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# Thermosolutal and binary fluid convection as a $2 \times 2$ matrix problem

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## Abstract

We describe an interpretation of convection in binary fluid mixtures as a superposition of thermal and solutal problems, with coupling due to advection and proportional to the separation parameter  $S$ . Many of the properties of binary fluid convection are then consequences of generic properties of  $2 \times 2$  matrices. The eigenvalues of  $2 \times 2$  matrices varying continuously with a parameter  $r$  undergo either *avoided crossing* or *complex coalescence*, depending on the sign of the coupling (product of off-diagonal terms). We first consider the matrix governing the stability of the conductive state. When the thermal and solutal gradients act in concert ( $S > 0$ , avoided crossing), the growth rates of perturbations remain real and of either thermal or solutal type. In contrast, when the thermal and solutal gradients are of opposite signs ( $S < 0$ , complex coalescence), the growth rates become complex and are of mixed type. Surprisingly, the kinetic energy of nonlinear steady states is also governed by an eigenvalue problem very similar to that governing the growth rates. More precisely, there is a quantitative analogy between the growth rates of the linear stability problem for infinite Prandtl number and the amplitudes of steady states of the minimal five-variable Veronis model for arbitrary Prandtl number. For positive  $S$ , avoided crossing leads to a distinction between low-amplitude solutal and high-amplitude thermal regimes. For negative  $S$ , the transition between real and complex eigenvalues leads to the creation of branches of finite amplitude, i.e. to saddle-node bifurcations. The codimension-two point at which the saddle-node bifurcations disappear, leading to a transition from subcritical to supercritical pitchfork bifurcations, is exactly analogous to the Bogdanov codimension-two point at which the Hopf bifurcations disappear in the linear problem. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Convection due to two competing or cooperating effects is realized in a number of different physical systems. In the thermosolutal or thermohaline problem, vertical thermal and concentration gradients are both externally imposed. In convection in binary fluids with Soret effect, only the temperature gradient is imposed, but cross diffusion induces a concentration gradient with similar properties. An electrically conducting magnetic fluid may be subjected to a vertical or a horizontal magnetic field, the fluid layer may be rotated, or two solutes may be introduced.

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In 1960s and 1970s, Veronis [1,4] and other researchers [2,3,5–12] recognized the variety of behavior manifested by double-diffusive convection; comprehensive texts and reviews [13–15] were written on the subject. One of the reasons for studying these double-diffusive systems is that all display a common basic set of phenomena. Both stationary and oscillatory instabilities occur, in other words pitchfork and Hopf bifurcations, the curves intersecting at parameter combinations which can be analytically calculated, at least approximately. In the 1980s, double-diffusive convection became the paradigm example in a renaissance in the study of bifurcation theory and dynamical systems, as attention was focused on precisely such intersections, re-interpreted as codimension-two points by Knobloch and Proctor [16] and then by Brand et al. [20].

There followed a divergence of efforts, roughly speaking between, on the one hand, detailed and rigorous mathematical analysis of the temporal complexity of thermosolutal convection and related problems in a small container, e.g. [17,18,26,46,47,58], and, on the other hand, realistic and physical exploration of the spatial complexity of the Soret and other problems in a large container for negative [19,21–23,25,28–36,39,42,43,48–57] and positive [24,27,29,37,38,40–42,44,45,51,53,58–60] values of the separation ratio. The first line of research has led to an understanding of the mechanisms producing global bifurcations, period-doubling, and chaos in two-dimensional convection in a confined geometry. The second line of research has resulted in bifurcation diagrams detailing the transitions in an extended periodic geometry, primarily between patterns of different spatio-temporal symmetry. For negative separation ratios, transitions occur between steady convection, standing waves, and traveling waves. For positive separation ratio, the transitions are between patterns of rolls and squares, and also between weakly and strongly convective regimes.

In the present paper, we will use an analytically tractable model of thermosolutal convection to examine the consequences of a simple idea. Consider a  $2 \times 2$  real matrix

$$\begin{pmatrix} \sigma_T & \beta \\ \gamma & \sigma_C \end{pmatrix}. \quad (1.1)$$

Its eigenvalues are

$$\sigma_{\pm} = \left( \frac{\sigma_T + \sigma_C}{2} \right) \pm \sqrt{\left( \frac{\sigma_T - \sigma_C}{2} \right)^2 + \beta\gamma}. \quad (1.2)$$

Each of the elements  $\sigma_T$ ,  $\sigma_C$ ,  $\beta$  and  $\gamma$  depend on parameters  $r$  (scaled Rayleigh number) and  $S$  (separation parameter). We assume that, for each  $S$ , there is a value  $r_{\text{int}}$  of  $r$  at which  $\sigma_T$  and  $\sigma_C$  coincide. We are interested in the behavior of the eigenvalues  $\sigma_{\pm}$  in this vicinity. There are three possibilities, depending on the sign of  $\beta\gamma$  at  $r_{\text{int}}$ . If  $\beta\gamma < 0$ , then  $\sigma_+$  and  $\sigma_-$  coalesce into a complex conjugate pair in the vicinity of  $r_{\text{int}}$ . If  $\beta\gamma = 0$ , then  $\sigma_+$  and  $\sigma_-$  intersect transversely. If  $\beta\gamma > 0$ , then  $\sigma_+$  and  $\sigma_-$  are real and  $\sigma_+ > \sigma_-$ , with a change of slope at  $r_{\text{int}}$ . This is the phenomenon known as avoided crossing. These three possibilities, shown in Fig. 1, correspond to the three scenarios observed when  $S < 0$ ,  $S = 0$ , and  $S > 0$ , respectively. Moreover, we claim that this interpretation applies to the nonlinear steady states as well as to the linear stability problem. The remainder of the paper is devoted to making these statements precise.

The paper is organized as follows. Section 2 sets out the idealized free-slip thermosolutal problem that we will study. Although our calculations are limited to this analytically tractable case, we will present evidence throughout the paper that our conclusions may apply to double-diffusive convection problems in general. Section 3 gives the classic linear stability analysis and the bifurcations undergone by the system. The behavior of the eigenvalues is interpreted in the framework of avoided crossing vs. complex coalescence and also as primarily thermal vs. solutal. Section 4 analyzes the minimal five-mode Veronis model. We show that the steady states of the Veronis model obey a two-variable eigenvalue problem entirely analogous to the linear stability

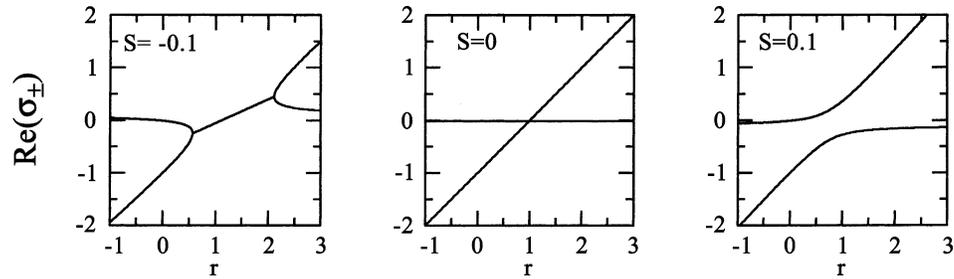


Fig. 1. Eigenvalues of thermosolutal convection as a function of scaled Rayleigh number  $r$  for different signs of separation parameter  $S$ .  $S = -0.1$ : formation of complex conjugate pair.  $S = 0$ : eigenvalues cross transversely.  $S = 0.1$ : avoided crossing.

problem, with the energy playing the role of the eigenvalue. We present the consequences of this analogy, particularly for the saddle-node bifurcations and codimension-two points that occur for negative  $S$ . For positive  $S$ , the transition between Soret and Rayleigh regimes is analyzed in detail as exemplifying the phenomenon of avoided crossing. Section 5 sets out the simplest time-dependent system reproducing the primary bifurcations and steady states and, additionally, standing waves terminating in a heteroclinic orbit. However, the traveling waves that have been the subject of so much interest in double-diffusive convection are not accessible by our simplified approach; nor is the competition between patterns of rolls and squares. Section 6 is a brief summary and discussion.

## 2. Thermosolutal problem

We study the thermosolutal problem, in which thermal and concentration gradients are imposed, instead of the more widely studied and experimentally accessible Soret-driven problem, in which the concentration gradient results from the temperature gradient, because its formulation is slightly simpler. The basic phenomena which we will investigate occur in both problems, which have in fact been shown by Knobloch [12] to be formally equivalent for the idealized boundary conditions, we consider here a two-dimensional geometry with free-slip upper and lower boundaries and imposed horizontal periodicity. Although all our calculations concern the idealized thermosolutal problem, we will also refer to results in the literature derived using rigid boundary conditions or concerning the Soret problem with various boundary conditions. We will not systematically refer to the related problems of rotating or magnetoconvection (although these are also discussed in [13–16,18,46,47]) nor to the equally vast experimental literature (for references, see, e.g. [49,51,52,57]).

In the thermosolutal problem, the flow is driven by vertical thermal and concentration differences  $\Delta T$  and  $\Delta C$  imposed across a layer of height  $h$ . In the Boussinesq approximation, the density is assumed constant except in the buoyancy term, where it is taken to vary linearly with temperature and concentration, with thermal and solutal expansion coefficients  $\rho_T$  and  $\rho_C$ . There exists a steady motionless conductive solution, with temperature and concentration profiles depending linearly on the vertical coordinate  $z$ .

The stability of the conductive solution and the subsequent evolution of the system is determined by four nondimensional parameters. The three diffusivities — the thermal diffusivity  $\kappa_T$ , the momentum diffusivity or kinematic viscosity  $\nu$ , and the solute diffusivity  $\kappa_C$  — are described by two nondimensional ratios, chosen to be the Prandtl number  $P = \nu/\kappa_T$  and the Lewis number  $L = \kappa_C/\kappa_T$ . For the linear problem, the Prandtl number is taken to be large;  $P = O(1)$  or  $O(10)$  suffices. The Lewis number is varied, but is considered to be small

and, for the figures, will be fixed at the frequently studied value of  $L = 0.01$ . The thermal Rayleigh number  $R \equiv g\rho_T \Delta T h^3 / \nu \kappa_T$  measures the imposed temperature gradient. The concentration gradient can be specified by an analogous solutal Rayleigh number, or by the separation parameter  $S \equiv (\rho_C \Delta C) / (\rho_T \Delta T)$ , which may be either positive or negative. Length, temperature, concentration, and time are nondimensionalized by  $h$ ,  $\Delta T$ ,  $\Delta C$ , and  $h^2 / \kappa_T$ .

It will be convenient to write the equations governing the evolution of the solution in terms of both the vertical velocity  $\hat{w}$  and a velocity streamfunction  $\hat{\phi}$ , and temperature and concentration deviations  $\hat{T}$  and  $\hat{C}$  from their conductive profiles:

$$\partial_t \hat{T} = \nabla^2 \hat{T} + \hat{w} + \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{T}), \quad (2.1a)$$

$$\partial_t \hat{C} = L \nabla^2 \hat{C} + \hat{w} + \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{C}), \quad (2.1b)$$

$$\partial_t \nabla^2 \hat{w} = P \nabla^4 \hat{w} + PR \partial_x^2 (\hat{T} + S \hat{C}) + \mathbf{e}_y \cdot \partial_x (\nabla \hat{\phi} \times \nabla \nabla^2 \hat{\phi}), \quad (2.1c)$$

$$\hat{w} = \partial_x \hat{\phi}. \quad (2.1d)$$

The boundary conditions are

$$\hat{T} = \hat{C} = \hat{\phi} = \partial_x^2 \hat{\phi} = \hat{w} = \partial_x^2 \hat{w} \quad \text{at } z = 0, 1, \quad (2.2a)$$

$$\hat{T}, \hat{C}, \hat{\phi}, \hat{w} \quad \frac{2\pi}{k} \text{ periodic in } x, \quad (2.2b)$$

where  $k$  may be fixed arbitrarily, or at the well-known value  $k_{\text{crit}} = \pi / \sqrt{2}$  which minimizes the convection threshold in the case of free-slip boundaries.

Following, e.g. [5,12], we additionally scale time and, consequently, velocity, by

$$q^2 \equiv k^2 + \pi^2 \quad (2.3a)$$

and introduce the geometrically scaled Rayleigh number

$$r \equiv \frac{Rk^2}{q^6}. \quad (2.3b)$$

Eq. (2.1) become

$$\partial_t \hat{T} = q^{-2} \nabla^2 \hat{T} + \hat{w} + \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{T}), \quad (2.4a)$$

$$\partial_t \hat{C} = Lq^{-2} \nabla^2 \hat{C} + \hat{w} + \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{C}), \quad (2.4b)$$

$$\partial_t q^{-2} \nabla^2 \hat{w} = Pq^{-4} \nabla^4 \hat{w} + Prk^{-2} \partial_x^2 (\hat{T} + S \hat{C}) + q^{-2} \mathbf{e}_y \cdot \partial_x (\nabla \hat{\phi} \times \nabla \nabla^2 \hat{\phi}), \quad (2.4c)$$

$$\hat{w} = \partial_x \hat{\phi}. \quad (2.4d)$$

### 3. Linear analysis

#### 3.1. Linear stability problem

We begin by discussing the linear stability of the conductive solution, governed by

$$\partial_t \hat{T} = q^{-2} \nabla^2 \hat{T} + \hat{w}, \quad (3.1a)$$

$$\partial_t \hat{C} = Lq^{-2} \nabla^2 \hat{C} + \hat{w}, \quad (3.1b)$$

$$\partial_t q^{-2} \nabla^2 \hat{w} = Pq^{-4} \nabla^4 \hat{w} + Prk^{-2} \partial_x^2 (\hat{T} + S\hat{C}). \quad (3.1c)$$

Solutions to (3.1) with boundary conditions (2.2) are of the form

$$\begin{pmatrix} \hat{T}(x, z, t) \\ \hat{C}(x, z, t) \\ \hat{w}(x, z, t) \end{pmatrix} = \begin{pmatrix} T(t) \\ C(t) \\ w(t) \end{pmatrix} \cos(kx) \sin(\pi z). \quad (3.2a)$$

$$(3.2b)$$

$$(3.2c)$$

The time dependence of the linear system (3.1) is

$$\begin{pmatrix} T(t) \\ C(t) \\ w(t) \end{pmatrix} = \exp(\lambda t) \begin{pmatrix} T \\ C \\ w \end{pmatrix}, \quad (3.3)$$

where  $T$ ,  $C$ ,  $w$  are scalars denoting the amplitudes of the corresponding fields. Substituting (3.2), (3.3) and (2.3a) into (3.1) yields the eigenvalue problem

$$\lambda \begin{pmatrix} T \\ C \\ w \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -L & 1 \\ Pr & PSr & -P \end{pmatrix} \begin{pmatrix} T \\ C \\ w \end{pmatrix}. \quad (3.4)$$

We simplify further by assuming that the Prandtl number  $P$  is infinite. Most quantitative results of interest to us depend only very weakly on  $P$  as long as  $P \gtrsim 1$ , e.g. [49]. (Another approach to reducing the  $3 \times 3$  problem to a  $2 \times 2$  problem, which does not rely on  $P$  large, is given in Appendix A.) The velocity amplitude  $w$  is then related to  $T$  and  $C$  by the algebraic equation

$$w = r(T + SC). \quad (3.5)$$

After eliminating the velocity via (3.5), the eigenvalue problem (3.4) becomes

$$\lambda \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r-1 & rS \\ r & rS-L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}. \quad (3.6)$$

The matrix

$$\begin{pmatrix} r-1 & rS \\ r & rS-L \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -L \end{pmatrix} + r \begin{pmatrix} 1 & S \\ 1 & S \end{pmatrix}, \quad (3.7)$$

$$M = M_0 + rM_1$$

is of the form discussed in Section 1. The matrix  $M_0$  describes diffusion and the matrix  $M_1$  describes advection via (3.5). The diagonal elements of  $M$ :

$$\sigma_T \equiv r - 1, \quad (3.8a)$$

$$\sigma_C \equiv Sr - L \quad (3.8b)$$

can be viewed as the eigenvalues of a pure thermal and a pure solutal problem, with a coupling of

$$\beta\gamma = Sr^2. \quad (3.8c)$$

The pure thermal convection problem for infinite Prandtl number is obtained from (3.6) by setting  $S = 0$ . Bifurcation to steady thermal convection occurs when  $\sigma_T = 0$ , i.e. at

$$r = r_T = 1. \quad (3.9a)$$

The pure solutal problem corresponds to convection driven exclusively by concentration gradients, i.e. incomplete mixing of the two species in the binary fluid. In the thermosolutal problem, the imposed concentration gradient is set to the value  $Sr$ , while in the Soret problem, the Soret coefficient  $S$  is a property of the fluid. Although we are studying the thermosolutal problem, it will be convenient for us to consider  $S$  as fixed and to vary the single control parameter  $r$  for all three problems: thermal, solutal, and thermosolutal. (In particular, all three problems have the same critical wavenumber for the idealized boundary conditions.) We therefore interpret (3.8b) as meaning that  $S$  is fixed and the onset of convection occurs at

$$r = r_C \equiv \frac{L}{S}. \quad (3.9b)$$

There are two cases, depending on the sign of  $S$  ( $L$ , the ratio of two diffusion coefficients, is always positive). If  $S > 0$ , then  $\sigma_C(r)$  has positive slope and the conductive state is unstable for  $r > L/S > 0$ . If  $S < 0$ , then  $\sigma_C(r)$  has negative slope and the conductive state is unstable for  $r < L/S < 0$ .

### 3.2. Bifurcations

We now return to the coupled thermosolutal system (3.6). We begin by giving some exact results concerning the eigenvalues and bifurcations of (3.6). Most of these results are well known, but we derive them here to illustrate our geometric interpretation and to prepare the analogy with the nonlinear problem of Section 4. The results are summarized in the two large Figs. 2 and 3.

The uncoupled thermal and solutal eigenvalues (3.8a) and (3.8b) intersect at

$$r_{\text{int}} = \frac{1 - L}{1 - S}. \quad (3.10)$$

The behavior of the thermosolutal eigenvalues of (3.6) near  $r_{\text{int}}$  is determined by the sign of the coupling strength (product of the off-diagonal terms)  $Sr^2$ , which is in turn determined by the sign of  $S$ . For  $S$  positive, the eigenvalues remain real for all  $r$ , with avoided crossing near  $r_{\text{int}}$ . For  $S$  negative, the eigenvalues form a complex conjugate pair over an interval surrounding  $r_{\text{int}}$ .

