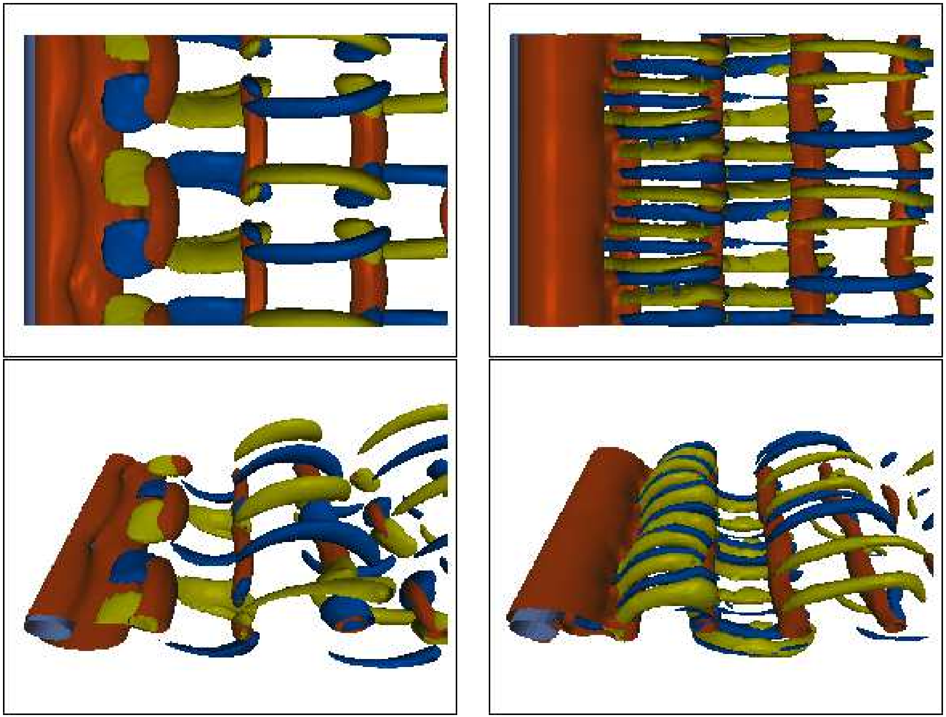
mode B: $\mathrm{Re_c} = 259$

$\beta_c = 7.64 \Longrightarrow \lambda_c \approx 1$

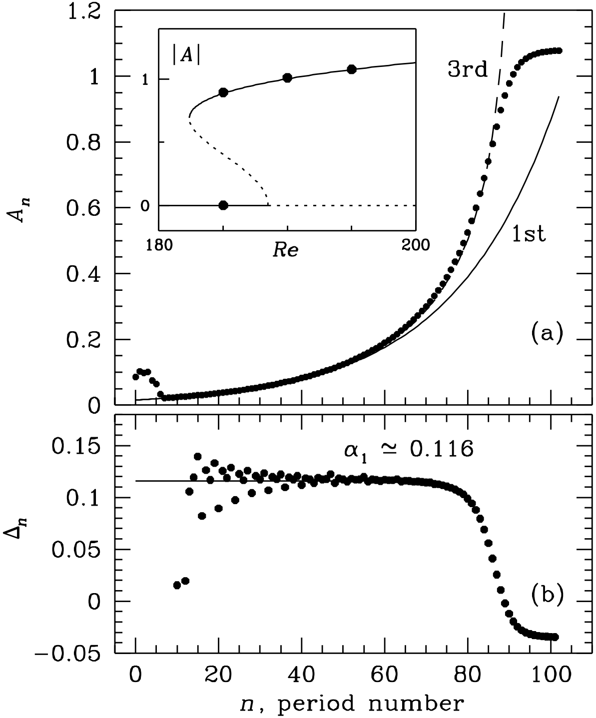
Temporally: $\mu = 1 \Longrightarrow$
steady bifurcation to limit cycle with same periodicity as U_{2D}

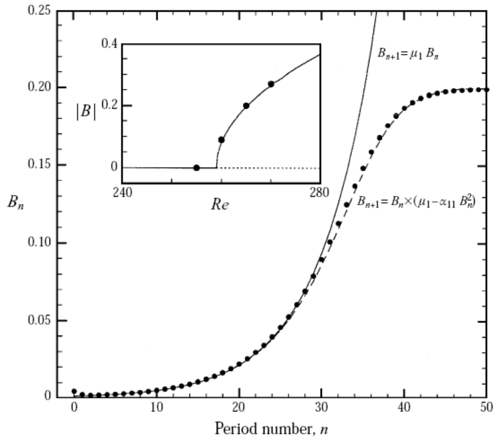
Spatially: circle pitchfork (any phase in z)

mode A at $\mathrm{Re} = 210$ **mode B at $\mathrm{Re} = 250$**



From M.C. Thompson, Monash University, Australia
(http://mec-mail.eng.monash.edu.au/∼mct/mct/docs/cylinder.html)



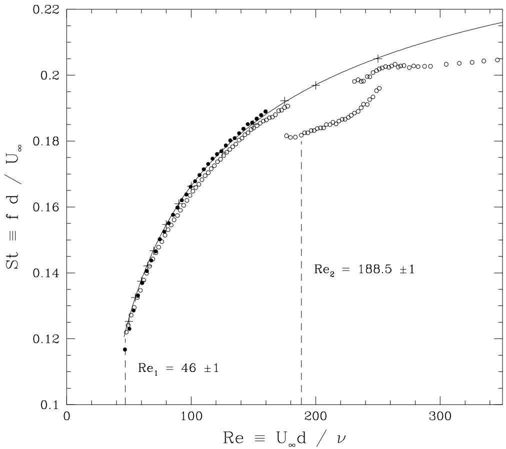


transition to mode A
initially faster than exp
$\Longrightarrow$ subcritical

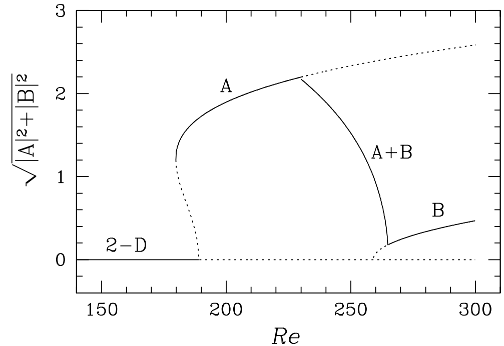
transition to mode B
initially slower than exp
$\Longrightarrow$ supercritical

Describe via complex amplitudes A_n, B_n
amplitude and phase of mode A, B at fixed instant of cycle

$$
\begin{aligned}
A_{n+1} &= \left(\mu_A + \alpha_A |A_n|^2 A_n - \beta_A |A_n|^4\right) A_n \\
B_{n+1} &= \left(\mu_B - \alpha_B |B_n|^2\right) B_n
\end{aligned}
$$



Numerical and experimental frequencies
From Barkley & Henderson, JFM (1996)



Solution to minimal model
From Barkley, Tuckerman
& Golubitsky, PRE (2000)

Interaction between A and B:

$$A_{n+1} = \left(\mu_A + \alpha_A|A_n|^2 + \gamma_A|B_n|^2 - \beta_A|A_n|^4\right) A_n$$
$$B_{n+1} = \left(\mu_B - \alpha_B|B_n|^2 + \gamma_B|A_n|^2\right) B_n$$

As Re is increased: mode $A \Longrightarrow A + B \Longrightarrow$ mode B