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Numerical Methods for Differential Equations in Physics

How to make a solenoidal (divergence-free) field

Influence matrix

Streamfunction-vorticity formulation

Sherman-Morrison-Woodbury formula

Poloidal-Toroidal formulation

$$\left(\begin{array}{c}\textbf{Influence}\\ \textbf{matrix}\end{array}\right) \iff \left(\begin{array}{c}\textbf{Green's}\\ \textbf{functions}\end{array}\right) \iff \left(\begin{array}{c}\textbf{Sherman-Morrison-}\\ \textbf{Woodbury formula}\end{array}\right)$$

2D Periodic-Bounded Geometry: Eliminating Pressure

$$\partial_t \mathbf{u} - \nu \Delta \mathbf{u} = -\nabla p - (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\mathbf{u}|_{\text{bdy}} = 0$$

$$\mathbf{u} = \nabla \times \psi \hat{\mathbf{e}}_z \qquad u = -\partial_y \psi$$
$$\omega = \hat{\mathbf{e}}_z \cdot \nabla \times \mathbf{u} = \hat{\mathbf{e}}_z \cdot \nabla \times \nabla \times \psi \hat{\mathbf{e}}_z = -\Delta \psi \qquad v = \partial_x \psi$$

Take $\hat{\mathbf{e}}_z \cdot \nabla \times$ of N-S:

$$\partial_t \omega - \nu \Delta \omega = \hat{\mathbf{e}}_z \cdot \nabla \times [-(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}]$$

Implicit-explicit time-integration:

$$(I - \nu \Delta t \Delta)\omega^{n+1} = \overbrace{\omega^n + \Delta t \hat{\mathbf{e}}_z \cdot \nabla \times [-(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}]}^{s^n}$$

Nested Helmholtz-Poisson systems:

$$(I - \epsilon \Delta)\omega = s \qquad \text{where } \epsilon \equiv \nu \Delta t$$
$$\Delta \psi = -\omega$$

• If y direction is periodic:

$$u = -\partial_y \psi = 0 \text{ at } x = \pm 1 \Longrightarrow \psi(x = \pm 1, y) = \psi_\pm$$
$$v = \partial_x \psi = 0 \text{ at } x = \pm 1$$

Flux in y direction:

$$\int dx\; v(x,y) = \int dx\; \partial_x \psi(x,y) = \psi(1,y) - \psi(-1,y) = \psi]_-^+$$

Fix flux $\Longrightarrow$ fix $\psi_+ - \psi_-$ $\Longrightarrow$ ψ_- is arbitrary constant

• If y direction is bounded:

$$u = -\partial_y \psi = 0 \quad \text{at } y = \pm 1$$
$$v = \partial_x \psi = 0 \quad \text{at } y = \pm 1 \Longrightarrow \psi = 0 \quad \text{everywhere on boundary}$$

In general:

$$\psi|_{\text{bdy}} = 0$$
$$\partial_n \psi|_{\text{bdy}} = 0$$

But too many boundary conditions on ψ, none on ω.

Superpose particular ($s \neq 0$) and homogeneous ($s = 0$) solutions.

For each $\mathrm{x}_i \in$ bdy, construct homogeneous soln (Green's function):

$$
\begin{aligned}
(I - \epsilon\Delta)\omega_i^{\text{hom}} &= 0 \\
\omega_i^{\text{hom}}(\mathrm{x}_j) &= \delta_{ij} \quad \mathrm{x}_j \in \textbf{boundary} \\
\Delta\psi_i^{\text{hom}} &= -\omega_i^{\text{hom}} \\
\psi_i^{\text{hom}}(\mathrm{x}_j) &= 0 \quad \mathrm{x}_j \in \textbf{boundary}
\end{aligned}
$$

Capacitance or influence matrix:

$$
C_{ji} \equiv \partial_n \psi_i^{\text{hom}}(\mathrm{x}_j) \quad \mathrm{x}_j \in \textbf{boundary}
$$