The eigenvalues of (3.6) are

$$\lambda_{\pm} = \frac{\sigma_T + \sigma_C}{2} \pm \sqrt{\left(\frac{\sigma_T - \sigma_C}{2}\right)^2 + Sr^2} \quad (3.11a)$$

$$= \frac{1}{2}[(1 + S)r - (1 + L)] \pm \frac{1}{2}\sqrt{(1 + S)^2 r^2 - 2(1 - S)(1 - L)r + (1 - L)^2} \quad (3.11b)$$

$$\equiv f(r) \pm \sqrt{g(r)}. \quad (3.11c)$$

The linear function  $f(r)$  and the quadratic function  $g(r)$  are half the trace and a quarter of the discriminant, respectively, of the matrix  $M$  of (3.7). The qualitative behavior of (3.11) depends on whether  $g$  is positive for all  $r$  ( $S > 0$ ) or of both signs ( $S < 0$ ) and on whether  $g$  is a linear ( $S = -1$ ) or a quadratic function of  $r$  ( $S \neq -1$ ). We write  $\lambda_{\pm} = \sigma_{\pm} \pm i\omega$  to denote the real and imaginary parts of  $\lambda$ , with  $\sigma_+ = \sigma_-$  if  $\omega \neq 0$ .

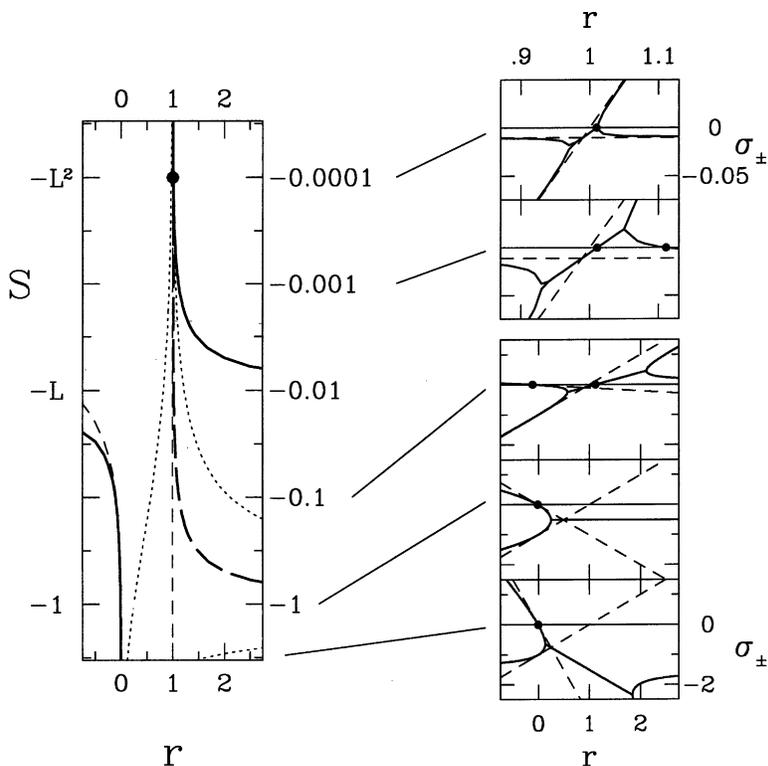


Fig. 2. Behavior of thermosolutal eigenvalues for negative  $S$ . Left: thresholds for negative values of  $S$  plotted on a logarithmic scale. Solid curves show the thresholds  $r_{PF} = L/(L + S)$  of steady bifurcations. Heavy long-dashed curve indicates the thresholds  $r_H = (1 + L)/(1 + S)$  of Hopf bifurcations. This curve appears from  $r = \infty$  at  $S = -1$  and disappears by meeting the steady bifurcation curve in a codimension-two point (Bogdanov bifurcation) at  $S_* = -L^2$ ,  $r_* = 1/(1 - L)$ , indicated by a heavy dot. Between the two dotted curves  $r = r_{\pm}$ , the eigenvalues are complex; the right boundary  $r = r_+$  goes to infinity at  $S = -1$ . Light dashed curves show the thresholds  $r_T = 1$  and  $r_C = L/S$  of the pure thermal and solutal problems, which are within 0.1 of  $r_{PF}$  for  $S \gtrsim -0.1L = 0.001$  and  $S \lesssim -L/\sqrt{0.1} = -0.03$ , respectively. Right: real part  $\sigma_{\pm}$  of the eigenvalues of the thermosolutal problem as a function of  $r$  for representative negative values of  $S$ . Straight segments show the real part of complex conjugate pairs of eigenvalues. Dots represent bifurcations. Dashed lines are the eigenvalues  $\sigma_T$  and  $\sigma_C$  of the pure thermal and solutal problems, with slopes 1 and  $S$ , respectively and intersection point  $r_{int}$ . For  $S = -3$ , representing  $S < -1$ , a steady bifurcation with  $\sigma_+ = 0$  occurs at  $r_{PF} = -0.00334 \approx -0.00333 = r_C$ . Complex eigenvalues all have negative real part. For the limiting case  $S = -1$ , the range of complex eigenvalues extends from  $r_- = \frac{1}{4}(1 - L)$  to  $r_+ = \infty$ . All have  $\sigma = f = -(1 + L)$  with  $\omega = \sqrt{-g} \rightarrow \infty$  as  $r \rightarrow \infty$ . For  $S = -0.1$ , representing  $-1 < S < -L$ , there is both a Hopf bifurcation at  $r_H = 1.12$  and a steady bifurcation at  $r_{PF} = -0.11$ . For the limiting case  $S = -L = -0.01$  (not shown), the pitchfork bifurcation has disappeared to  $+\infty$ . For  $S = -0.001$ , representing  $-L < S < -L^2$ , the pitchfork bifurcation has reappeared from  $-\infty$ , with  $\sigma_- = 0$  at  $r_{PF} = 1.11$ . The limiting case  $S = -L^2 = -0.0001$  is the codimension-two point. Note change of scale between upper two and lower three diagrams.

Appendix B derives properties of (3.11) based on the matrices  $M_0$  and  $M_1$  in (3.7). Assuming that  $S \neq -1$ , then the discriminant  $\text{Disc}_1 = (1 + S)^2$  of matrix  $M_1$  is positive, and so the real parts  $\sigma_{\pm}(r)$  of (3.11) describe a hyperbola given by

$$\left( \sigma + \frac{S + L}{1 + S} - (1 + S) \left( r - \frac{(1 - L)(1 - S)}{(1 + S)^2} \right) \right) \left( \sigma + \frac{S + L}{1 + S} \right) = S \frac{(1 - L)^2}{(1 + S)^2} \quad (3.12)$$

and the imaginary parts  $\pm\omega$  describe an ellipse given by

$$\omega^2 + \frac{(1 + S)^2}{4} \left( r - \frac{(1 - L)(1 - S)}{(1 + S)^2} \right)^2 = -S \frac{(1 - L)^2}{(1 + S)^2}. \quad (3.13)$$

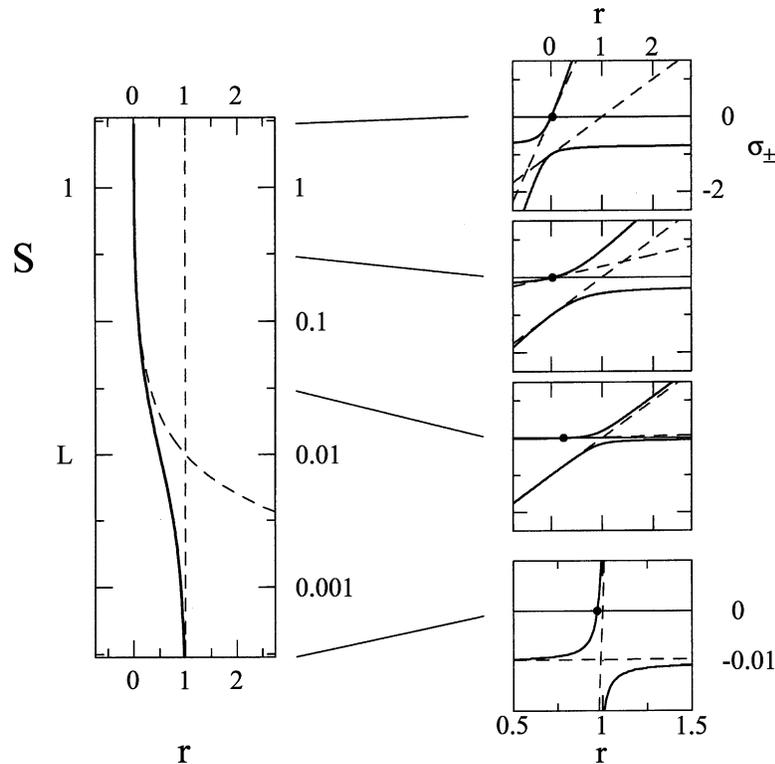


Fig. 3. Behavior of thermosolutal eigenvalues for positive  $S$ . Left: thresholds for positive values of  $S$  plotted on a logarithmic scale. Solid curve shows the threshold  $r_{PF} = L/(L + S)$  of steady bifurcations. Dashed curves show the thresholds  $r_T = 1$  and  $r_C = L/S$  of the pure thermal and solutal problems, which are within 0.1 of  $r_{PF}$  for  $S \lesssim 0.1L = 0.001$  and  $S \gtrsim L/\sqrt{0.1} = 0.03$ , respectively. Right: real part  $\sigma_{\pm}$  of the eigenvalues of the thermosolutal problem as a function of  $r$  for representative positive values of  $S$ . Dashed lines are the eigenvalues  $\sigma_T$  and  $\sigma_C$  of the pure thermal and solutal problems with slopes 1 and  $S$ , respectively and intersection point  $r_{int}$ . Dots represent bifurcations.  $S = 0.0003$  illustrates an extreme case of avoided crossing:  $\sigma_+$  is very close to  $\sigma_C$  for  $r < r_{int}$  and to  $\sigma_T$  for  $r > r_{int}$ . This is in the thermal-dominated regime:  $r_{PF} = 0.97 \approx 1 = r_T$ . As  $S$  increases,  $\sigma_{\pm}$  deviate more from the pure eigenvalues. For  $S = 0.03$ ,  $r_{PF} = 0.25$ , as compared to  $r_T = 1$  and  $r_C = 0.33$ . For  $S = 0.3$ ,  $r_{PF} = 0.032 \approx 0.033 = r_C$ . For  $S = 3$ , the slope of  $\sigma_C$  exceeds that of  $\sigma_T$ . The thermosolutal eigenvalues deviate substantially from the pure eigenvalues, but  $\sigma_+ \approx \sigma_C$  near the bifurcation point. Note change of scale between upper three and lower diagrams.

(In the exceptional case  $S = -1$ , when  $f(r)$  is constant and  $g(r)$  is a linear function, then the curves  $\sigma_{\pm}(r)$  and  $\pm\omega(r)$  are parabolas, see Fig. 2.)

The crucial quantity

$$\Delta \equiv S \frac{(1 - L)^2}{(1 + S)^2} \quad (3.14)$$

on the right-hand side of both (3.12) and (3.13), an invariant for second-degree equations (see Appendix B and [61]), distinguishes between *avoided crossing* and *complex coalescence*. The sign of  $\Delta$  is determined by that of  $S$ . If  $S > 0$ , the eigenvalues remain real: the branches  $\sigma_+$  and  $\sigma_-$  remain distinct and continuous over all values of  $r$ . If  $S < 0$ , the branches  $\sigma_{\pm}$  coalesce and there exists an interval of  $r$  over which the eigenvalues are complex.  $S$  also plays a role in determining parameters in (3.12) and (3.13) other than  $\Delta$ , e.g. the values  $(r_{mid}, \sigma_{mid})$  giving the intersection point of the asymptotes of the hyperbola or the center of the ellipse. But it is the dependence of  $\Delta$  on  $S$  which leads to the most striking results.

We now discuss each of (3.12) and (3.13) in turn. Expression (3.12) can be inverted as follows:

$$r = \frac{\sigma_{\pm}^2 + \sigma_{\pm}(1+L) + L}{\sigma_{\pm}(1+S) + (S+L)}. \quad (3.15)$$

Eq. (3.15) gives  $r$  as a single-valued function of  $\sigma$ , despite the fact that (3.11) would normally yield an equation quadratic in both  $\sigma$  and  $r$ . Geometrically, this can be understood as follows (see Appendix B). The roots of the two factors on the left-hand side of (3.12) are the asymptotes of the hyperbola whose slopes in the  $(r, \sigma)$  plane are the eigenvalues  $\lambda_{1\pm}$  of the matrix  $M_1$  in (3.7) with determinant  $\text{Det}_1$ . Here,  $\lambda_{1+} = 1+S$ ,  $\lambda_{1-} = 0$ , and  $\text{Det}_1 = 0$ . Physically,  $\lambda_{1-}$  and  $\text{Det}_1$  vanish because the velocity  $w$  advects both the conductive temperature and concentration profiles in the same way. Hence  $\sigma = -(S+L)/(1+S)$  is a horizontal asymptote. A line in the  $(r, \sigma)$  plane which is parallel but not equal to an asymptote intersects the hyperbola in exactly one point (see Appendix B). The consequence of this is that all real values (except  $-(S+L)/(1+S)$ ) of  $\sigma$  are achieved exactly once. (This distinctive feature of the eigenvalues arising from this problem will also have repercussions on the nonlinear problem, discussed in Section 4.) In particular (except when  $S = -L$ ),  $\sigma = 0$  for a unique value of  $r$ ,

$$r_{\text{PF}} = \frac{L}{S+L} \quad (3.16)$$

which is the location of the unique steady bifurcation from the basic state, well-known to be a pitchfork bifurcation. For  $S > (<) -L$ , the steady bifurcation occurs at positive (negative)  $r$ , with  $r_{\text{PF}} \rightarrow +(-)\infty$  and  $S \downarrow (\uparrow) -L$ .

If  $S > -L^2$  or if  $S < -L$ , then it is the upper branch  $\sigma_+$  which crosses zero at the bifurcation, i.e.  $r_{\text{PF}}$  satisfies  $f + \sqrt{g} = 0$  (see cases  $S = -0.01, -0.1, -1$  of Fig. 2). If  $S$  is in the range  $-L < S < -L^2$ , then  $\sigma_-$  crosses zero, i.e.  $r_{\text{PF}}$  satisfies  $f - \sqrt{g} = 0$  (see case  $S = -0.001$  of Fig. 2). At the endpoint  $S = -L$  of this range, the steady bifurcation goes to infinity, while at the other endpoint  $-L^2$ , it coalesces with a Hopf bifurcation (see below).

We now consider the ellipse (3.13) describing  $\omega(r)$ . If  $S < 0$ , the eigenvalues are complex over the range

$$\frac{1-L}{(1+\sqrt{-S})^2} \equiv r_- < r < r_+ \equiv \frac{1-L}{(1-\sqrt{-S})^2}. \quad (3.17)$$

The endpoints  $r_{\pm}$  are solutions to  $g = 0$ . In this range,  $\sigma_{\pm}$  is equal to half the trace of the matrix  $M$ :

$$\sigma_+ = \sigma_- = \frac{1}{2}[r - 1 + S(r-L)]. \quad (3.18)$$

A Hopf bifurcation occurs at  $r = r_{\text{H}}$  if  $\sigma_{\pm} = 0$  within the range (3.17) of complex eigenvalues. Eqs. (3.18) and (3.13) show that  $r_{\text{H}}$  and  $\omega_{\text{H}} \equiv \omega(r_{\text{H}})$  satisfy

$$r_{\text{H}} = \frac{1+L}{1+S}, \quad \omega_{\text{H}}^2 = -\frac{S+L^2}{1+S}. \quad (3.19)$$

The value  $r_{\text{H}}$  falls within the range (3.17) of complex eigenvalues if and only if

$$-1 < S < -L^2. \quad (3.20)$$

If  $S = -1$ , then  $r_{\text{H}}, r_+$ , and  $\omega_{\text{H}}$  all become infinite. This is the exceptional case in which  $r(\sigma)$  forms a leftward-opening parabola and  $r(\omega)$  a rightward-opening parabola (see Fig. 2). If

$$S = S_* = -L^2, \quad (3.21)$$

then

$$r_{\text{H}} = r_+ = r_{\text{PF}} = \frac{1}{1-L} \equiv r_* \quad (3.22)$$

and  $\omega_{\text{H}}$  vanishes. This is the well-known codimension-two point, e.g. [2,3,5,7,8,16,20,28].

A detailed representation of the bifurcations undergone by the thermosolutal system, made possible by logarithmic scaling, is given in Fig. 2 (for negative  $S$ ) and Fig. 3 (for positive  $S$ ).

### 3.3. Linear thermal and solutal regimes

From Figs. 2 and 3, it can be seen that the thermosolutal thresholds and eigenvalues are related to the pure thermal and solutal thresholds and eigenvalues. Such a resemblance is also clearly visible in a numerical study of Marangoni (surface-tension-driven) convection with Soret effect [58]. This is the relationship we wish to explore in this section.

The simplest classification is by proximity: a real eigenvalue is primarily thermal if it is closer to the pure thermal eigenvalue  $\sigma_T$  than to the pure solutal eigenvalue  $\sigma_C$ , i.e. if

$$|\sigma - \sigma_T| < |\sigma - \sigma_C|. \quad (3.23)$$

Another possible classification is based on the eigenvector, specifically on the magnitude of  $SC/T$  — the ratio of the solutal to the thermal contribution in the buoyancy force in (3.1c). These two criteria are equivalent. Indeed, the eigenvalue equation (3.6) states that the eigenvectors satisfy

$$\sigma - \sigma_T = \frac{SC}{T}r, \quad (3.24a)$$

$$\sigma - \sigma_C = \frac{T}{C}r. \quad (3.24b)$$

Thus (3.23) becomes

$$\left| \frac{SC}{T} \right| < \sqrt{|S|}. \quad (3.25)$$

Thus, an eigenvalue is thermal (solutal) if the corresponding eigenvector satisfies:  $|SC/T| < (>)\sqrt{|S|}$ .

For positive  $S$ ,  $r_{\text{int}}$  separates thermal from solutal portions of the eigenvalue curve. For  $0 < S < 1$ ,  $\sigma_+$  is thermal and  $\sigma_-$  is solutal for the range  $r > r_{\text{int}}$  and vice versa  $r < r_{\text{int}}$ . For  $S > 1$ , the opposite holds. For negative  $S$ ,  $r_{\text{int}}$  is the midpoint of the interval of complex eigenvalues. Complex eigenvalues, whose real part is equidistant between the pure thermal and solutal eigenvalues, cannot be classified in this way. Instead, it is  $r_{\pm}$  which serve as boundaries: for negative  $S$ , we classify the real eigenvalues  $\sigma_+$  as thermal and  $\sigma_-$  as solutal for the range  $r > r_+$  and vice versa for  $r < r_-$ . In Figs. 2 and 3, we can discern various regimes in which the thermosolutal eigenvalues  $\sigma_{\pm}$  adhere closely to the pure solutal and thermal eigenvalues  $\sigma_T$  and  $\sigma_C$ . For  $|S| \gg 1$ , e.g.,  $\sigma_+ \approx \sigma_C$  for  $r$  sufficiently small. For  $|S| \ll L$ ,  $\sigma_{\pm}$  resemble  $\sigma_T$  and  $\sigma_C$  over most of the range shown surrounding  $r_{\text{int}}$ . Detailed estimates, omitted here, justify these visual impressions.

