The Eckhaus Instability

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The Eckhaus instability can be analyzed by means of the real Ginzburg-Landau equation:

$$\partial_t A = \mu A + \partial_{xx} A - |A|^2 A \tag{1}$$

This equation governs pattern formation in a wide variety of systems. A is considered to be the complex amplitude of a pattern, via, e.g.:

$$w(x,t) = A(x,t)e^{iq_c x} + A^*(x,t)e^{-iq_c x} \tag{2}$$

Let us begin by assuming that A is horizontally periodic, with arbitrary periodicity length. One solution to (1) is the trivial solution, zero, implying the absence of spatial structure. Another set of solutions:

$$A_Q \equiv \sqrt{\mu - Q^2}\, e^{iQx}, \quad \text{for } |Q| < q_c \tag{3}$$

describes a pattern of spatial wavenumber $q_c + Q$. These are created by primary pitchfork bifurcations from the trivial state at:

$$\mu_Q \equiv Q^2 \tag{4}$$

and exist for $\mu > \mu_Q$. The bifurcations are supercritical, meaning that, as μ is increased past μ_Q, the trivial state is destabilized and the pattern A_Q is created.

The linear stability of the patterns A_Q is governed by the equation which results from replacing A by $A_Q + e^{\lambda t} a(x)$ in (1) and neglecting terms which are nonlinear in a:

$$\lambda a = \mu a + \partial_{xx} a - 2|A|^2 a - A^2 a^* \tag{5}$$

The solutions to (5) are the eigenpairs (λ_k, a_k) with eigenvectors:

$$a_k(x) \equiv \alpha_k e^{i(Q+k)x} + \beta_k e^{i(Q-k)x}, \quad k > 0 \tag{6}$$

and:

$$a_0(x) \equiv \alpha_0 e^{iQx} \tag{7}$$

with α_k, β_k, and α_0 real. Using:

$$\begin{aligned}
|A|^2 a_k &= (\mu - Q^2)\left(\alpha_k e^{i(Q+k)x} + \beta_k e^{i(Q-k)x}\right) \\
A^2 a_k^* &= (\mu - Q^2) e^{i2Qx}\left(\alpha_k e^{-i(Q+k)x} + \beta_k e^{-i(Q-k)x}\right) \\
&= (\mu - Q^2)\left(\alpha_k e^{i(Q-k)x} + \beta_k e^{i(Q+k)x}\right)
\end{aligned}$$

we find that eigenpairs (λ_k, a_k) satisfy:

$$\lambda_k \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \mu-(Q+k)^2-2(\mu-Q^2) & -(\mu-Q^2) \\ -(\mu-Q^2) & \mu-(Q-k)^2-2(\mu-Q^2) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

leading to eigenvalues:

$$\lambda_{k\pm} = -(\mu-Q^2)-k^2 \pm \sqrt{(2Qk)^2+(\mu-Q^2)^2} \tag{8}$$

Using

$$\begin{aligned} |A|^2 a_0 &= (\mu-Q^2)\alpha_0 e^{iQx} \\ A^2 a_0^* &= (\mu-Q^2)e^{i2Qx}\alpha_0 e^{-iQx} = (\mu-Q^2)\alpha_0 e^{iQx} \end{aligned}$$

we find that the eigenvalues λ_0 are:

$$\lambda_0 = \mu-Q^2-2(\mu-Q^2)-(\mu-Q^2) = -2(\mu-Q^2) \tag{9}$$

Secondary bifurcations, changing the stability of A_Q, occur whenever one of the eigenvalues λ_0 or $\lambda_{k\pm}$ crosses zero. We see that $\lambda_0 \le 0$ for $\mu > Q^2$, i.e. whenever A_Q is a solution; it is the eigenvalue responsible for the creation of branch A_Q from the trivial state. The eigenvalue λ_{k-} is always negative. Crossings of zero by λ_{k+} are determined by solving:

$$\begin{aligned} 0 &= -(\mu-Q^2)-k^2 \pm \sqrt{(2Qk)^2+(\mu-Q^2)^2} \\ (\mu-Q^2)+k^2 &= \pm\sqrt{(2Qk)^2+(\mu-Q^2)^2} \\ ((\mu-Q^2)+k^2)^2 &= (2Qk)^2+(\mu-Q^2)^2 \\ 2(\mu-Q^2)k^2+k^4 &= (2Qk)^2 \\ \mu-Q^2+\frac{1}{2}k^2 &= 2Q^2 \\ \mu_{Qk} &\equiv 3Q^2-\frac{1}{2}k^2 \end{aligned} \tag{10}$$