Particular solution:

$$
\begin{aligned}
(I - \epsilon\Delta)\omega^{\text{part}} &= s \\
\omega^{\text{part}}(\mathrm{x}_j) &= 0 \quad \mathrm{x}_j \in \textbf{boundary} \\
\Delta\psi^{\text{part}} &= -\omega^{\text{part}} \\
\psi^{\text{part}}(\mathrm{x}_j) &= 0 \quad \mathrm{x}_j \in \textbf{boundary}
\end{aligned}
$$

Superposing:

$$
\begin{aligned}
\omega &= \omega^{\text{part}} + \sum_i \alpha_i \omega_i^{\text{hom}} \\
\psi &= \psi^{\text{part}} + \sum_i \alpha_i \psi_i^{\text{hom}}
\end{aligned}
$$

Boundary conditions:

$$\begin{aligned} \psi(\mathrm{x}_j) &= 0 \\ \partial_n \psi(\mathrm{x}_j) &= \partial_n \psi^{\mathrm{part}}(\mathrm{x}_j) + \sum_i \alpha_i \partial_n \psi_i^{\mathrm{hom}}(\mathrm{x}_j) \\ &= \partial_n \psi^{\mathrm{part}}(\mathrm{x}_j) + \alpha_i C_{ji} \end{aligned}$$

Solve (C decouples into blocks by parity or Fourier mode)

$$\sum_i C_{ji} \alpha_i = -\partial_n \psi^{\mathrm{part}}(\mathrm{x}_j)$$

Requirements for preprocessing:

Geometry	Number of homog. solns.	Time for single homog. soln.	Size of C	Time for C^{-1}
y bounded	$2(N_x + N_y)$	$N_x N_y^2$	$4\ (N_x + N_y)/2 \times (N_x + N_y)/2$	$4((N_x + N_y)/2)^3$
y periodic	$2N_y$	$N_x N_y$	$2N_y(1 \times 1)$	$2N_y$

Requirements for each timestep:

Geometry	Time for particular soln.	Time for action by C^{-1}	Storage for homog. solns.
y bounded	$N_x N_y^2$	$4((N_x + N_y)/2)^2$	$2(N_x + N_y) N_x N_y$
y periodic	$N_x N_y$	$N_x N_y$	$2N_x N_y^2$

Sherman-Morrison-Woodbury Formula

Rank-1 matrix

$$A^{(1)}_{ik} = v_i w^T_k = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} & & & \end{bmatrix}$$

Only one non-zero eigenvalue and one linearly ind row (or column)

Rank-2 matrix

$$A^{(2)}_{ik} = v_i w^T_k + \tilde{v}_i \tilde{w}^T_k = \sum_{j=1}^{2} V_{ij} W^T_{jk}$$

Only two non-zero eigenvalues and two linearly ind rows (or columns)

Rank-J matrix

$$A^{(J)}_{ik} = \sum_{j=1}^{J} V_{ij} W^T_{jk} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \end{bmatrix}$$

J non-zero eigenvalues and J linearly independent rows or columns

Sherman-Morrison: relates inverses of matrices differing by rank-1 matrix

Woodbury: relates inverses of two matrices differing by a low-rank matrix

Changing J rows of a matrix is a rank-J change

$$
\begin{aligned}
(H + VW^T)^{-1} &= H^{-1} - H^{-1}VC^{-1}W^TH^{-1} \\
C &\equiv I + W^TH^{-1}V
\end{aligned}
$$

$$
\begin{aligned}
(H + VW^T) & \quad \left[H^{-1} - H^{-1}VC^{-1}W^TH^{-1}\right] \\
&= (H + VW^T)H^{-1} - (H + VW^T)H^{-1}VC^{-1}W^TH^{-1} \\
&= I + VW^TH^{-1} - V(I + W^TH^{-1}V)C^{-1}W^TH^{-1} \\
&= I + VW^TH^{-1} - VCC^{-1}W^TH^{-1} \\
&= I
\end{aligned}
$$