Similarity between thermosolutal and pure thermal or solutal eigenvalues implies similarity between the thresholds of these problems. The threshold of the steady bifurcation is  $r_{\text{PF}} = L/(L + S)$ . If  $|S| \ll L$ , then  $r_{\text{PF}} \approx 1 = r_T$ , i.e. the thermosolutal threshold approaches the pure thermal threshold. Conversely, if  $|S| \gg L$ , then  $r_{\text{PF}} \approx L/S = r_C$ , so that the thermosolutal threshold approaches the pure solutal threshold.

More precisely, we can calculate the difference between the thresholds:

$$|r_{\text{PF}} - r_T| = \left| \frac{L}{L + S} - 1 \right| = \left| \frac{L - (L + S)}{L + S} \right| = \left| \frac{S}{L + S} \right|, \quad (3.26a)$$

$$|r_{\text{PF}} - r_C| = \left| \frac{L}{L + S} - \frac{L}{S} \right| = \left| \frac{LS - L(L + S)}{S(L + S)} \right| = \left| \frac{L^2}{S(L + S)} \right|. \quad (3.26b)$$

Taking into account the different sign possibilities for  $S$  and  $L + S$ , we find that, for  $\epsilon \ll 1$ ,

$$|r_{\text{PF}} - r_{\text{T}}| < \epsilon \quad \text{if } |S| < L\epsilon, \quad (3.27a)$$

$$|r_{\text{PF}} - r_{\text{C}}| < \epsilon \quad \text{if } |S| < \frac{L}{\sqrt{\epsilon}}. \quad (3.27b)$$

Indeed, on the left diagrams of Figs. 2 and 3, we see that the thermosolutal threshold  $r_{\text{PF}}$  (heavy solid curve) is well approximated by the pure thermal threshold  $r_{\text{T}}$  (straight thin dashed curve) for  $|S| \lesssim 0.001$  and by the pure solutal threshold  $r_{\text{C}}$  (curved thin dashed curve) for  $|S| \gtrsim 0.03$ . These are precisely the estimates obtained from (3.27a) and (3.27b) with  $\epsilon = 0.1$  and  $L = 0.01$ . The domain  $|S| \lesssim 0.1L$  is analogous to what was termed the thermal-dominated regime by Bergeon et al. [58] while the  $|S| \gtrsim 3L$  is analogous to what was termed the solutal-dominated regime. We note further that the eigenvector at the steady bifurcation satisfies

$$\frac{SC}{T} = \frac{S}{L}. \quad (3.28)$$

Thus, at the bifurcation, either the solutal contribution  $SC$  or the thermal contribution  $T$  dominates the buoyancy force, according on whether one is in the solutal-dominated or thermal-dominated regime.

## 4. Nonlinear analysis

### 4.1. Derivation of nonlinear model

We return to the full nonlinear thermosolutal problem (2.4) and summarize the derivation of a minimal set of amplitude equations. This model was first introduced by Veronis [1] for the thermosolutal problem and later adapted by Platten and Chavepeyer [9] for the Soret problem.

The simple spatial dependence (3.2) is not preserved by the nonlinear terms in the governing equations (2.4). Substituting (3.2a) and

$$\hat{\phi}(x, z, t) = \phi(t) \sin(kx) \sin(\pi z), \quad (4.1)$$

where  $w = k\phi$ , into (2.4a) yields

$$\begin{aligned} \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{T}) &= \partial_z \hat{\phi} \partial_x \hat{T} - \partial_x \hat{\phi} \partial_z \hat{T} \\ &= -\pi k \phi T (\sin^2(kx) + \cos^2(kx)) \sin(\pi z) \cos(\pi z) = -\frac{\pi}{2} w T \sin(2\pi z) \end{aligned} \quad (4.2)$$

and similarly for  $\hat{C}$ . On the other hand, the dependence (4.1) is preserved by the nonlinear terms of (2.4c)

$$q^{-2} \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \nabla^2 \hat{\phi}) = -\mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{\phi}) = 0. \quad (4.3)$$

Expansions (3.2) are therefore generalized to include terms of the type (4.2)

$$\hat{T} = T(t) \cos(kx) \sin(\pi z) + T_2(t) \sin(2\pi z), \quad (4.4a)$$

$$\hat{C} = C(t) \cos(kx) \sin(\pi z) + C_2(t) \sin(2\pi z). \quad (4.4b)$$

The nonlinear term of (2.4a) on  $\hat{\phi}$  and  $\hat{T}_2$  yields

$$\begin{aligned} \mathbf{e}_y \cdot (\nabla \hat{\phi} \times \nabla \hat{T}_2) &= \partial_z \hat{\phi} \partial_x \hat{T}_2 - \partial_x \hat{\phi} \partial_z \hat{T}_2 \\ &= 0 - 2\pi w T_2 \cos(kx) \sin(\pi z) \cos(2\pi z) = \pi w T_2 \cos(kx) (\sin(\pi z) - \sin(3\pi z)). \end{aligned} \quad (4.5)$$

Expansions (4.4) are truncated by neglecting the term  $\cos(kx) \sin(3\pi z)$ . This procedure is quantitatively correct for small amplitudes and, as is often the case, qualitatively accurate even for moderate amplitudes. The validity and limitations of this truncation are discussed in [12,16,26,46,47].

Substituting (4.4), (4.1) and (3.2c) into (2.1) yields

$$\frac{d}{dt} \begin{pmatrix} T \\ C \\ w \\ T_2 \\ C_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -L & 1 & 0 & 0 \\ Pr & PSr & -P & 0 & 0 \\ 0 & 0 & 0 & -\frac{4\pi^2}{q^2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{4\pi^2 L}{q^2} \end{pmatrix} \begin{pmatrix} T \\ C \\ w \\ T_2 \\ C_2 \end{pmatrix} + \pi w \begin{pmatrix} T_2 \\ C_2 \\ 0 \\ -\frac{1}{2}T \\ -\frac{1}{2}C \end{pmatrix}. \quad (4.6)$$

In the linear part of (4.6),  $(T, C, w)$  is decoupled from  $(T_2, C_2)$ , and the eigenvalues of the latter system are always negative. Hence, the linear stability analysis of (4.6) is that already carried out in Section 3, assuming again that  $P \rightarrow \infty$ .

The remainder of Section 4 is devoted to analyzing the exact steady states of (4.6). Surprisingly, we will find that calculating these steady states reduces to calculating the eigenvalues of a  $2 \times 2$  matrix closely related to the linear stability matrix (3.6).

#### 4.2. Steady states of the nonlinear model

Although (4.6) is a five-dimensional nonlinear system, its form allows its steady solutions to be calculated analytically [1,2]. Steady solutions of (4.6) satisfy

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -L & 1 & 0 & 0 \\ Pr & PSr & -P & 0 & 0 \\ 0 & 0 & 0 & -\frac{4\pi^2}{q^2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{4\pi^2 L}{q^2} \end{pmatrix} \begin{pmatrix} T \\ C \\ w \\ T_2 \\ C_2 \end{pmatrix} + \pi w \begin{pmatrix} T_2 \\ C_2 \\ 0 \\ -\frac{1}{2}T \\ -\frac{1}{2}C \end{pmatrix}. \quad (4.7a)$$

$$(4.7b)$$

$$(4.7c)$$

$$(4.7d)$$

$$(4.7e)$$

Note that (4.7c), which was derived as (3.5) from the linear stability problem (3.4) under the assumption of large  $P$ , is here merely a consequence of the search for steady states and of (4.3). Using (4.7c)–(4.7e) to eliminate  $w$ ,  $T_2$ , and  $C_2$  yields

$$\begin{pmatrix} r-1 & Sr \\ r & Sr-L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} = \frac{q^2 w^2}{8} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{L} \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}. \quad (4.8)$$

Eq. (4.8) is of the form of an eigenvalue problem

$$\begin{pmatrix} r-1 & Sr \\ Lr & L(Sr-L) \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} = E \begin{pmatrix} T \\ C \end{pmatrix} \quad (4.9)$$

with

$$E \equiv \frac{1}{8} q^2 w^2 = \frac{1}{8} q^2 r^2 (T + SC)^2 \quad (4.10)$$

playing the role of an eigenvalue. After the eigenvalues  $E$  of (4.9) are found, the relative amplitude of components  $T$  and  $C$  is given by the eigenvectors. Note that the original five-dimensional steady-state problem (4.7) is also of this type. Then,  $-\pi w$  plays the role of a generalized eigenvalue and the generalized eigenvectors give the relative proportions of the five components.

A related version of this reduction has been shown [55] to be valid for the full PDEs governing binary fluid convection with Soret effect with realistic boundary conditions, and has been used [56] to derive a sophisticated and realistic 11-mode model. Hollinger et al. [55,56] demonstrate that the velocity field can be very well approximated by a single spatial mode and can be adiabatically eliminated. The nonlinearities are then only those which advect the temperature and concentration, and contain only the amplitude  $w$  of the vertical velocity. The temperature and concentration amplitudes are then solutions to a linear system depending on  $w$ . Hollinger et al.'s work provides evidence that our reformulation of (4.7) as an eigenvalue problem, while strictly valid only for the minimal five-mode Veronis model of thermosolutal convection, is a manifestation of a quite general property of double-diffusive convection.

$E$  can plausibly be called the energy of the convective state, since the kinetic energy density is

$$\begin{aligned}\mathcal{E} &= \frac{k}{2\pi} \int_0^{2\pi/k} dx \int_0^1 dz \frac{1}{2} (\hat{u}^2 + \hat{w}^2) \\ &= \frac{k}{2\pi} \int_0^{2\pi/k} dx \int_0^1 dz \frac{1}{2} \left( \left( \frac{\pi w}{k} \sin(kx) \cos(\pi z) \right)^2 + (w \cos(kx) \sin(\pi z))^2 \right) = \frac{q^2 w^2}{8k^2} = \frac{1}{k^2} E.\end{aligned}\quad (4.11)$$

The most common experimentally measured quantity is the convective heat transport  $\mathcal{N} - 1$ , which is related to  $E$  via

$$\begin{aligned}\mathcal{N} - 1 &= \int_0^{2\pi/k} dx \int_0^1 dz \hat{w} \hat{T} = \int_0^{2\pi/k} dx \int_0^1 dz w T \cos^2(kx) \sin^2(\pi z) \\ &= \frac{\pi}{2k} w T = \frac{\pi}{2k} \frac{w^2}{E + 1} = \frac{4\pi}{kq^2} \frac{E}{E + 1},\end{aligned}\quad (4.12)$$

where we have used  $w - T = ET$  from (4.9).

The matrix

$$\begin{aligned}\begin{pmatrix} r - 1 & Sr \\ Lr & L(Sr - L) \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -L^2 \end{pmatrix} + r \begin{pmatrix} 1 & S \\ L & LS \end{pmatrix}, \\ \tilde{M} &= \tilde{M}_0 + r \tilde{M}_1\end{aligned}\quad (4.13)$$

is very similar to the matrix (3.7).  $\tilde{M}_0$  again describes diffusion, and  $\tilde{M}_1$  now describes the combined effects of advection and nonlinear saturation. The interpretation of the behavior of the eigenvalues of (4.13) is, of course, different from that of (3.7). In the linear system, negative or complex eigenvalues characterize infinitesimal perturbations which decay and/or oscillate. For the nonlinear problem, negative or complex values of  $E$  are forbidden by virtue of definition (4.10), and imply nonexistence of steady solutions for certain ranges of  $r$ ,  $S$ ,  $L$ . Since only real positive values of  $E$  are meaningful, we will not introduce separate notation for real and imaginary parts of  $E$ . For  $E$  real and positive, we define  $A = \pm\sqrt{E}$  as the convection amplitude.

### 4.3. Bifurcations

Most of the results of Section 3 concerning the eigenvalues  $\sigma_{\pm}$  are easily modified to apply to the energy  $E$  of the nonlinear steady states, merely by substituting

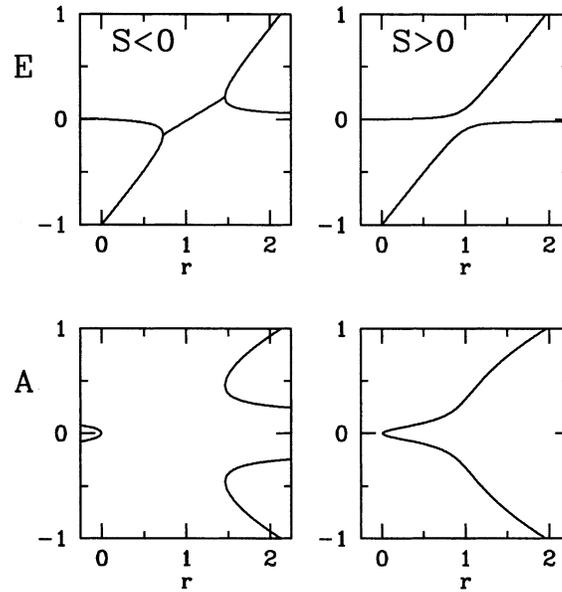


Fig. 4.  $E$  and  $A = \pm\sqrt{E}$  (for  $E$  real and positive) for  $S = -3$  and  $S = 1$ . For  $S = -3$ ,  $E$  undergoes complex coalescence. The point at which  $E$  becomes complex corresponds to a pair of saddle-node bifurcations in  $A$ . For  $S = 1$ ,  $E$  undergoes avoided crossing. This leads to a distinction in  $A$  between low-amplitude Soret and high-amplitude Rayleigh regimes.

$$L \rightarrow L^2, \quad (4.14a)$$

$$S \rightarrow LS. \quad (4.14b)$$

When we use the same notation for the nonlinear and linear problems, we add tildes to designate nonlinear quantities.

Just as we did for the linear problem, we can define nonlinear pure thermal and solutal solutions by setting the coupling terms in (4.9) to zero

$$E_T = r - 1 = r - r_T, \quad (4.15a)$$

$$E_C = L(Sr - L) = LS(r - r_C). \quad (4.15b)$$

Again, the coupling term

$$\tilde{\beta}\tilde{\gamma} = LSr^2 \quad (4.15c)$$

is proportional to, and has the same sign as,  $S$ . Hence, the eigenvalues  $E$  undergo avoided crossing near the intersection point of the two pure solutions

$$\tilde{r}_{\text{int}} = \frac{1 - L^2}{1 - LS} \quad (4.16)$$

if  $S$  is positive, and complex coalescence if  $S$  is negative. These possibilities are illustrated in Fig. 4, along with the resulting consequences for  $A = \pm\sqrt{E}$ .

The solutions to the eigenvalue problem (4.9) are

$$E = \frac{E_T + E_C}{2} \pm \sqrt{\left(\frac{E_T + E_C}{2}\right)^2 + SLr^2} \quad (4.17a)$$

$$= \frac{1}{2}[(1 + LS)r - (1 + L^2)] \pm \frac{1}{2}\sqrt{(1 + LS)^2r^2 - 2(1 - L^2)(1 - LS)r + (1 - L^2)^2}, \quad (4.17b)$$

$$\equiv \tilde{f}(r) \pm \sqrt{\tilde{g}(r)}. \quad (4.17c)$$

This expression for  $E$  is quite similar to (3.11) for the eigenvalues. Detailed results for representative values of  $S$  are shown in the two large Figs. 5 and 6.

Just as  $\lambda = 0$ ,  $(T, C) = (0, 0)$  is a solution of (3.6), it is also true that  $E = 0$ ,  $(T, C) = (0, 0)$  is always a solution to (4.9). The existence of the solution  $E = 0$  for all  $r$  reflects the fact that the conductive profile in a motionless fluid remains a solution for all Rayleigh numbers, although not necessarily a stable one. Eqs. (3.11), (4.17) give the nontrivial solutions for  $\lambda$  and  $E$ , which are zero only at isolated values of  $r$ .

The values (4.17) for  $E$  real form hyperbolas

$$\left(E + \frac{LS + L^2}{1 + LS} - (1 + LS)\left(r - \frac{(1 - L^2)(1 - LS)}{(1 + LS)^2}\right)\right)\left(E + \frac{LS + L^2}{1 + LS}\right) = SL\frac{(1 - L^2)^2}{(1 + LS)^2} \equiv \tilde{\Delta} \quad (4.18)$$

in the  $(r, E)$  plane. (In the exceptional case  $S = -1/L$ , the solutions form a parabola.) Just as we found that  $r$  was a single-valued function (3.15) of  $\sigma$ , here  $r$  is also a single-valued function of  $E$ . (This is again because the zero determinant of matrix  $\tilde{M}_1$  in (4.13) leads to a horizontal asymptote for the hyperbola.) We use the substitutions (4.14) to transform  $r(\sigma)$  into  $r(E)$

$$r = \frac{\sigma_{\pm}^2 + \sigma_{\pm}(1 + L) + L}{\sigma_{\pm}(1 + S) + (S + L)} \rightarrow r = \frac{E^2 + E(1 + L^2) + L^2}{E(1 + LS) + L(S + L)} \quad (4.19)$$

a result which can be verified by inverting (4.17). Thus each value of  $E$  except  $-L(S + L)/(1 + LS)$  is achieved exactly once, a feature mentioned in Section 3 for the growth rates  $\sigma$ , but more important in the context of amplitudes of steady states; see Figs. 5 and 6. It is remarkable that this property continues to hold even for the far more complicated Soret problem with rigid boundaries and adequate spatial resolution. In [56], it is also found that  $r$  is a single-valued function of the energy, as measured by the square of the vertical velocity amplitude. This function, like (4.19), is the ratio of a quadratic to a linear function of  $E$ , although with coefficients more complicated than those of (4.19).

Steady bifurcations from the conductive profile occur at values of  $r$  at which  $E = 0$ . Eq. (3.16) gives the values  $r_{PF}$  at which  $\sigma = 0$ . The substitutions (4.14) leave expression (3.16) unchanged

$$r_{PF} = \frac{L}{S + L} \rightarrow \tilde{r}_{PF} = \frac{L^2}{LS + L^2} = \frac{L}{S + L} = r_{PF}. \quad (4.20)$$

This result can be verified by setting  $E = 0$  in (4.17) or (4.19). The fact that the “linear” ( $\sigma = 0$ ) and “nonlinear” ( $E = 0$ ) values of  $r_{PF}$  are identical reflects the fundamental bifurcation-theoretic fact that a change of sign of a real eigenvalue signals the transverse intersection of two or more steady branches, i.e. a steady bifurcation. The fact that  $E = A^2$  (rather than  $A$ ) changes sign at  $r_{PF}$  identifies these as pitchfork bifurcations, since either sign of  $A$  is permitted. Eqs. (4.12) and (4.19) can be used to calculate the slope of the convective heat transport at the bifurcation

$$\left.\frac{d(\mathcal{N} - 1)}{dr}\right|_{r=r_{PF}} = \frac{4\pi}{kq^2} \frac{1}{(E + 1)^2} \left.\left(\frac{dr}{dE}\right)^{-1}\right|_{E=0} = \frac{4\pi}{kq^2} \frac{L(S + L)^2}{S + L^3} \quad (4.21)$$

very similar to the analogous formula derived for the Soret problem in [20].