Secondary pitchfork bifurcations from the branch of states A_Q occur at μ_{Qk}. These bifurcations are subcritical, meaning that, as μ is increased past successive values μ_{Qk}, the solution A_Q is stabilized against eigenvector a_k as λ_{k+} decreases through zero. Simultaneously, a mixed-mode state combining different wavenumbers is created. Since these mixed-mode states are always unstable, they are never realized by the system, but serve as *basin boundaries*, separating the domains of attraction of the periodic states A_Q.

The pattern described by A_Q is stable when *all* of its eigenvalues have become negative. For a domain of infinite length, this criterion yields the classic Eckhaus curve given by:

$$\mu_\infty = \max_{k>0}\left(3Q^2-\frac{1}{2}k^2\right) = 3Q^2 \tag{11}$$

Now consider a domain of any finite horizontal length L, with q_c and Q given in units of $2\pi/L$. Recalling (2), the periodic patterns A_Q permitted in the domain must satisfy $q_c + Q = n$, with n integer. The potentially unstable eigenvectors a_k, with $k > 0$, must also fit in the domain, so that k must satisfy $q_c + Q + k = n + k$ integer. Thus k must be a positive integer. The pattern A_Q is stable in the finite domain if:

$$\mu_{\text{finite}} = \max_{k=1,2,\ldots} \left(3Q^2 - \frac{1}{2}k^2\right) = 3Q^2 - \frac{1}{2} \tag{12}$$

Thus:

$$\mu_{\text{finite}} = \mu_\infty - \frac{1}{2} \tag{13}$$

independently of the actual size of the domain.

Figure 1 shows the finite-domain Eckhaus curves μ_{Qk} of equation (10) along which the patterns A_Q are stabilized and secondary bifurcations occur. The highest of these, corresponding to $k = 1$, is the Eckhaus curve μ_{finite} of equation (12) above which the patterns A_Q are stable in a finite domain. Also shown is the marginal stability curve μ_Q of equation (4) along which the trivial state loses stability and the primary bifurcations to patterns A_Q take place. The portions of the Eckhaus curves for $\mu < \mu_Q$ have no significance, since, where A_Q does not exist, it cannot undergo a bifurcation. These curves are universal: they do not depend on the size of the domain. Specific bifurcation points located on these curves do, however, depend on the fractional part of q_c. In the figure, we have fixed $q_c - [q_c] = -1/4$ in order to indicate the primary and secondary bifurcations by dots. The pattern which is the first to be created as μ is increased is that whose wavenumber Q_0 is closest to q_c (Q closest to 0). It is stable when it is created. The pattern which is next created has a wavenumber Q_1 which is second closest to q_c (Q closest to -1) and undergoes one restabilizing Eckhaus bifurcation. Those which are third and fourth closest (Q_2 and Q_3) to q_c (Q closest to $+1$ and to -2) undergo two and three restabilizing Eckhaus bifurcations, and so on.

The classic Eckhaus curve μ_∞ of equation (11), which is tangent to the marginal stability curve and above which patterns A_Q are stable in a domain of infinite length, is also included in the diagram. This curve cannot describe the stability of patterns in a finite domain since there exists a range of μ over which the trivial state is unstable to a wavenumber Q fitting in the domain, but the resulting pattern A_Q is below the infinite-domain Eckhaus curve and hence would also be unstable. The absence of any stable solution is inconsistent with the variational character for all values of μ of the Ginzburg-Landau equation (1), which implies that any initial condition approaches asymptotically a stable steady state. Hence the classic infinite-domain curve μ_∞ of equation (11) cannot be the Eckhaus boundary for any finite-length domain. Figure 1 remains valid for domains of any finite length.