Streamfunction-vorticity problem: boundary conditions couple ψ and ω:

$$\left[\begin{array}{cc} (I-\epsilon\Delta)_{\text{int}} & 0_{\text{int}} \\ \hline 0 & \partial_n|_{\text{bdy}} \\ \hline I_{\text{int}} & \Delta_{\text{int}} \\ \hline 0 & |_{\text{bdy}} \end{array}\right] \left[\begin{array}{c} \omega_{\text{int}} \\ \omega_{\text{hi}} \\ \psi_{\text{int}} \\ \psi_{\text{hi}} \end{array}\right] = \left[\begin{array}{c} s_{\text{int}} \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Particular problem: uncoupled (soluble) matrix H

$$\left[\begin{array}{cc} (I-\epsilon\Delta)_{\text{int}} & 0_{\text{int}} \\ \hline |_{\text{bdy}} & 0 \\ \hline I_{\text{int}} & \Delta_{\text{int}} \\ \hline 0 & |_{\text{bdy}} \end{array}\right] \left[\begin{array}{c} \omega_{\text{int}} \\ \omega_{\text{hi}} \\ \psi_{\text{int}} \\ \psi_{\text{hi}} \end{array}\right] = \left[\begin{array}{c} s_{\text{int}} \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Subtract (Exact Matrix) – (Particular Matrix):

$$\left[\begin{array}{cc} 0 & 0 \\ \hline -|_{\text{bdy}} & \partial_n|_{\text{bdy}} \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ I \\ 0 \\ 0 \end{array}\right] \left[\begin{array}{cc} -|_{\text{bdy}} & \partial_n|_{\text{bdy}} \end{array}\right] \equiv VW^T$$

Preprocessing to construct C:

$$\partial_n \psi^{\text{hom}}|_{\text{bdy}} \;\leftarrow\; (-\omega + \partial_n \psi)^{\text{hom}}|_{\text{bdy}} \;\leftarrow\; \begin{bmatrix} \psi^{\text{hom}} \\ \\ \omega^{\text{hom}} \\ \\ \end{bmatrix} \;\leftarrow\; \begin{bmatrix} 0 \\ \omega^{\text{hom}}|_{\text{bdy}} \\ 0 \\ 0 \end{bmatrix} \;\leftarrow\; \omega^{\text{hom}}|_{\text{bdy}}$$

$$\boldsymbol{I} \;+\; \qquad \boldsymbol{W}^T \qquad \boldsymbol{H}^{-1} \qquad \boldsymbol{V}$$

One timestep:

$$\begin{bmatrix} \psi^{\text{hom}} \\ \\ \omega^{\text{hom}} \\ \\ \end{bmatrix} \;\leftarrow\; \begin{bmatrix} 0 \\ \omega^{\text{hom}}|_{\text{bdy}} \\ 0 \\ 0 \end{bmatrix} \;\leftarrow\; \omega^{\text{hom}}|_{\text{bdy}} \;\leftarrow\; \partial_n \psi^{\text{part}}|_{\text{bdy}} \;\leftarrow\; \begin{bmatrix} \psi^{\text{part}} \\ \\ \omega^{\text{part}} \\ \\ \end{bmatrix} \;\leftarrow\; \begin{bmatrix} s_{\text{int}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{H}^{-1} \qquad \boldsymbol{V} \qquad \boldsymbol{C}^{-1} \qquad -\boldsymbol{W}^T \qquad \boldsymbol{H}^{-1}$$

Adding another component (not dimension)

Difficulty comes from independent variables (x, y, z)
not from dependent variables (u, v, w)

$$\mathbf{u} = \nabla \times \psi(x,y)\hat{\mathbf{e}}_z + w(x,y)\hat{\mathbf{e}}_z \quad \Longrightarrow \hat{\mathbf{e}}_z \cdot \mathbf{u} = w$$

$$\nabla \times \mathbf{u} = -\hat{\mathbf{e}}_z \Delta\psi + \hat{\mathbf{e}}_x \partial_y w - \hat{\mathbf{e}}_y \partial_x w \quad \Longrightarrow \hat{\mathbf{e}}_z \cdot \nabla \times \mathbf{u} = -\Delta\psi$$

$$p = p_0 z + p(x,y) \quad \Longrightarrow \hat{\mathbf{e}}_z \cdot \nabla p = p_0$$

$$\Longrightarrow \hat{\mathbf{e}}_z \cdot \nabla \times p = 0$$

Navier-Stokes:

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p - (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}$$

Governing equations from $\hat{\mathbf{e}}_z \cdot$ and $\hat{\mathbf{e}}_z \cdot \nabla \times$

vertical velocity: $$(\partial_t - \nu\Delta)w = -p_0 + \hat{\mathbf{e}}_z \cdot [-(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}]$$

vertical vorticity: $$(\partial_t - \nu\Delta)\Delta\psi = -\hat{\mathbf{e}}_z \cdot \nabla \times [-(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{f}]$$

Called 2D (two dimensions) 3C (three components)

Influence matrix: velocity-pressure formulation

$$(I - \epsilon\Delta)\mathrm{u}^{n+1} = -\Delta t \nabla p^{n+1} + \underbrace{u^n + \Delta t(-(\mathrm{u}\cdot\nabla)\mathrm{u} + f)^n}_{s^n}$$

$$\nabla \cdot \mathrm{u} = 0$$

$$\mathrm{u}|_{\mathrm{bdy}} = 0$$

Divergence $\Longrightarrow$ Pressure-Poisson Equation (PPE):

$$\nabla \cdot (I - \epsilon\Delta)\mathrm{u}^{n+1} = -\Delta t \nabla \cdot \nabla p^{n+1} + \nabla \cdot s^n \qquad \textbf{where} \qquad \epsilon \equiv \nu\Delta t$$

$$\Longrightarrow \quad \Delta p^{n+1} = \nabla \cdot s^n / \Delta t$$

Correct boundary condition for PPE is:

$$(\nabla \cdot \mathrm{u})|_{\mathrm{bdy}} = 0$$

Calculate homogeneous solutions or Green's functions:

$$\Delta p_i^{\mathrm{hom}} = 0$$

$$p_i^{\mathrm{hom}}(\mathrm{x}_j) = \delta_{ij} \qquad \mathrm{x}_j \in \textbf{boundary}$$

$$(I - \epsilon\Delta)\mathrm{u}_i^{\mathrm{hom}} = -\Delta t \nabla p_i^{\mathrm{hom}}$$

$$\mathrm{u}_i^{\mathrm{hom}}(\mathrm{x}_j) = 0 \qquad \mathrm{x}_j \in \textbf{boundary}$$

Calculate capacitance or influence matrix:

$$C_{ji} \equiv \nabla \cdot \mathbf{u}_i^{\mathrm{hom}}(\mathbf{x}_j) \qquad \mathbf{x}_j \in \textbf{boundary}$$

Particular solution:

$$\begin{aligned}
\Delta p^{\mathrm{part}} &= \nabla \cdot s/\Delta t \\
p^{\mathrm{part}}(\mathbf{x}_j) &= 0 && \mathbf{x}_j \in \textbf{boundary} \\
(I - \epsilon\Delta)\mathbf{u}^{\mathrm{part}} &= s - \Delta t(\nabla p^{\mathrm{part}}) \\
\mathbf{u}_i^{\mathrm{hom}}(\mathbf{x}_j) &= 0 && \mathbf{x}_j \in \textbf{boundary}
\end{aligned}$$

Superposition:

$$\mathbf{u} = \mathbf{u}^{\mathrm{part}} + \sum_i \alpha_i \mathbf{u}_i^{\mathrm{hom}}$$

$$(\nabla \cdot \mathbf{u})(\mathbf{x}_j) = u^{\mathrm{part}}(\mathbf{x}_j) + \sum_i \alpha_i C_{ji}$$

Must solve:

$$\sum_i \alpha_i C_{ji} = -u^{\mathrm{part}}(\mathbf{x}_j)$$

Scaling:

C decouples into blocks by parity or Fourier mode.
Requirements for one timestep are quadratic in:

line	rectangle	3D rectangle
2	$2(N_x + N_y)$	$2(N_xN_y + N_yN_z + N_xN_z)$

but linear in number of points in any periodic direction

$\Longrightarrow$ Very fast for bounded-periodic or bounded-periodic-periodic
OK for bounded-bounded-periodic
Too large and expensive for bounded-bounded-bounded

$\Longrightarrow$ Pressure boundary conditions $\partial_n p =$ RHS of NS often used in 3D
$\Longrightarrow \nabla \cdot u \sim (\Delta t)^n$ with $n = 1$ or 2
Acceptable results obtained nonetheless in most circumstances

Similar problems with 3D vorticity formulation

$$\mathbf{u} = \nabla \times A$$

Stokes' Theorem in 2D or 3D:

$$[\text{N-S without grad}] = \nabla p \quad \Leftrightarrow \quad \mathrm{g} \equiv \nabla \times [\text{N-S without grad}] = 0$$

Reduce to two scalar equations via potential theory:

$$\mathrm{g} = 0 \quad \Leftrightarrow \quad \begin{cases} \hat{\mathrm{e}} \cdot \mathrm{g} = 0 & \text{in } \Omega \\ \hat{\mathrm{e}} \cdot \nabla \times \mathrm{g} = 0 & \text{in } \Omega \\ \nabla \cdot \mathrm{g} = 0 & \text{in } \Omega \\ \mathrm{n} \cdot \mathrm{g} = 0 & \text{on } \partial\Omega_h \end{cases}$$

$\hat{\mathrm{e}}$ = unit vector, $\partial\Omega_h$ = boundary of slices Ω_h of $\Omega \perp \hat{\mathrm{e}}$, n = normal to boundary

Two cases without any $\partial\Omega_h$:

$\hat{\mathrm{e}} = \hat{\mathrm{e}}_{\mathrm{z}}$: domain between two infinite horizontal plates (= torus)

$\hat{\mathrm{e}} = \hat{\mathrm{e}}_{\mathrm{r}}$: sphere

Fields which are divergence-free by construction:

Poloidal-Toroidal Decomposition

$$\mathbf{u} = \boldsymbol{\nabla} \times \psi \hat{\mathrm{e}} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi \hat{\mathrm{e}}$$

Cartesian/torus $\hat{\mathrm{e}} = \hat{\mathrm{e}}_{\mathrm{z}}$ **(if mean field must include** $b_x(z)\hat{\mathrm{e}}_{\mathrm{x}} + b_y(z)\hat{\mathrm{e}}_{\mathrm{y}}$**)**

$$\boldsymbol{\nabla} \times \psi \hat{\mathrm{e}}_{\mathrm{z}} = \hat{\mathrm{e}}_{\mathrm{x}} \partial_y \psi - \hat{\mathrm{e}}_{\mathrm{y}} \partial_x \psi$$

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi \hat{\mathrm{e}}_{\mathrm{z}} &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \phi \hat{\mathrm{e}}_{\mathrm{z}}) && - \boldsymbol{\Delta}\phi\, \hat{\mathrm{e}}_{\mathrm{z}} \\
&= \boldsymbol{\nabla} \partial_z \phi && - \boldsymbol{\Delta}\phi\, \hat{\mathrm{e}}_{\mathrm{z}} \\
&= (\hat{\mathrm{e}}_{\mathrm{x}}\, \partial_x + \hat{\mathrm{e}}_{\mathrm{y}}\, \partial_y + \hat{\mathrm{e}}_{\mathrm{z}}\, \partial_z) \partial_z \phi && - (\partial_x^2 + \partial_y^2 + \partial_z^2)\phi\, \hat{\mathrm{e}}_{\mathrm{z}} \\
&= (\hat{\mathrm{e}}_{\mathrm{x}}\, \partial_x + \hat{\mathrm{e}}_{\mathrm{y}}\, \partial_y) \partial_z \phi && - \hat{\mathrm{e}}_{\mathrm{z}}\, (\partial_x^2 + \partial_y^2)\phi \\
&\equiv \boldsymbol{\nabla}_h \partial_z \phi && - \hat{\mathrm{e}}_{\mathrm{z}}\, \boldsymbol{\Delta}_h \phi
\end{aligned}$$