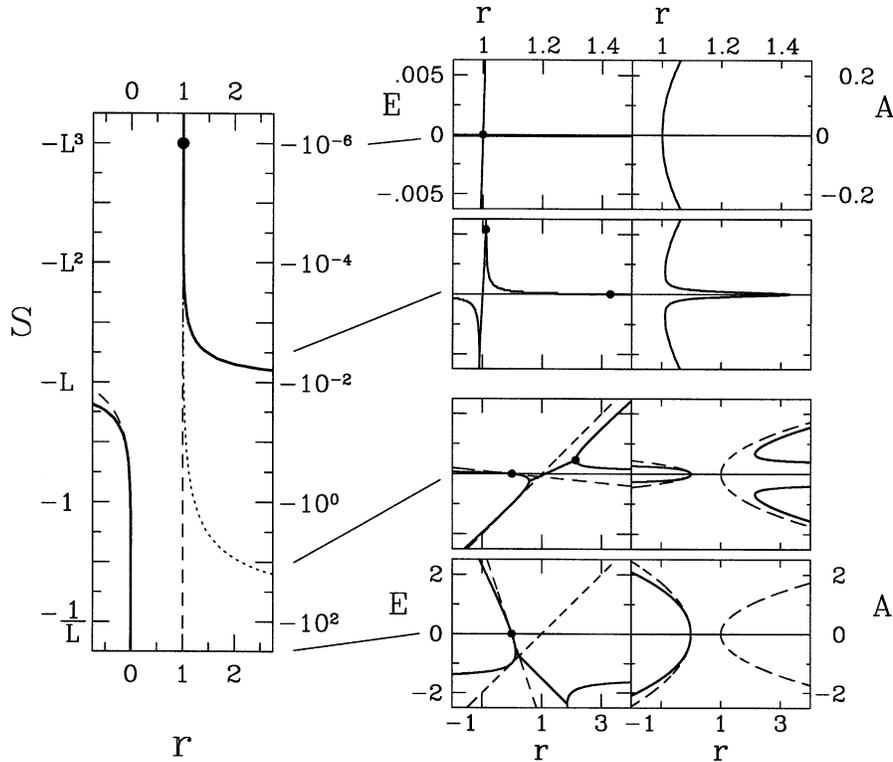


Fig. 5. Bifurcation diagrams for negative  $S$ . Left: thresholds for negative values of  $S$  plotted on a logarithmic scale. Solid curves show the thresholds  $r_{PF} = L/(L + S)$  of steady bifurcations. Dotted curve shows the thresholds  $r_{SN}$  for saddle-node bifurcations. This curve appears from  $r = +\infty$  at  $S = -1/L$  and disappears by meeting the steady bifurcation curve in a codimension-two point (degenerate pitchfork) at  $\tilde{S}_* = -L^3, \tilde{r}_* = 1/(1 - L^2)$ , indicated by a heavy dot. Dashed curves show the pitchfork bifurcation thresholds  $r_T = 1$  and  $r_C = L/(L + S)$  of the pure thermal and solutal problems. Right: energy  $E$  and amplitude  $A$  of steady nonlinear solutions to thermosolutal problem as a function of  $r$  for representative negative values of  $S$ . Dashed curves on lower two sets of diagrams show energy and amplitude of pure thermal and solutal solutions. For  $S = -300$ , representing  $S < -1/L$ , there is one branch of real solutions  $A$ , bifurcating towards negative  $r$  and resembling the pure solutal branch. At  $S = -1/L$  (not shown), the set  $(r, E)$  is a parabola and a pair of saddle-node bifurcations descends from  $r = \infty$ . For  $S = -10$ , representing  $-1/L < S < -L$ , the resulting pairs of disconnected branches can be seen inside and rather close to the pure thermal branch. The other solutions, branching towards negative  $r$ , are still present. They, and the pure solutal solutions which they resemble, are of greatly decreased amplitude. At  $S = -L$  (not shown), the pitchfork bifurcation disappears at  $r = -\infty$  to reappear at  $r = +\infty$ . (The pure solutal branch, increasingly small and distant, continues to exist until  $S = 0$ .) For  $S = -0.003$ , representing  $-L < S < -L^3$ , the pitchfork bifurcation has descended to  $r = 1.429$  and connects the two pairs of branches arising from the saddle-node bifurcations. Here, and in the next case, the thermal branch resembles the upper branch of solutions too closely to be distinguished from it in the figure, whereas the solutal branch cannot be distinguished from the  $r$ -axis. At  $S = -L^3 = -10^{-6}$ , the saddle-nodes and pitchfork coalesce in a codimension-two point separating subcritical from supercritical pitchfork bifurcations. Note change of scale between upper two and lower two diagrams.

If  $S < 0$ , complex solutions to (4.17) are possible. The interval of  $r$ -values over which this is so, delimited by  $\tilde{g} = 0$ , is easily obtained using substitutions (4.14) in (3.17)

$$\frac{1 - L^2}{(1 + \sqrt{-LS})^2} \equiv \tilde{r}_- < r < \tilde{r}_+ \equiv \frac{1 - L^2}{(1 - \sqrt{-LS})^2}. \tag{4.22}$$

For the linear problem, the endpoints of the complex interval (3.17) for  $\lambda$  divided oscillatory evolution from monotonically growing (if  $\sigma > 0$ ) or decaying (if  $\sigma < 0$ ) evolution. For the nonlinear problem, the endpoints of the complex interval (4.22) mark the appearance or disappearance of real solutions  $A$ ; see Fig. 4 for  $S < 0$  and Fig. 5

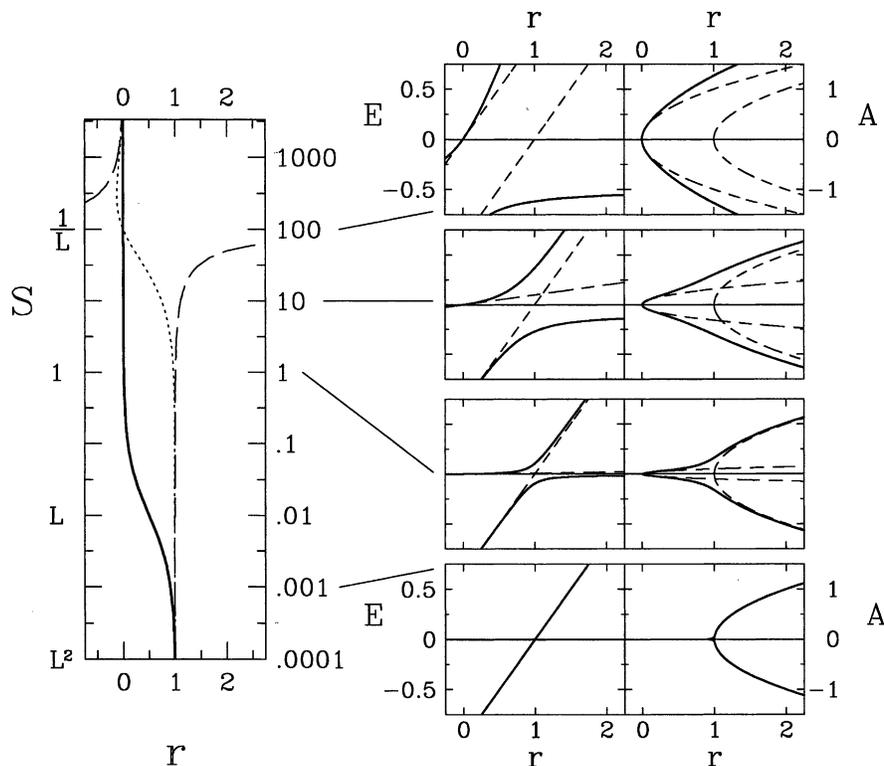


Fig. 6. Bifurcation diagrams for positive  $S$ . Left: thresholds for positive values of  $S$  plotted on a logarithmic scale. Solid curve show the thresholds  $r_{PF} = L/(L + S)$  of steady bifurcations. Long-dashed curve shows value  $\tilde{r}_{int}$  at which the pure thermal and solutal branches intersect. Dotted curve shows value  $\tilde{r}_{mid}$  at which the asymptotes of the hyperbola  $E$  intersect, and at which the slope of  $E$  and the curvature of  $A$  change most rapidly. Right: energy  $E$  and amplitude  $A$  of steady nonlinear solutions to thermosolutal problem as a function of  $r$  for representative positive values of  $S$ . Dashed curves show energy and amplitude of pure thermal and solutal solutions. At  $S = 0.001$ , representing  $L^3 < S < L$ , the thermosolutal branch hugs the thermal branch. At  $S = 1$  and  $S = 10$ , representing  $L < S < 1/L$ , the pure solutal branch has descended below the pure thermal branch, “pulling” the thermosolutal branch with it. At  $S = 100$ , representing  $S \geq 1/L$ , the solutal branch lies above the thermal branch. The thermosolutal branch, initially tangent to the solutal branch, lies above both pure branches. All bifurcation diagrams use the same scale.

for  $S = -10$ . These are saddle-node bifurcations: the simultaneous creation of four branches of steady solutions  $A$ , corresponding to two different values of  $E = A^2$ . Since  $E$  must be nonnegative as well as real, the occurrence of saddle-node bifurcations at  $\tilde{r}_-$  or  $\tilde{r}_+$  requires that  $\mathcal{R}(E) = \tilde{f} \geq 0$ . Substituting (4.22) into

$$\tilde{f}(\tilde{r}_{\pm}) = \frac{1}{2}[\tilde{r}_{\pm} - 1 + LS(\tilde{r}_{\pm} - L^2)], \tag{4.23}$$

we calculate that saddle-node bifurcations occur at  $r_{SN} \equiv \tilde{r}_+$  if

$$-\frac{1}{L} < S < -L^3. \tag{4.24}$$

Fig. 7 characterizes the saddle-node bifurcation over the domain  $-1/L < S < -L^3$  of its existence. Over most of the range, the value  $E_{SN} = E(r_{SN})$  varies with  $S$  like  $1/\sqrt{-SL}$  and  $r_{SN}$  varies like  $1 + 2\sqrt{-SL}$ . When the saddle-node bifurcation appears at  $S = -1/L$ , we have  $r_{SN} = E_{SN} = \infty$ . When the saddle-node bifurcation disappears at the degenerate pitchfork at  $S = -L^3$ , we have  $r_{SN} = 1/(1 - L^2)$ ,  $E_{SN} = 0$ . The curvature  $d^2r/dA^2(r_{SN})$  at the

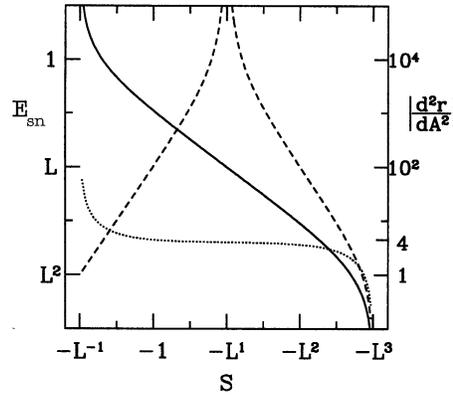


Fig. 7. Saddle-node bifurcation for  $-1/L < S < -L^3$ . The solid curve shows  $E_{\text{SN}}$ , the energy at the saddle-node bifurcation at  $r_{\text{SN}}$ .  $E_{\text{SN}} \approx \sqrt{|LS|}$  except very near  $-1/L$ , where it diverges, and near  $-L^3$ , where it goes to zero. The dotted curve shows  $d^2r/dA^2(r_{\text{SN}})$ , the curvature of the saddle-node bifurcation, which remains near 4 except near the endpoints  $-1/L$  and  $-L^3$ . For contrast, the dashed curve shows  $d^2r/dA^2(r_{\text{PF}})$ , the curvature of the pitchfork bifurcation, which varies greatly everywhere over this interval, especially near  $-L$ , where  $r_{\text{PF}}$  and  $d^2r/dA^2(r_{\text{PF}})$  approach  $\pm\infty$ .

saddle-node bifurcation is also shown.  $d^2r/dA^2(r_{\text{SN}}) \approx 4$  over much of the range. In contrast, the curvature at the pitchfork bifurcation  $d^2r/dA^2(r_{\text{PF}}) \sim (d(\mathcal{N} - 1)/dr)^{-1}(r_{\text{PF}})$  (see Eqs. (4.19) and (4.21)) varies over many orders of magnitude (even away from its divergence at  $S = -L$ ). This great difference in curvatures is one of the factors giving the thermosolutal bifurcation diagrams, such as the  $S = -0.003$  case in Fig. 5, their characteristic appearance.

We note that (4.24) is the transformation via substitutions (4.14) of the range of existence (3.20) of Hopf bifurcations of the linear problem. Yet the criterion for a Hopf bifurcation, where  $\lambda$  has zero real part ( $f = 0$ ) and finite imaginary part ( $g \leq 0$ ) has no significance for a nonlinear steady state  $E$ . We explain the correspondence between the range of existence of Hopf and saddle-node bifurcations as follows.

Although a Hopf bifurcation has no analog for nonlinear steady states, the codimension-two point of the linear analysis, at which the pitchfork and Hopf bifurcations coalesce, does have a relevant nonlinear analog. This is because at the codimension-two point, the pitchfork and Hopf bifurcations necessarily coalesce with a third point: that at which the eigenvalues become complex (see  $S = -L^2$  case of Fig. 2), i.e., the conditions defining a steady bifurcation,  $f \pm \sqrt{g} = 0$ ,  $g \geq 0$  and a Hopf bifurcation,  $f = 0$ ,  $g \leq 0$  together imply  $f = g = 0$ . For the nonlinear analysis, the same conditions mark the coalescence of the pitchfork and saddle-node bifurcations, i.e., the nonlinear codimension-two point satisfies both  $\tilde{f} \pm \sqrt{\tilde{g}} = 0$ ,  $\tilde{g} \geq 0$  (pitchfork) and  $\tilde{f} \geq 0$ ,  $\tilde{g} = 0$  (saddle-node) which together again imply  $\tilde{f} = \tilde{g} = 0$ . We use (4.14) to transform the codimension-two point of the linear problem to the codimension-two point of the nonlinear problem:

$$S_* = -L^2 \rightarrow L\tilde{S}_* = -L^4, \quad \text{i.e.} \quad \tilde{S}_* = -L^3, \quad (4.25)$$

$$r_* = \frac{1}{1-L} \rightarrow \tilde{r}_* = \frac{1}{1-L^2}. \quad (4.26)$$

Whereas the linear codimension-two point  $S_* = -L^2$  marks the limit of existence of the Hopf bifurcation, the nonlinear codimension-two point  $\tilde{S}_* = -L^3$  marks the limit of existence of the saddle-node bifurcations. For  $S > -L^3$ , the pitchfork bifurcation is forwards, with a pair of nontrivial solutions branching right towards  $r > r_{\text{PF}}$ . For  $S < -L^3$ , the pitchfork bifurcation is backwards, with nontrivial solutions branching left towards  $r < r_{\text{PF}}$ .

The correspondence between the lower endpoint of ranges (3.20) and (4.24) is also easily explained. As  $S \downarrow -1$ , the Hopf bifurcation point  $r_H$  diverges to  $+\infty$ , as seen in Fig. 2. This event coincides with the divergence of the right endpoint  $r_+$  of the interval of complex eigenvalues  $\lambda$ , as the set  $(r, \pm\omega(r))$  evolves from an ellipse to a rightward-opening parabola. Similarly, the right endpoint  $\tilde{r}_+ = r_{SN}$  of the interval of complex (forbidden) values of  $E$  diverges to  $+\infty$  as  $LS \downarrow -1$  or  $S \downarrow -1/L$ , as seen in Fig. 5.

The saddle-node bifurcation for the nonlinear thermosolutal and the Soret problems is well-known, as are the Hopf bifurcation and the codimension-two point of the linear problem. Indeed, these are the basic features that originally inspired the great interest evoked by binary fluid convection. However, the relationship between these phenomena has not been previously formulated. One of the advantages of the idealized free-slip thermosolutal model is that the codimension-two points have the simple forms  $S_* = -L^2$  and  $\tilde{S}_* = -L^3$ . However, this *scaling* is quite general. For the Soret problem with free-slip permeable boundaries and finite Prandtl number  $P$  [7,20,28],

$$S_* = -L^2 \frac{P+1}{P+L(1+L)(P+1)}, \quad \tilde{S}_* = -\frac{L^3}{1+L+L^2+L^3}. \quad (4.27)$$

Schöpf and Zimmermann [49] observe the scaling  $S_* = -L^2$  and  $\tilde{S}_* = -L^3$  in amplitude equations calculated for the Soret problem with rigid impermeable boundaries. Using their 11-mode minimal model, Hollinger et al. [56] calculate

$$S_* = -L^2 \frac{0.35P+0.18}{P+0.28L}, \quad \tilde{S}_* = -L^3 \frac{1.96}{1+1.04L+0.37L^2+0.97L^3}. \quad (4.28)$$

Exact expressions for the lower bounds for the existence of the Hopf ( $S > -1$  in our case) and saddle-node ( $S > -1/L$  in our case) bifurcations are not as readily available, but also continue to hold approximately. This provides empirical evidence that the relationship between the complex eigenvalues of the linear growth-rate problem and the saddle-node bifurcation of the nonlinear steady-state problem continues to hold, at least approximately, even for a more complicated and realistic case, and thus that this relationship is a fundamental feature of binary fluid convection.

#### 4.4. Soret and Rayleigh regimes

In Section 3.3, we saw that the eigenvalues could be characterized as primarily thermal or primarily solutal according to their distance from the pure thermal or solutal eigenvalues. Here, we will discuss various ways of classifying the nonlinear solution branches in this way. This classification is more significant since we are interested in entire branches of nonlinear steady states, whereas eigenvalues are of interest primarily at the thresholds.