For details and justifications of the analysis above, see references.

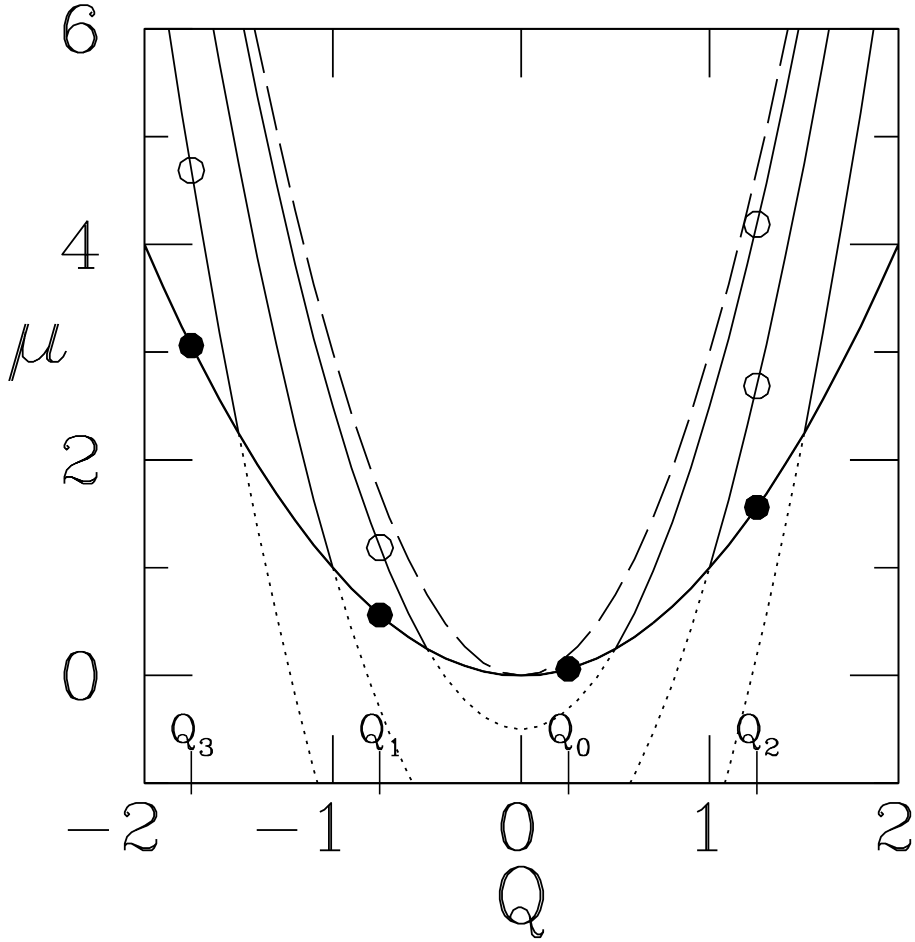

Figure 1: Stability curves. The thick parabola shows the marginal stability curve $\mu_Q = Q^2$ along which the trivial state is destabilized by primary bifurcations to periodic patterns A_Q. Thin parabolas show the finite-domain Eckhaus curves $\mu_{Qk} = 3Q^2 - k^2/2$ for $k = 1, 2, \ldots$ along which the periodic patterns are stabilized by successive secondary bifurcations to unstable mixed-mode states. The highest of these, $\mu_{\text{finite}} = \mu_{Q1} = 3Q^2 - 1/2$, is the finite-domain Eckhaus boundary above which pattern A_Q is stable. The dotted portions of the Eckhaus curves below the marginal stability curve have no significance, since states A_Q do not exist in this region. Primary and secondary bifurcations for the specific case $q_c - [q_c] = -1/4$ are shown as solid and hollow dots, respectively. The infinite-domain Eckhaus curve $\mu_\infty = 3Q^2$ is shown for contrast as a dashed curve.