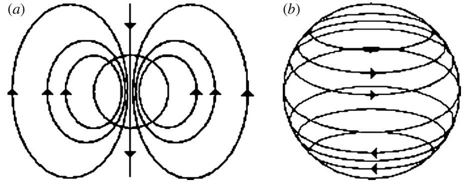
Spherical $\hat{\mathrm{e}} = \hat{\mathrm{e}}_{\mathrm{r}}$

$$\boldsymbol{\nabla} \times \psi \hat{\mathrm{e}}_{\mathrm{r}} = \frac{1}{r \sin\theta} \partial_\varphi \psi \hat{\mathrm{e}}_\theta - \frac{1}{r} \partial_\theta \psi \hat{\mathrm{e}}_\phi$$

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \phi \hat{\mathrm{e}}_{\mathrm{r}} &= \left(\frac{\hat{\mathrm{e}}_\theta}{r} \partial_\theta + \frac{\hat{\mathrm{e}}_\phi}{r \sin\theta} \partial_\varphi \right) \partial_r \phi && - \frac{\hat{\mathrm{e}}_{\mathrm{r}}}{r^2 \sin\theta} \left(\partial_\theta^2 \sin\theta + \frac{1}{\sin\theta} \partial_\varphi^2 \right) \phi \\
&= \boldsymbol{\nabla}_h \partial_r \phi && - \hat{\mathrm{e}}_{\mathrm{r}} \boldsymbol{\Delta}_h \phi
\end{aligned}$$

Poloidal-Toroidal Decomposition

Poloidal **Toroidal**



$$\mathbf{u} = \overbrace{\nabla \times \psi \hat{\mathbf{e}}}^{\textbf{Toroidal}} + \overbrace{\nabla \times \nabla \times \phi \hat{\mathbf{e}}}^{\textbf{Poloidal}}$$

Navier-Stokes:

$$(\partial_t - \nu \Delta)\mathbf{u} = -\nabla p + \mathbf{s}$$

Take $\hat{\mathbf{e}} \cdot \nabla \times$ and $\hat{\mathbf{e}} \cdot \nabla \times \nabla \times$ of N-S:

	Order	
	Vertical ($\hat{\mathbf{e}}$)	**Horizontal (h)**
$(\partial_t - \nu \Delta)\Delta_h \psi = -\hat{\mathbf{e}} \cdot \nabla \times \mathbf{s}$	2	4
$(\partial_t - \nu \Delta)\Delta \Delta_h \phi = \hat{\mathbf{e}} \cdot \nabla \times \nabla \times \mathbf{s}$	4	6
Conditions at each boundary point	3	5
	3 velocity	**3 velocity $+2$ more**

No horizontal boundary conditions needed for:

Domain between two infinite horizontal plates:

$$\mathrm{u}(x,y,z) = \sum_{k_x,k_y} \nabla \times \left(\psi_{k_x,k_y}(z)\hat{\mathrm{e}}_\mathrm{z} + \nabla \times \phi_{k_x,k_y}(z)\hat{\mathrm{e}}_\mathrm{z}\right) \; e^{ik_x x} e^{ik_y y}$$

$$\left[\partial_t - \nu\left(\partial_z^2 - (k_x^2 + k_y^2)\right)\right] \qquad \psi_{k_x,k_y} = -(\hat{\mathrm{e}} \cdot \nabla \times \mathrm{s})_{k_x,k_y}$$

$$\left[\partial_t - \nu\left(\partial_z^2 - (k_x^2 + k_y^2)\right)\right](-(k_x^2 + k_y^2))\,\phi_{k_x,k_y} = (\hat{\mathrm{e}} \cdot \nabla \times \nabla \times \mathrm{s})_{k_x,k_y}$$