We can apply the classification by proximity analogous to that we used for the eigenvalues: a nonlinear steady state is primarily thermal if it is closer to the pure thermal than to the solutal branch, i.e. if

$$|E - E_T| < |E - E_C| \quad (4.29)$$

and primarily solutal otherwise. Just as we did for the eigenvalues and eigenvectors, we can show that this criterion is equivalent to one based on the magnitude of  $SC/T$ , which is the ratio of the solutal to the thermal contribution in the definition of the convective amplitude  $A \propto T + SC$  as well as in the buoyancy force. The eigenvalue equation (4.9) states that the nonlinear steady states satisfy

$$E - E_T = \frac{SC}{T} r, \quad (4.30a)$$

$$E - E_C = \frac{LT}{C} r. \quad (4.30b)$$

Thus (4.29) becomes

$$\left| \frac{SC}{T} \right| < \sqrt{|LS|} \quad (4.31)$$

and so a nonlinear steady state is thermal (solutal) if  $SC/T < (>)\sqrt{LS}$ .

We first consider negative  $S$ . Four ranges of  $S$  can be distinguished, as can be seen on Fig. 5. For  $-L^3 < S < 0$ , a pair of branches bifurcates towards positive  $r$ ; these nonlinear steady states are all thermal. For  $-L < S < -L^3$ , the pitchfork bifurcation is backwards and both  $E_+$  and  $E_-$  are real and positive for  $r_{SN} < r < r_{PF}$ . Thermal and solutal branches are separated by  $\tilde{r}_+ = r_{SN}$ , where  $SC/T = \sqrt{LS}$ . The lower branch  $E_-$  is solutal (but there exist no corresponding pure solutal steady states, since  $E_C$  is negative) and the upper branch  $E_+$  is thermal. For  $-1/L < S < -L$ , the pitchfork bifurcation occurs at negative  $r_{PF}$  and branches towards lower  $r$ ; the resulting branches are solutal. (These do correspond to pure solutal steady states.) The branches that exist for  $r > r_{SN}$  are isolated. The high-amplitude branches  $E_+$  are primarily thermal and the low-amplitude branches  $E_-$  solutal. For  $S < -1/L$ , the isolated branches no longer exist, leaving only the solutal branches, whose amplitude increases with  $|S|$ .

We now focus on positive  $S$ . Although less studied than negative  $S$ , this case has nonetheless received substantial attention for the Soret problem. Qualitatively, in experiments or three-dimensional calculations, a striking pattern of squares is produced [24,27,37,38,40,44,45,53,57,59,60], possibly alternating with rolls of different orientation. Quantitatively, a fairly abrupt transition is observed between a low-amplitude and a high-amplitude convective regime, as can be seen in Fig. 4. This abrupt transition was first derived for the five-variable model by Platten and Chavepeyer [10], first observed experimentally by Le Gal et al. [24], and the two regimes identified and named the Soret and Rayleigh regimes by Moses and Steinberg [27,44]. The Soret-to-Rayleigh transition has also been reproduced experimentally in [29,41,53] and numerically in [41,42,51,58]. We will interpret this transition as a manifestation of the avoided crossing phenomenon which occurs at the separation between the solutal and thermal regimes.

The physically significant real and positive values of  $E$  are those for  $r > r_{PF}$ . Thus, a steady state branch has physically significant solutal and thermal portions if  $\tilde{r}_{int} > r_{PF}$ , which occurs if  $L^3 < S < 1/L$ . In this case, the solutal regime comprises

$$\frac{L}{L+S} = r_{PF} < r < \tilde{r}_{int} = \frac{1-L^2}{1-LS}. \quad (4.32)$$

The transition between solutal and thermal steady states is manifested by the related increase in slope (for  $E$ ) or curvature (for  $A$ ). For  $S \geq 1/L$ , the entire solution branch is solutal while for  $0 < S \leq L^3$  the entire solution branch is thermal (see Fig. 6).

What is actually observed is more complicated than the analysis given above. Recall that the asymptotes of the hyperbolas (4.18) describing  $E$  differ from the lines  $E_T(r)$  and  $E_C(r)$ . The increase in slope of  $E$  occurs, not at the intersection point  $\tilde{r}_{int}$  between the pure solutal and thermal branches, but at the intersection point  $\tilde{r}_{mid}$  between the two asymptotes. (For  $0 < S < 1$  and for  $S > 10^4$ ,  $\tilde{r}_{mid}$  and  $\tilde{r}_{int}$  are so close as to be indistinguishable on Fig. 9.) The same proviso holds for  $\tilde{r}_{mid}$  as for  $\tilde{r}_{int}$ : the transition is observed only if it occurs at a real and positive value of  $E$ , i.e. only if  $\tilde{r}_{mid} > r_{PF}$ . In this case, the Soret regime comprises

$$\frac{L}{L+S} = r_{PF} < r < \tilde{r}_{mid} = \frac{(1-L^2)(1-LS)}{(1+LS)^2}. \quad (4.33)$$

For  $S = 0$ , we have  $\tilde{r}_{mid} = 1 - L^2 < 1 = r_{PF}$ , so the entire solution branch is in the Rayleigh regime. For  $S$  large, it can also be shown that  $\tilde{r}_{mid} < r_{PF}$ ; the entire branch is then in the Soret regime. But there is an intermediate range

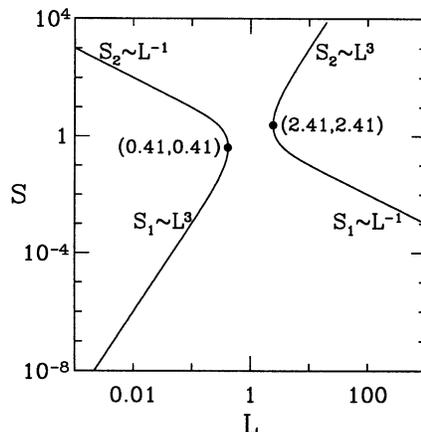


Fig. 8. Range  $(S_1, S_2)$  of values for which  $\tilde{r}_{\text{mid}} > r_{\text{PF}}$  as a function of  $L$ . This range exists for  $L < 0.41$  or if  $L > 2.41$ .

of  $S$  over which the transition can be observed, as can be seen from setting  $r_{\text{PF}}$  equal to  $\tilde{r}_{\text{mid}}$

$$\frac{L}{L+S} = \frac{(1-L^2)(1-LS)}{(1+LS)^2}, \quad 0 = LS^2 - (1-4L^2+L^4)S + L^3,$$

$$S = \frac{1}{2L} [1 - 4L^2 + L^4 \pm (1-L^2)\sqrt{1-6L^2+L^4}]. \quad (4.34)$$

Eq. (4.34) has two real solutions  $S_1, S_2$  if  $1-6L^2+L^4$  is positive, which occurs if  $L < \sqrt{3-2\sqrt{2}} = 0.41$  or if  $L > \sqrt{3+2\sqrt{2}} = 2.41$  as illustrated in Fig. 8. For  $L$  satisfying these conditions, and for  $S_1 < S < S_2$ , there exists a low-amplitude Soret regime over interval (4.33). For  $L \ll 0.41$ , we have  $S_1 \sim L^3$  and  $S_2 \sim 1/L$  and vice versa for  $L \gg 2.41$ . (Recall that  $\tilde{r}_{\text{int}} > r_{\text{PF}}$  for  $L^3 < S < 1/L$ .) Fig. 9 shows the variation with  $S$  of  $\tilde{r}_{\text{mid}}$ ,  $r_{\text{PF}}$ , and  $\tilde{r}_{\text{int}}$  for the case  $L = 0.01$ . We see that while the size of the Soret range  $\tilde{r}_{\text{mid}} - r_{\text{PF}}$  is positive for  $L^3 \lesssim S \lesssim 1/L$ , it is appreciable only over the smaller interval  $L^{5/4} \lesssim S \lesssim L^{-3/4}$ .

Another factor that blurs the transition from Soret-to-Rayleigh-regime is the fact that as  $S$  increases above 0, the hyperbola (4.18) separates from its asymptotes, and the change in slope becomes more gradual as seen, e.g., in Fig. 6 for  $S = 10$ . (This is somewhat counterbalanced by the fact that the angle between the two asymptotes increases from  $\pi/4$  at  $S = 0$  to  $\pi/2$  at  $S = \infty$ .) The change in slope of  $E$  is measured by  $E''$ , which is maximal at  $\tilde{r}_{\text{mid}}$  (see Appendix B). Normalizing  $E''(\tilde{r}_{\text{mid}})$  by  $E(\tilde{r}_{\text{mid}})$  and taking the inverse square root defines a length in  $r$  over which the change in slope occurs. Dividing this length by  $\tilde{r}_{\text{mid}} - r_{\text{PF}}$  yields a ratio which compares this length to the extent of the Soret regime. Thus, we define

$$\delta \equiv \frac{1}{\tilde{r}_{\text{mid}} - r_{\text{PF}}} \left[ \frac{E(\tilde{r}_{\text{mid}})}{E''(\tilde{r}_{\text{mid}})} \right]^{1/2}, \quad (4.35)$$

where

$$E''(\tilde{r}_{\text{mid}}) = \frac{(1+LS)^3}{4\sqrt{SL}(1-L^2)} \quad (4.36)$$

and  $r_{\text{PF}}$ ,  $\tilde{r}_{\text{mid}}$ , and  $E(\tilde{r}_{\text{mid}})$  are given in (4.33) and (4.37). Fig. 9 shows that  $\delta$  is smallest — i.e., the change is most abrupt — near  $S = L$  and rises steeply for  $S > 1$  and  $S < L^2$ .

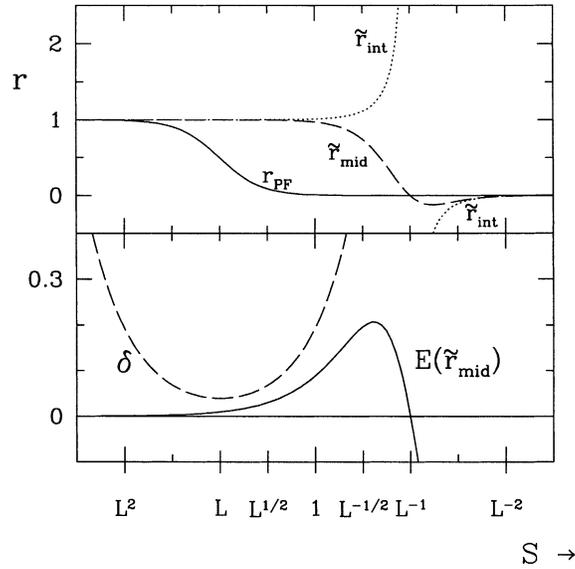


Fig. 9. Quantities determining existence of Soret-to-Rayleigh-regime transition for positive  $S$ . Upper diagram: Soret regime exists for  $r_{PF} < r < \tilde{r}_{mid}$ .  $\tilde{r}_{mid} - r_{PF}$  is positive for  $L^3 \lesssim S \lesssim 1/L$  and appreciable for  $L^{5/4} \lesssim S \lesssim L^{-3/4}$ . Lower diagram: abruptness of transition is measured by the ratio  $\delta$  (dashed curve) defined in (4.35) which remains less than 0.2 for  $L^2 \lesssim S \lesssim 1$ . Energy  $E(\tilde{r}_{mid})$  at transition (solid curve) is maximal at  $1/4L$  and nonnegligible for  $L \lesssim S \lesssim 1/L$ .

Finally, for the transition to be observed, the energy must also be sufficiently large for convection to be detected.

$$E(\tilde{r}_{mid}) = E_{mid} + \sqrt{\Delta} = \frac{-L(S+L) + \sqrt{SL}(1-L^2)}{1+LS} \quad (4.37)$$

is maximal at  $S = (1-L)^2/4L \approx 1/4L$  for  $L \ll 1$  and is negative or small outside the range  $L \lesssim S \lesssim 1/L$  (see, e.g., Fig. 6 for  $S = 0.001$ ). The interplay of counterbalancing criteria demonstrates the multiple roles played by  $S$  in this geometric analysis.

Combining all of these criteria, we finally obtain  $L \lesssim S \lesssim 1$  as the separation parameter range for the Soret-to-Rayleigh transition to be observable. The experimental observations of the Soret regime have indeed been approximately in this range. Moses and Steinberg [44], who have carried out the most extensive experimental investigation, observe the Soret regime for  $L^{0.85} < S < L^{0.27}$ . Other experimental observations are at  $S = L^{1.12}$  [29], at  $S = L^{1.15}$  [41], and at  $S = L^{0.99}$  [53]. Numerical observations should be possible over a larger range: since  $r$  and  $E$  ranges can effectively be magnified as required, the thresholds for  $\tilde{r}_{mid} - r_{PF}$  and  $E(\tilde{r}_{mid})$  are not as constraining. Platten and Chavepeyer [10] observe the Soret regime for  $L^{1.96} < S < L^{0.5}$ ; other subsequent investigators reported a transition at  $S = L^{1.13}$  [58], at  $S = L^{0.21}$  [41], at  $S = L^{1.46}$  [42], and at  $S = L^{0.5}$  [51]. For all values  $S < 1$ , the transition point  $\tilde{r}_{mid}$  is indistinguishable from the thermal threshold  $r_T = 1$  (see Fig. 9) and thus the Soret-to-Rayleigh transition is invariably described as coinciding with the onset of thermal convection in a pure fluid.

All of the references cited above have investigated the Soret problem. There, the no-flux boundary conditions on  $C$  lead to  $k = 0$  as yielding the lowest threshold for linear instability for sufficiently large  $S$ , e.g. [3,36]. This zero-wavenumber instability is sometimes invoked as part of the explanation for the weak heat transport in the Soret regime [41,44]. However, the wavenumber actually realized in full nonlinear simulations [51] for the Soret problem with rigid boundaries in a two-dimensional geometry is close to  $\pi$ , as is the wavenumber for the square patterns observed experimentally, e.g. [44,53]. Considerations of pattern selection clearly play no role in the mechanism,

we have discussed since, in the thermosolutal problem, the boundary conditions on  $T$  and  $C$  are identical and we have fixed  $k$  at  $k_{\text{crit}} = \pi/\sqrt{2}$ .

Finally, we consider the asymptotic behavior of the thermosolutal solution branches. For  $|r|$  large, we have, for the upper branch  $E_+$ ,

$$\frac{SC}{T} \approx LS \text{ with } |LS| \leq \sqrt{|LS|} \text{ if } |S| \leq \frac{1}{L}, \tag{4.38a}$$

$$\frac{E}{E_T} \approx \frac{(1+LS)r}{r-1} \approx 1+LS \approx \left\{ \frac{1}{LS} \right\} \text{ if } |S| \left\{ \begin{array}{l} \ll \\ \gg \end{array} \right\} \frac{1}{L}, \tag{4.38b}$$

$$\frac{E}{E_C} \approx \frac{(1+LS)r}{LS(r-S/L)} \approx \frac{(1+LS)}{LS} \approx \left\{ \frac{1}{LS} \right\} \text{ if } |S| \left\{ \begin{array}{l} \ll \\ \gg \end{array} \right\} \frac{1}{L}. \tag{4.38c}$$

Relation (4.38a) shows that the upper branch is thermal if  $|S| < 1/L$  and solutal if  $|S| > 1/L$ . Relations (4.38b) and (4.38c) suggest another, more stringent, criterion for classification: a steady state is thermal (solutal) if the ratio  $E/E_T(E/E_C)$  is close to one, which is true for the upper branch if  $|S| \ll (\gg) 1/L$ .

For the lower branch  $E_-$ ,

$$\frac{SC}{T} \approx -1 \text{ with } 1 \geq \sqrt{|LS|} \text{ if } |S| \leq \frac{1}{L} \tag{4.39a}$$

so the lower branch is solutal if  $|S| < 1/L$  and thermal if  $|S| > 1/L$ , by the criterion (4.29) but it fails to meet the more stringent criterion since the ratios  $E/E_T$  and  $E/E_C$  both tend to zero:

$$\frac{E}{E_T} \approx \frac{-L(L+S)/(1+LS)}{r-1}, \tag{4.39b}$$

$$\frac{E}{E_C} \approx \frac{-L(L+S)/(1+LS)}{LS(r-S/L)}. \tag{4.39c}$$

The asymptotic behavior (4.38a) and (4.39a) is illustrated in Fig. 10.

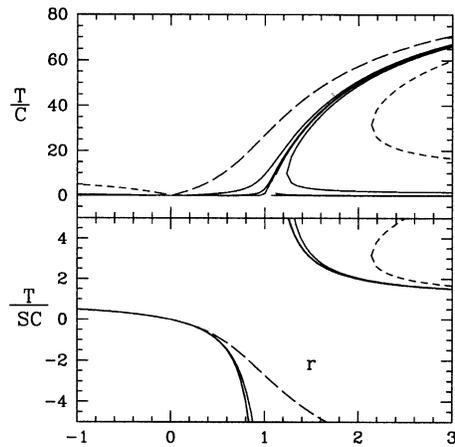


Fig. 10. Solid curves show  $T/C$  (above) and  $T/(SC)$  (below) for  $S = \pm 1, \pm 0.1, \pm 0.01$ ; long-(short-) dashed curves show  $T/C$  or  $T/(SC)$  for  $S = +(-)10$ . As  $r$  becomes large (and along the upper of the two branches for  $S$  negative),  $T/C$  approaches its asymptotic value of  $1/L$ , independent of  $S$ . For low-amplitude states existing for  $S$  negative,  $T/(SC)$  approaches  $-1$ .

## 5. Time-dependent model

The linear and nonlinear equations of Sections 3 and 4

$$\frac{d}{dt} \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r-1 & Sr \\ r & Sr-L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}, \quad (5.1a)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} r-1 & Sr \\ r & Sr-L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} - \frac{1}{2} \left( \frac{rq}{2} \right)^2 (T+SC)^2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{L} \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} \quad (5.1b)$$

each lack an essential feature of the bifurcation diagram for binary fluid convection: the linear stability problem (5.1a) cannot contain saddle-node bifurcations, whereas the nonlinear steady-state problem (5.1b) cannot describe Hopf bifurcations. We can combine (5.1a) and (5.1b) to form a single time-dependent two-variable system containing all of these features

$$\frac{d}{dt} \begin{pmatrix} T \\ C \end{pmatrix} = \begin{pmatrix} r-1 & Sr \\ r & Sr-L \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix} - \frac{1}{2} \left( \frac{rq}{2} \right)^2 (T+SC)^2 \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{L} \end{pmatrix} \begin{pmatrix} T \\ C \end{pmatrix}. \quad (5.2)$$

The conditions for the validity of this two-dimensional time-dependent nonlinear system combine those required for the two systems (5.1a) and (5.1b): large Prandtl number and small amplitudes. In addition, the spatial representation (3.2) imposes a fixed phase on the solutions. Beyond these two statements, we make no claim for the accuracy of system (5.2) as a representation of the partial differential equations (2.1).