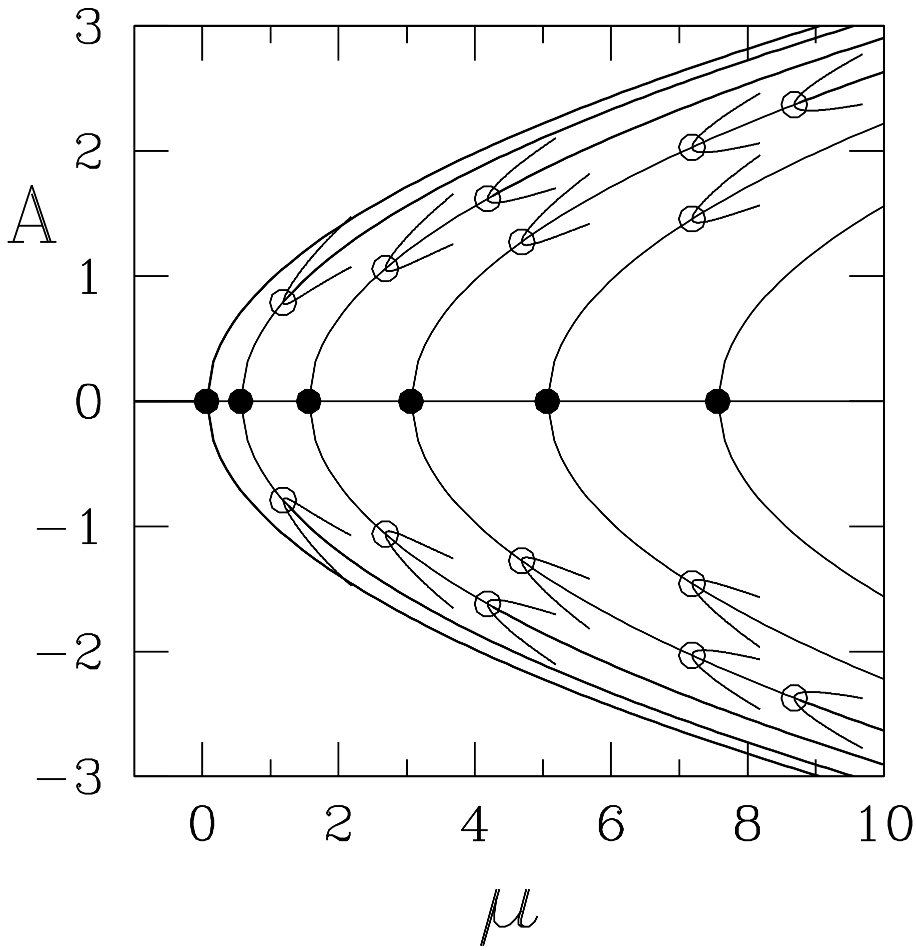

Figure 2: Bifurcation diagram. Branches with wavenumbers $Q_0, Q_1, Q_2 \cdots$ are created at successive primary pitchfork bifurcations (solid dots) as μ is increased through the values $Q_0^2, Q_1^2, Q_2^2, \cdots$. All but the first ($Q_0$) branch is unstable; each branch is restabilized by successive secondary Eckhaus bifurcations (hollow dots) at $\mu = 3Q_n - k^2$. For clarity, only the lowest-μ portions of the mixed-mode branches created at the Eckhaus bifurcations are shown. Thick curves indicate stable portions of the trivial and primary branches.

Bibliography

The analysis presented above is an condensed version of:
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The Eckhaus instability was first described in:
• W. Eckhaus, *Studies in Nonlinear Stability Theory* (Springer, 1965).

The stabilizing effect of finite boundaries was first noted in:
• G. Ahlers, D.S. Cannell, M.A. Dominguez-Lerma & R. Heinrichs, *Wavenumber selection and Eckhaus instability in Couette-Taylor flow*, Physica D **23**, 202 (1986).

See also:
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