Boundary conditions at $z = z_1$ and $z = z_2$

Sphere: $u(r,\theta,\phi) = \sum_{\ell,m} \left(\nabla \times \psi_{\ell,m}(r)\hat{\mathrm{e}}_\mathrm{r} + \nabla \times \phi_{\ell,m}(r)\hat{\mathrm{e}}_\mathrm{r}\right) \; Y_\ell^m(\theta,\phi)$

$$\left[\partial_t - \nu\left(\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{\ell(\ell+1)}{r^2}\right)\right] \qquad \psi_{\ell,m} = -(\hat{\mathrm{e}} \cdot \nabla \times \mathrm{s})_{\ell,m}$$

$$\left[\partial_t - \nu\left(\frac{1}{r^2}\partial_r r^2 \partial_r - \frac{\ell(\ell+1)}{r^2}\right)\right]\left(\frac{-\ell(\ell+1)}{r^2}\right)\phi_{\ell,m} = (\hat{\mathrm{e}} \cdot \nabla \times \nabla \times \mathrm{s})_{\ell,m}$$

Boundary at $r = r_1$ (or regularity conditions at $r = 0$) and $r = r_2$

Magnetohydrodynamics

$$(\partial_t - \nu\Delta)\mathrm{u} = -\nabla\left(p + \tfrac{1}{2}B^2\right) - (\mathrm{u}\cdot\nabla \mathrm{u}) + (B\cdot\nabla)B$$

$\nabla\cdot\mathrm{u} = 0$ **Incompressibility**

$(\partial_t - \eta\Delta)B = \nabla\times(\mathrm{u}\times B)$ **η = magnetic diffusivity**

$\nabla\cdot B = 0$ **No magnetic monopoles**

Equation	Nonlinear term		Gradient term	
Navier-Stokes	$(\mathrm{u}\cdot\nabla)\mathrm{u}$	creates div	$\nabla(p + \frac{1}{2}B^2)$	removes div
Induction	$\nabla\times(\mathrm{u}\times B)$	div-free	none	

Stokes' Theorem in 2D or 3D:

[N-S without grad] $= \nabla\left(p + \tfrac{1}{2}B^2\right) \quad \Leftrightarrow \quad \mathrm{g} \equiv \nabla\times$ [N-S without grad] $= 0$

[Induction eqn] $= 0 \quad \Leftrightarrow \quad \mathrm{g} \equiv$ [Induction eqn] $= 0$

Reduce to two scalar equations via potential theory:

$$\mathrm{g} = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{llll} \hat{\mathrm{e}} \cdot \mathrm{g} = 0 & \textbf{in } \Omega & \nabla \cdot \mathrm{g} = 0 & \textbf{in } \Omega \\ \hat{\mathrm{e}} \cdot \nabla \times \mathrm{g} = 0 & \textbf{in } \Omega & \mathrm{n} \cdot \mathrm{g} = 0 & \textbf{on } \partial\Omega_h \end{array} \right\}$$

Poloidal-Toroidal Decomposition

$$B = \nabla \times \psi_B \hat{\mathrm{e}} + \nabla \times \nabla \times \phi_B \hat{\mathrm{e}}$$

Induction equation

$$(\partial_t - \eta\Delta) B = \nabla \times (\mathrm{u} \times B) \equiv s$$

Take $\hat{\mathrm{e}} \cdot \nabla \times$ and $\hat{\mathrm{e}} \cdot$ of Induction:

	Order	
	Vertical ($\hat{\mathrm{e}}$)	Horizontal (h)
$(\partial_t - \eta\Delta)\Delta_h \psi_B = -\hat{\mathrm{e}} \cdot \nabla \times \mathrm{s}$	2	4
$(\partial_t - \eta\Delta)\Delta_h \phi_B = \hat{\mathrm{e}} \cdot \mathrm{s}$	2	4
Conditions at each boundary point	2	4