By construction, system (5.2) undergoes a pitchfork bifurcation at  $r = r_{PF} = L/(L+S)$  and a Hopf bifurcation at  $r_H = (1+L)/(1+S)$ , both from the trivial state, and saddle-node bifurcations at  $r = r_{SN} = (1-L^2)/(1-\sqrt{-LS})^2$ . In addition to reproducing the linear stability of the conductive state and the nonlinear steady states, this system also displays an interesting phenomenon that can occur in actual binary fluid convection: the limit cycle disappears via a global bifurcation by colliding with the saddles on the lower branch of steady states. In Fig. 11, we show numerically computed phase portraits of (5.2) for  $S = -L = -0.01$ . Initial conditions are  $(T, C) = \pm(0, 0.2), \pm(0, 2.0)$ . For this value of  $S$ , the Hopf and saddle-node bifurcations occur simultaneously at  $r = 1.02$ . For  $r$  slightly less than  $r = 1.02$ , the origin is a stable spiral node; all trajectories spiral into  $(0, 0)$ . At  $r = 1.02$ , a limit cycle and two pairs of steady states (stable nodes and unstable saddles) are created. For  $r$  slightly exceeding  $r = 1.02$ , trajectories approach either the limit cycle or the stable steady state, depending on the initial condition. At  $r = r_G$  with  $1.05 < r_G < 1.06$ , the heteroclinic bifurcation destroys the limit cycle. For  $r \gtrsim 1.06$ , all trajectories terminate on one of the stable steady states. The influence of the saddles on the trajectories is clearly seen.

In Fig. 12, we show the thresholds for the bifurcations undergone by model (5.2). The accompanying bifurcation diagrams are schematic, unlike those of Figs. 2, 3, 5 and 6. There are six qualitatively different diagrams, for  $S$  in ranges  $S < -1/L$ ,  $-1/L < S < -1$ ,  $-1 < S < -L$ ,  $-L < S < -L^2$ ,  $-L^2 < S < -L^3$ , and  $-L^3 < S$ . This illustrates the advantage of the simplified model: each change in the qualitative dynamics occurs exactly at a power of  $L$ . Each steady branch is labeled with the number of eigenvectors to which it is unstable, i.e. the number of eigenvalues with positive real part. In Section 4, we characterized the pitchfork bifurcations as forward or backward facing, according to whether the new solutions created branch towards  $r > r_{PF}$  or  $r < r_{PF}$ , rather than as supercritical or subcritical. The reason for this is that the criterion we use for super or subcriticality depends on a combination of linear and nonlinear information: a bifurcation is supercritical if the new solutions branch in the direction of increasing instability of the parent branch. In this sense, the pitchfork bifurcations are supercritical for all cases except  $-L^2 < S < -L^3$ , despite being backward-facing for the five cases  $S < -L^3$ . For the three cases  $S < -1/L$ ,  $-1/L < S < -1$  and  $-1 < S < -L$ , a real eigenvalue becomes and remains positive as  $r$  is

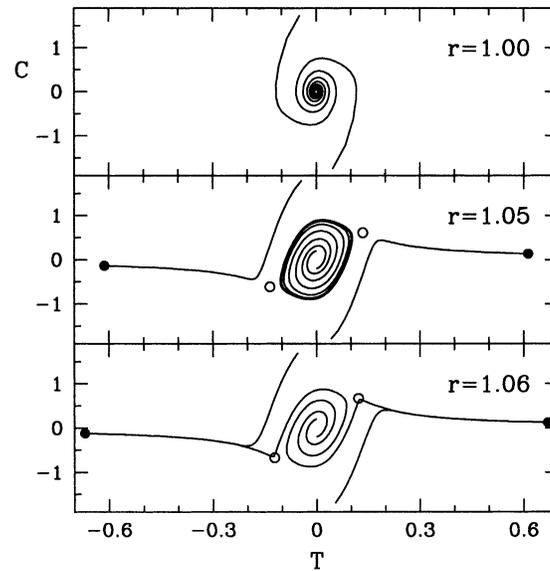


Fig. 11. Phase portraits illustrating heteroclinic bifurcation for  $S = -L = -0.01$ . For  $r = 1.00$ , all trajectories spiral into  $(T, C) = (0, 0)$ . For this value of  $S$ , the Hopf and saddle-node bifurcations occur simultaneously at  $r = 1.02$ . For  $r = 1.05$ , trajectories originating close to zero spiral out to a limit cycle, while trajectories originating sufficiently far from zero terminate on one of the stable steady states (solid dots), possibly after being deflected by one of the saddle points (hollow dots). For  $r = 1.06$ , the limit cycle has been destroyed by colliding with the saddle points in a heteroclinic bifurcation and all trajectories terminate on one of the stable steady states.

decreased below  $r_H$ . For the fourth case  $-L < S < -L^2$ , the positive eigenvalue coalesces with another positive eigenvalue to form a complex conjugate pair whose real part then reverses direction and becomes negative as  $r$  is decreased, resulting in the Hopf bifurcation at  $r_H$ , as was shown in Fig. 2 for  $S = -0.001$ . Although we have shown the Hopf bifurcation as supercritical in Fig. 12, we do not exclude the possibility of a subcritical Hopf bifurcation accompanied by a stabilizing saddle-node bifurcation.

In a two-variable system, the heteroclinic bifurcation is a natural consequence of Hopf bifurcations and pitchfork bifurcations which branch towards each other. Starting from the Hopf bifurcation and approaching the pitchfork bifurcation, the amplitude of the limit cycle increases while that of the steady branch (of saddle points) decreases. Confined to a plane, the limit cycle and steady states then collide at some intermediate value of  $r$ .

More generally, this bifurcation sequence has been studied extensively [11,16–18,26,46,47]. The approach to the heteroclinic bifurcation has been observed in numerical simulations of the full system of governing partial differential equations for thermosolutal convection [11,26], and for the Marangoni–Soret problem [58]. The existence of the heteroclinic bifurcation has been proven using the techniques of normal form reduction and amplitude expansions, and analytic expressions calculated for the limit cycles and the bifurcation [16,18,47]. In addition, simulations of the five-mode Veronis model exhibit interesting complex dynamical phenomena such as period-doubling and chaos [17]; this is not possible for a two-variable model such as (5.2). Mathematical analyses have rigorously derived reduced models of binary fluid convection, analyzed their domains of validity, and determined when period-doubling and chaos may occur [46,47].

Referring to (3.2), the limit cycle of Fig. 11 is a standing wave solution of the partial differential equations (2.1). If the phases of the various components are allowed to vary, then the Hopf bifurcation gives rise to a branch of stable traveling waves which disappears via a drift bifurcation by meeting (and stabilizing) the branch of stable

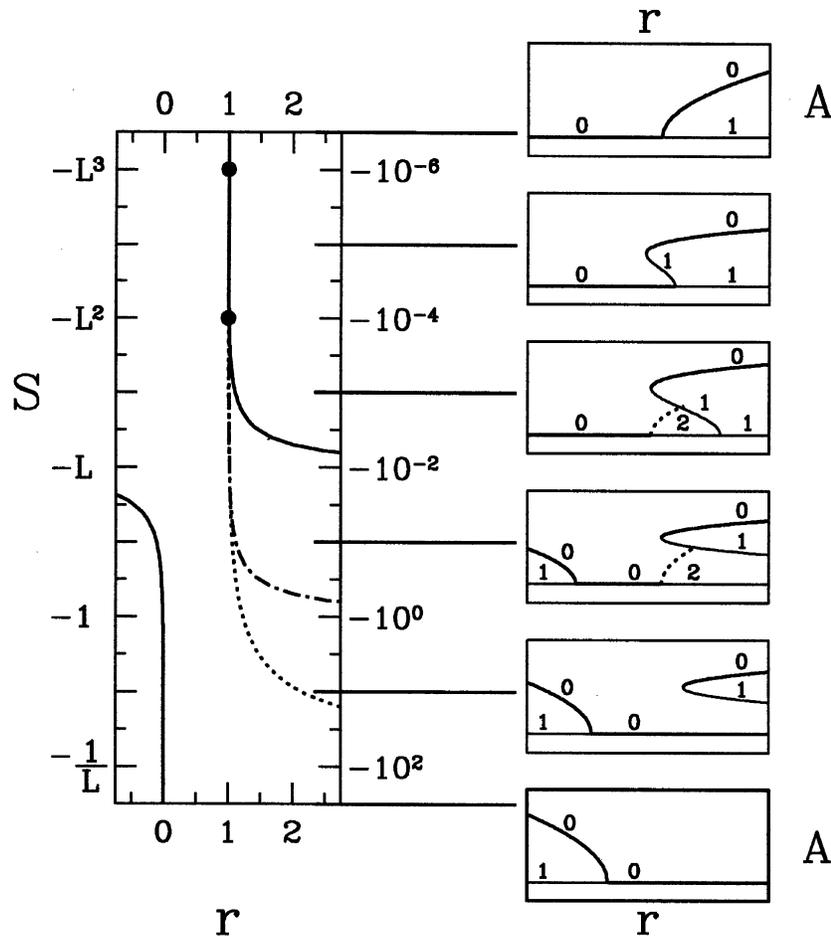


Fig. 12. Behavior of the hybrid model for negative  $S$ . Left: thresholds for negative values of  $S$  plotted on a logarithmic scale. Solid curves show the thresholds  $r_{PF} = L/(L+S)$  of pitchfork bifurcations, negative for  $S < -L$  and positive for  $S > -L$ . Dash-dotted curve indicates the thresholds  $r_H = (1+L)/(1+S)$  of Hopf bifurcations, appearing from  $r = \infty$  at  $S = -1$  and disappearing by meeting the pitchfork bifurcation curve in a codimension-two (Bogdanov) bifurcation at  $S = -L^2$  (large dot). Dotted curve indicates the thresholds  $r_{SN}$  of saddle-node bifurcations, appearing from  $r = \infty$  at  $S = -1/L$  and disappearing by meeting the pitchfork bifurcation curve in a codimension-two (degenerate pitchfork) bifurcation at  $S = -L^3$  (large dot). Right: schematic bifurcation diagrams for the six qualitatively different cases. Numbers above branches indicate the number of eigenvectors to which the branch is unstable. Stable branches (0 unstable eigenvectors) are additionally shown as heavy curves. Dotted curves indicate limit cycles. For  $S < -1/L$ , the pitchfork bifurcation is supercritical and branches backwards from negative  $r_{PF}$ . For  $-1/L < S < -1$ , additional disconnected branches are created via saddle-node bifurcations. For  $-1 < S < -L$ , a Hopf bifurcation creates a limit cycle which terminates via a global heteroclinic bifurcation. For  $-L < S < -L^2$ , the supercritical pitchfork bifurcation branches backwards from positive  $r_{PF}$  and connects the branches emanating from the saddle-node bifurcations. For  $-L^2 < S < -L^3$ , the Hopf bifurcation no longer exists and the pitchfork bifurcation is now subcritical. For  $-L^3 < S$ , the saddle-node bifurcations no longer exist and the pitchfork bifurcation is again supercritical.

steady states, as explained by Knobloch [28]. The standing waves described above continue to exist but are unstable [19,22,23,28]. In both the thermosolutal problem [30,35] and in the Soret problem [25,51], traveling waves are indeed observed in containers which are large or periodic in the horizontal direction [25,30,35,51]. The situation is in fact far more complicated: the traveling or standing wave branches may bifurcate subcritically, e.g. [11,15,28,49],

the traveling wave branch undergoes a secondary bifurcation to modulated traveling waves [28,35,43], and, at least for the Soret problem, the traveling wave branch can undergo several saddle-node bifurcations between slow and fast branches [54]. As a further complication, in larger two-dimensional domains, localized traveling waves and pulses predominate [33,50,52].

Yet, certain large-scale aspects of steady-state convection examined in Section 4 have counterparts for standing and traveling waves. Schöpf and Zimmermann [49] have found that the degenerate Hopf bifurcations for the standing and traveling wave branches are located at  $S \sim -L^2$ . Hollinger et al. [54–56] have shown that  $r$  is a simple function of the amplitude and frequency of traveling waves and that the traveling wave branch can be divided into low-amplitude Soret and high-amplitude Rayleigh regimes.

Many attempts have been made to reduce the governing-fluid dynamical equations to minimal models which describe traveling waves. The most obvious approach is to extend the five-mode free-slip Veronis model (4.6) by adding additional modes proportional to  $\sin(kx)$  to expansions (4.1), (4.2) [31,32]. However, the resulting eight-mode model proves to be singular, because the Hopf bifurcation to traveling waves in binary fluid convection with free-slip boundary conditions is always degenerate [19,22,28,30,50], as a consequence of (4.3). This means that the truncation (4.1), (4.4) is insufficient for even a qualitative description of traveling waves in the full thermosolutal problem (2.4). Numerous other models have ensued [34,39,42,43,48,49,56], using other boundary conditions, additional field variables or different theoretical approaches. Understanding the diverse aspects of traveling waves in binary fluid convection is an extremely challenging problem.

## 6. Conclusions

We have examined the well-known idealized thermosolutal problem from a variety of different perspectives.

For infinite or large Prandtl number, the linear stability problem for temperature and concentration perturbations  $(T, C)$  is governed by a  $2 \times 2$  matrix  $M$  whose entries depend linearly on the reduced Rayleigh number  $r$ , and whose eigenvalues provide the growth rates of perturbations to the motionless conductive state.

We interpret the diagonal terms of this thermosolutal matrix as growth rates of two “pure” convection problems driven exclusively by a thermal gradient or by a concentration gradient, which we term the thermal and the solutal eigenvalues. Without coupling, the thermosolutal eigenvalues merely cross transversely as  $r$  is varied through the intersection point  $r_{\text{int}}$  of the thermal and the solutal eigenvalues. Otherwise, the two eigenvalues of the coupled problem either undergo avoided crossing (the eigenvalues appear to deflect each other and remain real) or complex coalescence (the two eigenvalues join into a complex conjugate pair and then become real again). Which possibility is realized depends on the sign of the coupling: the product of off-diagonal terms. For the thermosolutal problem the coupling is proportional to, and has the same sign as, the separation parameter  $S$ .

In the equivalent language of conic sections, the eigenvalues  $\sigma + i\omega$  and reduced Rayleigh number  $r$  satisfy a second-degree equation. Hence, the sets  $(r, \sigma)$  and  $(r, \omega)$  form hyperbolas, parabolas, or ellipses, according to the values of two invariants. The first is the discriminant of the matrix responsible for advection, which here is positive (except for the single value  $S = -1$ ). The curves  $(r, \sigma)$  are thus hyperbolas and the curves  $(r, \omega)$  ellipses. The second invariant  $\Delta$  is, for the thermosolutal problem, proportional to the separation constant  $S$ .  $\Delta = 0$  is the limiting case of a hyperbola consisting of two intersecting lines, or of an ellipse whose radii are 0. As  $\Delta$  changes sign, the quadrants occupied by the hyperbola shift, and the ellipse becomes empty.

Both of these equivalent formulations underline the organizing role played by the delimiting case  $S = 0$ . Most studies of binary fluid convection treat the codimension-two point [16]  $S_* = -L^2$ , where the pitchfork and Hopf bifurcation curves meet at  $r_{\text{PF}} = r_{\text{H}} = r_* = (1 - L)^{-1}$ , as a distinguished point in the  $(S, r)$  plane, and expand

around it. Our complementary point of view focuses on  $S = 0$ , where the pure thermal and solutal eigenvalues intersect at  $r_{\text{int}} = 1 - L$ , as a different kind of distinguished point.

Turning to the nonlinear problem, the minimal model of thermosolutal convection incorporating the lowest-order nonlinear effects was first derived by Veronis [1] and has since been extensively studied. We find that the system of nonlinear equations satisfied by the steady states of the minimal model is of the special form

$$\tilde{M}v = E(v_1, v_2, \dots)v, \quad (6.1)$$

where  $\tilde{M}$  is a matrix,  $v$  a vector, and  $E$  a scalar function of the components of  $v$ . The solution of such systems reduces to that of diagonalizing a matrix and solving a single nonlinear equation of one variable. The eigenvalues of  $\tilde{M}$  are the possible values of  $E$ . Its eigenvectors are used to reduce the number of arguments of  $E$  to one by rewriting all but one component of  $v$  as multiples of the remaining component.

The five-variable system governing the steady states of the minimal thermosolutal model can be further reduced to a two-variable system in  $(T, C)$  of type (6.1) in which the scalar function  $E$  is proportional to the kinetic energy and the  $2 \times 2$  matrix  $\tilde{M}$  bears a striking resemblance to the linear stability matrix  $M$ . This leads to a remarkable analogy between the linear stability problem and the nonlinear steady state problem. The energy  $E$  also undergoes avoided crossing or complex coalescence, again according to the sign of  $S$ . The curves  $(r, E)$  are hyperbolas. Complex coalescence for  $E$  must be interpreted as the disappearance of solution branches, rather than as the onset of oscillatory behavior. Quantitative results concerning the growth rates of perturbations to the conductive state can be translated to results concerning the kinetic energy of nonlinear steady states merely by transforming  $S \rightarrow LS$  and  $L \rightarrow L^2$ , where  $L$  is the Lewis number, the ratio of solutal to thermal diffusivities.

For some results — the pitchfork bifurcation  $r_{\text{PF}}$  — the analogy leaves the linear result unchanged. This reflects the bifurcation-theoretic fact that a change in sign in eigenvalue signals a bifurcation, i.e. an intersection between solution branches. For other results — the Hopf bifurcation  $r_{\text{H}}$  — the analogy is valid, but not meaningful for the nonlinear problem. For yet other results — the point at which eigenvalues become complex  $r_+$  and the codimension-two point  $(S_*, r_*)$  — the nonlinear analogs are both significant and different from the linear phenomena. In particular, for negative  $S$ , the analogy indicates that there is a fundamental relationship between the onset of oscillation via the complex coalescence at  $r_+$  of two real eigenvalues in the linear problem, and the termination of two finite-amplitude solution branches via saddle-node bifurcation at  $\tilde{r}_+ = r_{\text{SN}}$  for the nonlinear problem. As a corollary, there also exists a fundamental relationship between the merging of the pitchfork with the Hopf bifurcation in the codimension-two point (Bogdanov bifurcation) at  $(S_*, r_*)$  for the linear problem, and the merging of the pitchfork with the saddle-node bifurcation in a different kind of codimension-two point (degenerate pitchfork) at  $(\tilde{S}_*, \tilde{r}_*)$  for the nonlinear problem.

The relationship is a consequence of the exact analogy between the growth rates of the linear stability problem at infinite  $P$  and the energies of the nonlinear steady states of the minimal five-mode Veronis model both for the thermosolutal problem with imposed horizontally periodicity and free-slip vertical boundaries. We would not expect this analogy to persist when any of these assumptions are relaxed, i.e. when the boundary conditions are realistic, when the concentration gradient occurs via the Soret effect, when the amplitudes or the Prandtl number are moderate rather than infinitesimal or infinite. Yet, evidence [5,7,20,28,49,56] suggests that the analogy must hold at least approximately, since the leading behavior of the two codimension-two points,  $S_* \sim -L^2$  and  $\tilde{S}_* \sim -L^3$ , and the domains of existence of the Hopf and saddle-node bifurcations continue to be related in approximately the same way as for the idealized thermosolutal problem. This indicates that the analogy between growth rates and energies of steady states could be a fundamental unifying feature of double-diffusive problems.

Another provocative feature of binary fluid convection is the fact that the Rayleigh number is a single-valued function of any one of the following variables: growth rate, steady state energy, traveling wave amplitude and frequency, i.e. for any of these variables, each value is achieved at most once when varying  $r$ . We have explained this dependence for the growth rate and energy by the way in which  $r$  enters the problem via advection. Hollinger

et al. [55,56] provide a related explanation invoking the reduction of the velocity field to one mode and the resulting simplification of the nonlinear terms.

Finally, we have proposed a classification of eigenvalues and of steady states as primarily thermal or primarily solutal, based on their proximity to the eigenvalues or steady states of the pure thermal and pure solutal problems and on the relative proportions of solutal and thermal contributions to the buoyancy force. For many  $S$  values of interest, the coupling term is small in the vicinity of the bifurcations, and so the convection threshold is very close to that of pure thermal or pure solutal convection. The classification is particularly useful for the nonlinear problem for positive  $S$ . We have shown that the relatively abrupt transition between small amplitude and large amplitude convection called the Soret and Rayleigh regimes [24,27,29,41,44,51,53] corresponds to the change in slope seen as a hyperbola adheres to first one and then the other asymptote as  $r$  is increased. This transition may be masked, because it occurs in a regime not corresponding to a real steady state, or muted, because it is too gradual. We have been able to give precise conditions under which the transition from Soret-to-Rayleigh regimes can be observed.

We emphasize that our goal has not been to reproduce all of the spatio-temporal dynamics of double-diffusive convection, nor even to investigate the temporal dynamics of the five-mode Veronis model. Indeed, these goals have been admirably pursued and accomplished in previous research using other approaches, e.g. [17,28,43,46,47,49,51,55,56]. Nor can this purpose be accomplished by the idealized two-variable models, we have investigated. Instead, our goal has been to extract certain universal large-scale features of double-diffusive convection in as simple a context as possible, and to re-examine these features in light of the avoided crossing/complex coalescence dichotomy and the linear/nonlinear analogy that we have put forth.

Convection in binary fluids has previously provided a testbed for the discovery and realization of many fascinating phenomena in dynamical systems. Our hope is that these new perspectives continue this tradition.

## Acknowledgements

I am very grateful to Alain Bergeon and Daniel Henry for introducing me to the mysteries of binary fluid convection. I am indebted to Edgar Knobloch for his interest and encouragement and to Fritz Busse for suggesting the infinite Prandtl number limit. I also thank Dwight Barkley, John Guckenheimer, Manfred Lücke, Ehouarn Millour, Hermann Riecke, and Alastair Rucklidge for helpful discussions and references.

## Appendix A. Finite Prandtl number model

Although the results of Section 4 of concerning the nonlinear steady states are independent of Prandtl number  $P$ , those of Section 3 concerning the linear stability problem are derived by taking  $P$  to be infinite. In this appendix, we describe our reduction of the thermosolutal linear stability problem to a  $2 \times 2$  matrix in the case when  $P$  is finite. The key step in our interpretation is to decompose the velocity field into “thermal” and “solutal” velocity fields induced by the thermal and concentration gradients, with vertical components  $\hat{w}_T$  and  $\hat{w}_C$ , respectively. Referring to Eq. (3.1), the linearized equations governing this augmented set of fields are

$$\partial_t \hat{T} = \hat{w}_T + \hat{w}_C + q^{-2} \nabla^2 \hat{T}, \quad (\text{A.1a})$$

$$\partial_t q^{-2} \nabla^2 \hat{w}_T = Prk^{-2} \partial_x^2 \hat{T} + Pq^{-4} \nabla^4 \hat{w}_T, \quad (\text{A.1b})$$

$$\partial_t \hat{C} = \hat{w}_T + \hat{w}_C + Lq^{-2} \nabla^2 \hat{C}, \quad (\text{A.1c})$$

$$\partial_t q^{-2} \nabla^2 \hat{w}_C = PSrk^{-2} \partial_x^2 \hat{C} + Pq^{-4} \nabla^4 \hat{w}_C. \quad (\text{A.1d})$$

Using the spatial and temporal dependence and notation defined in (3.3), (A.1) becomes

$$\mu \begin{pmatrix} T \\ w_T \\ C \\ w_C \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 1 \\ Pr & -P & 0 & 0 \\ 0 & 1 & -L & 1 \\ 0 & 0 & PSr & -P \end{pmatrix} \begin{pmatrix} T \\ w_T \\ C \\ w_C \end{pmatrix}. \quad (\text{A.2})$$

This  $4 \times 4$  system has exactly the same eigenvectors and eigenvalues as system (3.4) with the additional eigenvector  $(T \ w_T \ C \ w_C) = (0 \ 1 \ 0 \ -1)$  and eigenvalue  $\mu = -P$ .

Note that the temperature and concentration fields are each advected by both the “thermal velocity” and the “solutorial velocity”. It is this cross-advection which couples the thermal and solutorial problem. Neglecting it leads to the decoupled thermal and solutorial problems discussed below.

### A.1. Thermal problem

The upper left  $2 \times 2$  submatrix of (A.2) describes the onset of thermal convection in a simple fluid of finite Prandtl number. This is perhaps the prototypical problem in hydrodynamic stability theory (e.g. [13–15]). The corresponding thermal eigenvalues satisfy

$$\mu_T \begin{pmatrix} T \\ w_T \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ Pr & -P \end{pmatrix} \begin{pmatrix} T \\ w_T \end{pmatrix}, \quad (\text{A.3})$$

$$\mu_{T\pm} = \left[ -\left(\frac{P+1}{2}\right) \pm \sqrt{\left(\frac{P-1}{2}\right)^2 + Pr} \right] \quad (\text{A.4})$$

and are plotted in Fig. 13.

Fig. 13 shows that for  $r < -(P-1)^2/4P$ , eigenvalues are complex; perturbations to the conductive state oscillate as they decay. (In the terms introduced in Appendix B, the discriminant of the  $r$ -dependent component of (A.3) is

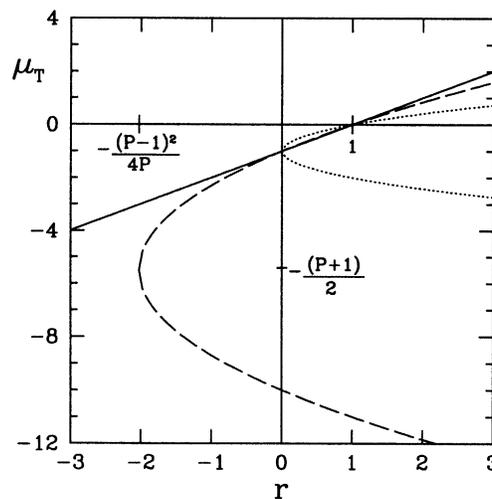


Fig. 13. Eigenvalues of pure thermal problem for  $L = 0.01$ . Dotted and dashed curves show  $\mu_T$  for  $P = 1$  and  $P = 10$ , respectively. Solid curve shows  $\sigma_T$ , valid for  $P = \infty$ , and also  $\mu_T$  for  $P = 100$ , from which it is indistinguishable in this range. Coordinates of vertex are shown for  $P = 10$ .

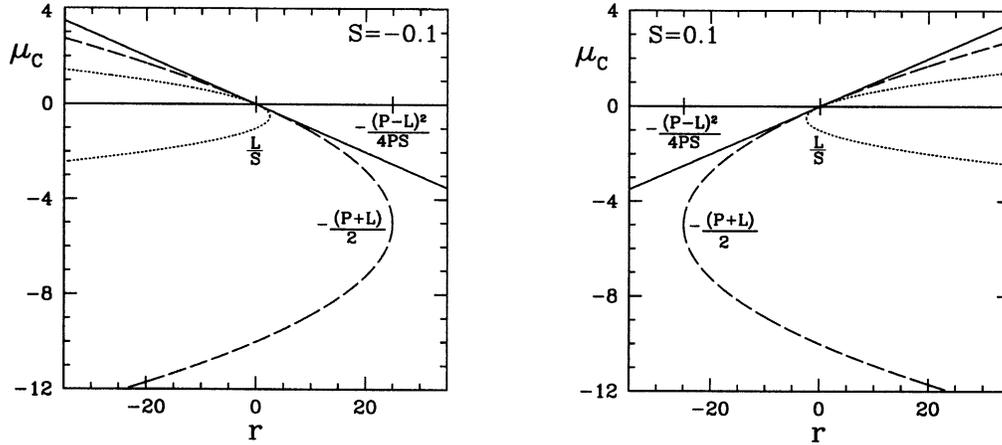


Fig. 14. Eigenvalues of pure solutal problem for  $L = 0.01$  and  $S = -0.1$  (left) and  $S = 0.1$  (right). Dotted and dashed curves shows  $\mu_C$  for  $P = 1$  and  $P = 10$ , respectively. Solid curve shows  $\mu_C$  for  $P = 100$ , which is indistinguishable from  $\sigma_C$  valid for  $P = \infty$ . Coordinates of vertex are shown for  $P = 10$ .

zero, so the curves of real values and imaginary parts of  $\mu_{T\pm}$  are both parabolas.) We consider only the regime in which the eigenvalues are real. Since we will only require the larger of the two eigenvalues, we will write  $\mu_T \equiv \mu_{T+}$ . For  $P$  large,  $\mu_T$  becomes  $\sigma_T$  of (3.8a), as expected. The threshold of  $\mu_T$  is  $r_T = 1$ , that of  $\sigma_T$ . Indeed, as is well known, the threshold of convection is independent of  $P$ . The slope of  $\mu_T$  at threshold is  $P/(P + 1)$ , which also approaches that of  $\sigma_T$  for  $P$  large. We will also require the normalized right and left eigenvectors corresponding to  $\mu_T$

$$\mathcal{T}^R \equiv \begin{pmatrix} T^R \\ w_T^R \end{pmatrix} = \frac{1}{N_T} \begin{pmatrix} 1 \\ 1 + \mu_T \end{pmatrix}, \quad (\text{A.5a})$$

$$\mathcal{T}^L \equiv (T^L \quad w_T^L) = \frac{1}{N_T} (P + \mu_T \quad 1), \quad (\text{A.5b})$$

where  $N_T^2 \equiv 2\sqrt{(\frac{1}{2}(P - 1))^2 + Pr}$ . Thus

$$\mathcal{T}^L M_T \mathcal{T}^R = \mu_T, \quad (\text{A.5c})$$

where  $M_T$  is the pure thermal matrix in (A.3).

## A.2. Solutal problem

The pure solutal problem, described by the lower right  $2 \times 2$  submatrix of (A.2) is completely analogous to the thermal case, with the inclusion of the Lewis number  $L$  and the separation parameter  $S$

$$\mu_C \begin{pmatrix} C \\ w_C \end{pmatrix} = \begin{pmatrix} -L & 1 \\ PSr & -P \end{pmatrix} \begin{pmatrix} C \\ w_C \end{pmatrix}. \quad (\text{A.6})$$

The solutal eigenvalues, plotted in Fig. 14, are

$$\mu_{C\pm} = \left[ -\left(\frac{P+L}{2}\right) \pm \sqrt{\left(\frac{P-L}{2}\right)^2 + PSr} \right]. \quad (\text{A.7})$$

We again require only the larger of the two eigenvalues  $\mu_C \equiv \mu_{C+}$ , which approaches  $\sigma_C$  for large  $P$ . The onset of convection occurs at  $r_C = L/S$ , at which  $\mu_C$  has slope  $PS/(P+L)$ . We again consider only the regime in which the eigenvalues are real, i.e.  $(\frac{1}{2}(L-P))^2 + PSr > 0$ .

The corresponding normalized right and left solutal eigenvectors are

$$\mathcal{C}^R \equiv \begin{pmatrix} C^R \\ w_C^R \end{pmatrix} = \frac{1}{N_C} \begin{pmatrix} 1 \\ L + \mu_C \end{pmatrix}, \quad (\text{A.8a})$$

$$\mathcal{C}^L \equiv (C^L \quad w_C^L) = \frac{1}{N_C} (P + \mu_C \quad 1), \quad (\text{A.8b})$$

where  $N_C^2 \equiv 2\sqrt{(\frac{1}{2}(P-L))^2 + PSr}$ . Thus

$$\mathcal{C}^L M_C \mathcal{C}^R = \mu_C, \quad (\text{A.8c})$$

where  $M_C$  is the pure solutal matrix in (A.6).

### A.3. Thermosolutal coupling

We now project the  $4 \times 4$  thermosolutal problem onto the most unstable thermal and solutal modes to form the  $2 \times 2$  matrix which constitutes our approximation. We do so by multiplying the matrix of (A.2) by left and right eigenvectors as follows:

$$\left( \begin{array}{cc|cc} \mathcal{T}^L & w_T^L & 0 & 0 \\ 0 & 0 & C^L & w_C^L \end{array} \right) \left( \begin{array}{cc|cc} -1 & 1 & 0 & 1 \\ Pr & -P & 0 & 0 \\ 0 & 1 & -L & 1 \\ 0 & 0 & PSr & -P \end{array} \right) \left( \begin{array}{c|c} \mathcal{T}^R & 0 \\ w_T^R & 0 \\ 0 & C^R \\ 0 & w_C^R \end{array} \right) \quad (\text{A.9a})$$

$$= \begin{pmatrix} \mathcal{T}^L & 0 \\ 0 & C^L \end{pmatrix} \begin{pmatrix} M_T & B \\ \Gamma & M_C \end{pmatrix} \begin{pmatrix} \mathcal{T}^R & 0 \\ 0 & C^R \end{pmatrix} \quad (\text{A.9b})$$

$$= \begin{pmatrix} \mathcal{T}^L M_T \mathcal{T}^R & \mathcal{T}^L B C^R \\ C^L \Gamma \mathcal{T}^R & C^L M_C C^R \end{pmatrix} \quad (\text{A.9c})$$

$$= \begin{pmatrix} \mu_T & \beta \\ \gamma & \mu_C \end{pmatrix} \quad (\text{A.9d})$$

In (A.9b) and (A.9c),  $M_T$ ,  $M_C$  are the  $2 \times 2$  pure thermal and solutal matrices and  $\mathcal{T}^L$ ,  $\mathcal{T}^R$ ,  $C^L$ ,  $C^R$  the corresponding left and right eigenvectors defined in (A.5), (A.8).  $B$ ,  $\Gamma$  are the  $2 \times 2$  off-diagonal submatrices in (A.9a). The calculation of the off-diagonal elements  $\beta$ ,  $\gamma$  in (A.9d) is tedious but straightforward. Their product is

$$\beta\gamma = \frac{P^2 S r^2}{4\sqrt{(\frac{1}{2}(P-1))^2 + Pr}(\frac{1}{2}(P-L))^2 + PSr)}. \quad (\text{A.10})$$

This expression is not singular in the regime we consider here; the assumption that the eigenvalues  $\mu_T$ ,  $\mu_C$  of the thermal and the solutal problems be real requires that both factors inside the square root be positive. The coupling (A.10), plotted in Fig. 15 for  $P = 10$ , reduces to the far simpler coupling  $Sr^2$  of (3.8c) in the limit of  $P = \infty$  and shares its salient feature: its sign is that of  $S$ , leading to avoided crossing if  $S > 0$  and complex coalescence if  $S < 0$ .

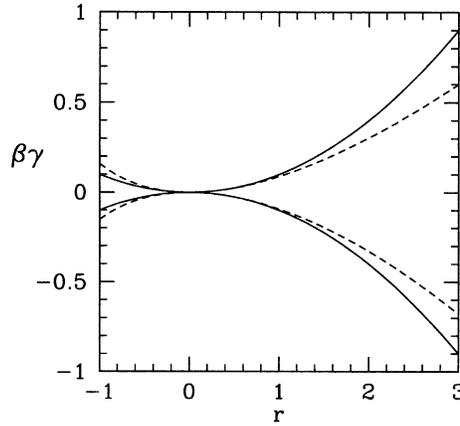


Fig. 15. Product  $\beta\gamma$  of off-diagonal terms. Dashed curves show expression (A.10) for  $P = 10$ , solid curves show  $Sr^2$ , valid for  $P = \infty$ . Positive values correspond to  $S = 0.1$ , negative to  $S = -0.1$ .

The eigenvalues of the  $2 \times 2$  approximate matrix are

$$\mu_{\pm} = \frac{\mu_T + \mu_C}{2} \pm \sqrt{\left(\frac{\mu_T - \mu_C}{2}\right)^2 + \beta\gamma} \tag{A.11}$$

with  $\mu_T = \mu_{T+}$ ,  $\mu_C = \mu_{C+}$ , and  $\beta\gamma$  given by (A.4), (A.7) and (A.10). In Figs. 16 and 17, we compare results from:

- the  $2 \times 2$  approximate matrix (A.9d) whose eigenvalues  $\mu_{\pm}$  are given by (A.11) for  $P = 10$  (dashed curves);
- the  $3 \times 3$  exact matrix (3.4) for  $P = 10$  (dotted curves);
- the  $2 \times 2$  matrix (3.6) whose eigenvalues  $\sigma_{\pm}$  are given by (3.11) for  $P = \infty$  (solid curves).

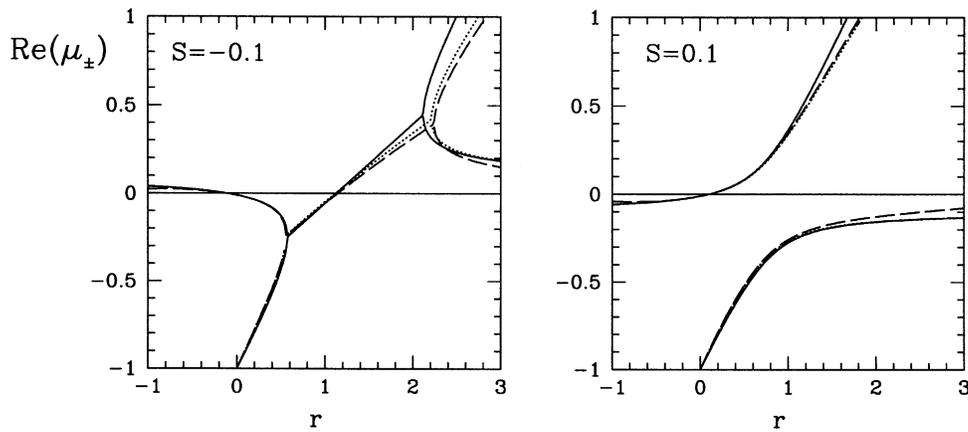


Fig. 16. Real parts of the eigenvalues  $\mu_{\pm}$  of the approximate  $2 \times 2$  matrix (dashed curves) for  $P = 10$  and for  $S = -0.1$  (left) and  $S = 0.1$  (right). Shown for comparison are the real parts of the eigenvalues of the exact  $3 \times 3$  matrix (dotted curves) for  $P = 10$  and of the eigenvalues  $\sigma_{\pm}$  (solid curves) for  $P = \infty$ .

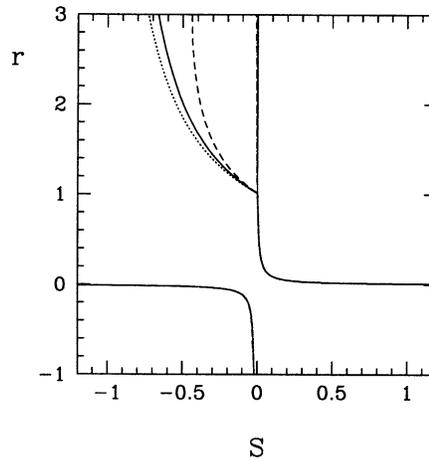


Fig. 17. Thresholds for pitchfork and Hopf bifurcations. Dashed curves are thresholds calculated from the approximate  $2 \times 2$  matrix for  $P = 10$ . Dotted curves are thresholds calculated from the exact  $3 \times 3$  matrix for  $P = 10$ . Solid curves are thresholds for  $P = \infty$ . The three thresholds for the pitchfork bifurcations cannot be distinguished; all lie on solid curves.

Specifically, in Fig. 16 we compare the real parts of the eigenvalues (A.11) with those of (3.11) and with those of the two eigenvalues of (3.4) with largest real part. For the parameter values plotted,  $P = 10$  and  $S = \pm 0.1$ , all three expressions give very similar results. For both  $S = \pm 0.1$ ,  $\mu_+$  gives a slightly better approximation of the  $3 \times 3$  eigenvalue than  $\sigma_+$ , but  $\sigma_-$  is slightly more accurate than  $\mu_-$ .

In Fig. 17, we compare the thresholds for pitchfork and Hopf bifurcations derived from these three matrices. The thresholds for the pitchfork bifurcations are so close as to be indistinguishable on the figure, whereas the Hopf bifurcation threshold is overestimated by the  $2 \times 2$  approximate matrix.

Aside from these quantitative comparisons, Figs. 16 and 17 demonstrate that the exact finite Prandtl number linear stability problem and our reduced model both exhibit the essential qualitative features of thermosolutal convection: complex coalescence for  $S < 0$  and avoided crossing for  $S > 0$ ; pitchfork, Hopf, and codimension-two bifurcations.

## Appendix B. Conic sections and eigenvalues

Consider a  $2 \times 2$  matrix whose elements depend linearly on a parameter  $r$

$$M = M_0 + rM_1, \quad (B.1)$$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} + r \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}.$$

We wish to describe the dependence of the eigenvalues of  $M$  on  $r$ . The equation obeyed by the eigenvalues  $\lambda(r)$  is

$$\lambda^2 - T\lambda + \text{Det} = 0. \quad (B.2)$$

Here, and throughout this appendix,  $T$  refers to the trace and not to the temperature deviation;  $\text{Det}$  is the determinant of  $M$ . We have

$$T = (\alpha_1 + \delta_1)r + \alpha_0 + \delta_0 = T_1r + T_0, \quad (B.3)$$

$$\text{Det} = (\alpha_1 \delta_1 - \beta_1 \gamma_1) r^2 + (\alpha_0 \delta_1 + \alpha_1 \delta_0 - \beta_0 \gamma_1 - \beta_1 \gamma_0) r + \alpha_0 \delta_0 - \beta_0 \gamma_0 \quad (\text{B.4a})$$

$$= \text{Det}_1 r^2 + 2 \text{Det}_{1/2} r + \text{Det}_0, \quad (\text{B.4b})$$

where  $T_0, T_1, \text{Det}_0, \text{Det}_1$  are the traces and determinants of the matrices  $M_0$  and  $M_1$  in (B.1) and  $\text{Det}_{1/2}$  is defined by (B.4a) and (B.4b). The eigenvalues of  $M$  are complex where the discriminant  $\text{Disc}$  is negative, where

$$\text{Disc} \equiv T^2 - 4 \text{Det} = (T_1^2 - 4 \text{Det}_1) r^2 + 2(T_0 T_1 - 4 \text{Det}_{1/2}) r + (T_0^2 - 4 \text{Det}_0) \quad (\text{B.5a})$$

$$= \text{Disc}_1 r^2 + 2 \text{Disc}_{1/2} r + \text{Disc}_0. \quad (\text{B.5b})$$

Here  $\text{Disc}_0, \text{Disc}_1$  are the discriminants of  $M_0, M_1$  and  $\text{Disc}_{1/2}$  is defined by (B.5a) and (B.5b). Whether and where  $\text{Disc}$  is negative is in turn seen from (B.5b) to be determined by  $\text{Disc}_1$  and by

$$\Delta \equiv \frac{\text{Disc}_{1/2}^2 - \text{Disc}_1 \text{Disc}_0}{4 \text{Disc}_1}. \quad (\text{B.6})$$

$\text{Disc}_1$  and  $\Delta$  are both *invariants* under translation and rotation of  $(r, \lambda)$  of Eq. (B.2); see, e.g. [61].

Writing  $\lambda = \sigma + i\omega$ , substituting (B.3) and (B.4b) into (B.2), and separating into real and imaginary parts, we obtain

$$\sigma^2 - \omega^2 - T_1 \sigma r + \text{Det}_1 r^2 - T_0 \sigma + 2 \text{Det}_{1/2} r + \text{Det}_0 = 0, \quad (\text{B.7a})$$

$$(2\sigma - T_1 r - T_0) \omega = 0. \quad (\text{B.7b})$$

According to Eq. (B.7b),

$$\text{either } \omega = 0 \quad (\text{B.8a})$$

$$\text{or } \sigma = \frac{1}{2}(T_0 + T_1 r). \quad (\text{B.8b})$$

If  $\omega = 0$ , then (B.7a) becomes

$$\sigma^2 - T_1 \sigma r + \text{Det}_1 r^2 - T_0 \sigma + 2 \text{Det}_{1/2} r + \text{Det}_0 = 0. \quad (\text{B.9})$$

If  $\omega \neq 0$ , then substituting (B.8b) into (B.7a) yields

$$\omega^2 + \frac{1}{4} \text{Disc}_1 r^2 + \frac{1}{2} \text{Disc}_{1/2} r + \frac{1}{4} \text{Disc}_0 = 0. \quad (\text{B.10})$$

The qualitative nature of the solutions to the second-degree equations (B.9) and (B.10) depends on the sign of  $\text{Disc}_1$  and  $\Delta$ . If the discriminant  $\text{Disc}_1$  is positive, then the set  $(r, \sigma)$  satisfying (B.9) is a hyperbola and the set  $(r, \omega)$  satisfying (B.10) is an ellipse, and vice versa if  $\text{Disc}_1$  is negative. If  $\text{Disc}_1$  vanishes, then both sets are parabolas. In almost all the cases we shall study,  $\text{Disc}_1$  is positive and we shall assume this from now on.

We rewrite Eq. (B.10) as

$$\omega^2 + \frac{\text{Disc}_1}{4} \left( r + \frac{\text{Disc}_{1/2}}{\text{Disc}_1} \right)^2 = -\frac{1}{4 \text{Disc}_1} (\text{Disc}_0 \text{Disc}_1 - \text{Disc}_{1/2}^2), \quad (\text{B.11a})$$

$$\omega^2 + \frac{1}{4} \text{Disc}_1 (r - r_{\text{mid}})^2 = -\Delta, \quad (\text{B.11b})$$

where  $\Delta$  is defined by (B.6) and  $r_{\text{mid}}$  by (B.11a) and (B.11b). The sign of  $\Delta$  is also crucial: if  $\Delta > 0$ , then (B.11) describes a degenerate ellipse containing no points: there are no complex eigenvalues. If  $\Delta = 0$ , then the ellipse

contains just the point  $(r_{\text{mid}}, 0)$ . If  $\Delta < 0$ , then (B.11) describes an ellipse whose two semiaxes are  $r = r_{\text{mid}}$  of length  $\sqrt{-\Delta}$  and  $\omega = 0$  of length  $\sqrt{-4\Delta/\text{Disc}_1}$ ; complex eigenvalues exist over the interval

$$|r - r_{\text{mid}}| < \sqrt{\frac{4\Delta}{\text{Disc}_1}}. \quad (\text{B.12})$$

At  $r = r_{\text{mid}}$ ,  $\omega$  attains its maximum value of  $\sqrt{-\Delta}$  and (B.8b) implies  $\sigma(r_{\text{mid}}) = \sigma_{\text{mid}}$ , where

$$\sigma_{\text{mid}} \equiv \frac{1}{2} \left( T_0 - T_1 \frac{\text{Disc}_{1/2}}{\text{Disc}_1} \right) = \frac{T_0 \text{Disc}_1 - T_1 \text{Disc}_{1/2}}{2 \text{Disc}_1} = \frac{-2T_0 \text{Det}_1 + 2T_1 \text{Det}_{1/2}}{\text{Disc}_1}. \quad (\text{B.13})$$

We now turn to the hyperbola described by (B.9) when  $\text{Disc}_1 > 0$ ; in particular we seek to characterize it by its asymptotes. The first three terms of (B.9) imply that the sum of the slopes of the asymptotes is  $T_1$  and their product is  $\text{Det}_1$ , i.e. the slopes are the eigenvalues  $\lambda_{1\pm} = (T_1 \pm \sqrt{\text{Disc}_1})/2$  of  $M_1$ . These are real and distinct by the assumption  $\text{Disc}_1 > 0$ . Some more algebra shows that (B.9) is equivalent to

$$(\sigma - \sigma_{\text{mid}} - \lambda_{1+}(r - r_{\text{mid}}))(\sigma - \sigma_{\text{mid}} - \lambda_{1-}(r - r_{\text{mid}})) = \Delta, \quad (\text{B.14})$$

where  $r_{\text{mid}}$ ,  $\Delta$ , and  $\sigma_{\text{mid}}$  are defined in (B.11a), (B.11b) and (B.13).

The asymptotes are the roots of the two factors in (B.14). They intersect at  $(r_{\text{mid}}, \sigma_{\text{mid}})$  and divide the plane into four quadrants. The magnitude of  $\Delta$  measures the distance of closest approach of the two portions of the hyperbola; its sign determines which two of the four quadrants are occupied by the hyperbola. If  $\Delta > 0$ , then one branch  $\sigma_+$  of the hyperbola lies above both asymptotes (in the sense of greater  $\sigma$ ) and the other branch  $\sigma_-$  lies below them. Each branch exists for all  $r$ . This is the situation called *avoided crossing*. When  $\Delta = 0$ , the hyperbola is degenerate and consists precisely of the two intersecting asymptotic lines. In this case, the two branches  $\sigma_+$  and  $\sigma_-$  can be considered to either behave nonsmoothly or to exchange identities at  $r = r_{\text{mid}}$ . If  $\Delta < 0$ , then both values  $\sigma_{\pm}$  lie between the asymptotes, i.e. they both lie above one asymptote and below the other. In this case, there are no real solutions to (B.14) in the range (B.12) surrounding  $r_{\text{mid}}$ . At the endpoints of the interval in (B.12), the curves  $\sigma_{\pm}$  join, to be replaced within this interval by the single linear segment (B.8b). This is the situation we call complex coalescence.

The phenomenon of avoided crossing can be quantified by differentiating (B.14) implicitly with respect to  $r$ :

$$(\sigma' - \lambda_{1+})(\sigma - \sigma_{\text{mid}} - \lambda_{1-}(r - r_{\text{mid}})) + (\sigma' - \lambda_{1-})(\sigma - \sigma_{\text{mid}} - \lambda_{1+}(r - r_{\text{mid}})) = 0, \quad (\text{B.15a})$$

$$\sigma''(2(\sigma - \sigma_{\text{mid}}) - (\lambda_{1-} + \lambda_{1+})(r - r_{\text{mid}})) + 2(\sigma' - \lambda_{1-})(\sigma' - \lambda_{1+}) = 0, \quad (\text{B.15b})$$

$$\sigma'''(2(\sigma - \sigma_{\text{mid}}) - (\lambda_{1-} + \lambda_{1+})(r - r_{\text{mid}})) + 3\sigma''(2\sigma' - (\lambda_{1-} + \lambda_{1+})) = 0 \quad (\text{B.15c})$$

and then evaluating  $\sigma$ ,  $\sigma'$ ,  $\sigma''$  and  $\sigma'''$  successively at  $r_{\text{mid}}$ :

$$\sigma_{\pm} = \sigma_{\text{mid}} \pm \sqrt{\Delta}, \quad (\text{B.16a})$$

$$\sigma'_{\pm} = \frac{1}{2}(\lambda_{1+} + \lambda_{1-}), \quad (\text{B.16b})$$

$$\sigma''_{\pm} = \frac{(\lambda_{1+} - \lambda_{1-})^2}{\pm 4\sqrt{\Delta}}, \quad (\text{B.16c})$$

$$\sigma'''_{\pm} = 0. \quad (\text{B.16d})$$

We see from (B.16) that the change in slope undergone by  $\sigma_{\pm}$  at  $r_{\text{mid}}$  corresponds to an extremum in  $\sigma''_{\pm}$ , whose magnitude measures the abruptness of the change.

Table 1

Growth rate matrix		
$\begin{pmatrix} r-1 & rS \\ r & rS-L \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -L \end{pmatrix} + r \begin{pmatrix} 1 & S \\ 1 & S \end{pmatrix}$		
$T_0 = -(1+L)$	$\text{Det}_0 = L$	$\text{Disc}_0 = (1-L)^2$
	$2\text{Det}_{1/2} = -(S+L)$	$\text{Disc}_{1/2} = -(1-L)(1-S)$
$T_1 = 1+S$	$\text{Det}_1 = 0$	$\text{Disc}_1 = (1+S)^2$
$\Delta = S \frac{(1-L)^2}{(1+S)^2}$		
$r_{\text{mid}} = \frac{(1-L)(1-S)}{(1+S)^2} \quad \sigma_{\text{mid}} = -\frac{S+L}{1+S}$		
Ellipse : $\omega^2 + \frac{(1+S)^2}{4} \left( r - \frac{(1-L)(1-S)}{(1+S)^2} \right)^2 = -S \frac{(1-L)^2}{(1+S)^2}$		
Hyperbola : $\left( \sigma + \frac{S+L}{1+S} - (1+S) \left( r - \frac{(1-L)(1-S)}{(1+S)^2} \right) \right) \left( \sigma + \frac{S+L}{1+S} \right) = S \frac{(1-L)^2}{(1+S)^2}$		

Another fact which we shall use is that a line in the  $(r, \sigma)$  plane which is parallel, but not equal, to one of the asymptotes intersects the hyperbola in exactly one point,

We briefly discuss the exceptional case  $\text{Disc}_1 = 0$ . Eq. (B.10) for the imaginary part of the eigenvalues becomes

$$0 = \omega^2 + \frac{1}{2}\text{Disc}_{1/2}r + \frac{1}{4}\text{Disc}_0. \quad (\text{B.17})$$

Eq. (B.9) for the real part of the eigenvalues becomes

$$\begin{aligned} 0 &= (\sigma - \frac{1}{2}T_1r - \frac{1}{2}T_0)^2 + (2\text{Det}_{1/2} - \frac{1}{2}T_0T_1)r + \text{Det}_0 - \frac{1}{4}T_0^2 \\ &= (\sigma - \frac{1}{2}T_1r - \frac{1}{2}T_0)^2 - \frac{1}{2}\text{Disc}_{1/2}r - \frac{1}{4}\text{Disc}_0. \end{aligned} \quad (\text{B.18})$$

Table 2

Energy matrix		
$\begin{pmatrix} r-1 & Sr \\ Lr & L(Sr-L) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -L^2 \end{pmatrix} + r \begin{pmatrix} 1 & S \\ L & LS \end{pmatrix}$		
$T_0 = -(1+L^2)$	$\text{Det}_0 = L^2$	$\text{Disc}_0 = (1-L^2)^2$
	$2\text{Det}_{1/2} = -L(S+L)$	$\text{Disc}_{1/2} = -(1-L^2)(1-LS)$
$T_1 = 1+LS$	$\text{Det}_1 = 0$	$\text{Disc}_1 = (1+LS)^2$
$\tilde{\Delta} = SL \frac{(1-L^2)^2}{(1+LS)^2}$		
$\tilde{r}_{\text{mid}} = \frac{(1-L^2)(1-LS)}{(1+LS)^2} \quad E_{\text{mid}} = -\frac{L(S+L)}{1+LS}$		
Ellipse : $\omega^2 + \frac{(1+LS)^2}{4} \left( r - \frac{(1-L^2)(1-LS)}{(1+LS)^2} \right)^2 = -LS \frac{(1-L^2)^2}{(1+LS)^2}$		
Hyperbola : $\left( E + \frac{L(S+L)}{1+LS} - (1+LS) \left( r - \frac{(1-L^2)(1-LS)}{(1+LS)^2} \right) \right) \left( E + \frac{L(S+L)}{1+LS} \right) = SL \frac{(1-L^2)^2}{(1+LS)^2}$		

Both (B.17) and (B.18) describe parabolas. The parabola of (B.17) is oriented along the  $r$ -axis, while the axis of (B.18) is the line  $\sigma = \frac{1}{2}(T_1 r + T_0)$ . If  $T_1 = 0$ , then (B.18) is also oriented along the  $r$ -axis. The two parabolas are oriented in opposite directions: (B.17) opens towards positive values of  $r$  if  $\text{Disc}_{1/2}$  is negative and vice versa for (B.18). The vertex of both parabolas is located at  $r = -\text{Disc}_0/(2 \text{Disc}_{1/2})$  with  $\omega = 0$  for (B.17) and  $\sigma = \frac{1}{2}(-\text{Disc}_0 T_1/(2 \text{Disc}_{1/2}) + T_0)$  for (B.18).

Our treatment of binary fluid convection leads to two matrices of type (B.1), one whose eigenvalues  $\sigma$  govern the linear stability of the conductive state and the other whose eigenvalues  $E$  are the kinetic energy of nonlinear steady states.  $L$  is the Lewis number, which is necessarily positive and usually small,  $S$  is the separation parameter, which may have either sign, and  $r$  is the reduced Rayleigh number. In Tables 1 and 2, we give the quantities we have defined above for each of these two matrices.

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