Polar coordinates and rotating frames

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Consider a trajectory:

$$x(t)\mathbf{e}_x + y(t)\mathbf{e}_y = (x(t), y(t)) = x(t) + iy(t) = r(t)e^{i\theta(t)} \tag{1}$$

To translate from Cartesian to polar coordinates, we write:

$$\partial_t(re^{i\theta}) = (\dot{r} + ri\dot{\theta})e^{i\theta} = (u_r + iu_\theta)e^{i\theta} \tag{2a}$$

$$\begin{aligned}\partial_t^2(re^{i\theta}) &= (\ddot{r} + \dot{r}i\dot{\theta} + ri\ddot{\theta})e^{i\theta} + (\dot{r} + ri\dot{\theta})i\dot{\theta}e^{i\theta} \\ &= \left[(\ddot{r} - r\dot{\theta}^2) + i(r\ddot{\theta} + 2\dot{r}\dot{\theta})\right] e^{i\theta}\end{aligned} \tag{2b}$$

where we have defined:

$$u_r \equiv \dot{r} \qquad\qquad u_\theta \equiv r\dot{\theta} \tag{3}$$

In the absence of forces or acceleration, we have $\partial_t^2(x + iy) = \partial_t^2(re^{i\theta}) = 0$. This leads to:

$$0 = \ddot{r} - r\dot{\theta}^2 = \partial_t \dot{r} - \frac{(r\dot{\theta})^2}{r} \qquad \Longrightarrow \partial_t u_r = \frac{u_\theta^2}{r} \tag{4a}$$

$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \begin{cases} r\ddot{\theta} + \dot{r}\dot{\theta} + \dot{r}\dot{\theta} = \partial_t(r\dot{\theta}) + \dfrac{\dot{r}(r\dot{\theta})}{r} & \Longrightarrow \partial_t u_\theta = \dfrac{-u_r u_\theta}{r} \\ \dfrac{r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}}{r} = \dfrac{\partial_t(r^2\dot{\theta})}{r} & \Longrightarrow \partial_t(ru_\theta) = 0 \end{cases} \tag{4b}$$

Note the extra terms that appear in the evolution equations for u_r and u_θ, *even in the absence of forces and in a non-rotating frame*. The second version of equations (4b) can be regarded as stating that angular momentum is conserved in the absence of forces and thus of torques. Taking this into account, we can rewrite (4b) and (4a) as:

$$h \equiv r^2\dot{\theta} = ru_\theta \qquad\qquad \ddot{r} = \partial_t u_r = \frac{(ru_\theta)^2}{r^3} = \frac{h^2}{r^3} \tag{5}$$

In a rotating frame, we define θ' and $u_{\theta'}$:

$$\theta' \equiv \theta - \Omega t \implies \dot{\theta}' = \dot{\theta} - \Omega \tag{6a}$$

$$r\dot{\theta}' = r\dot{\theta} - \Omega r \implies u_{\theta'} = u_\theta - \Omega r \tag{6b}$$

We then write equations (4) in the rotating frame in terms of (r, θ') and $(u_r, u_{\theta'})$:

$$\partial_t u_r = \frac{u_\theta^2}{r} = \frac{(u_{\theta'} + \Omega r)^2}{r} = \underbrace{\frac{u_{\theta'}^2}{r}}_{\text{polar coords}} + \underbrace{\Omega^2 r}_{\text{centrifugal}} + \underbrace{2\Omega u_{\theta'}}_{\text{Coriolis}} \tag{7a}$$

$$\begin{aligned}\partial_t u_{\theta'} &= \partial_t(u_\theta - \Omega r) = \partial_t u_\theta - \Omega\,(\partial_t r) = \frac{-u_r u_\theta}{r} - \Omega u_r \\ &= \frac{-u_r(u_{\theta'} + \Omega r)}{r} - \Omega u_r = \underbrace{\frac{-u_r u_{\theta'}}{r}}_{\text{polar coords}} - \underbrace{2\Omega u_r}_{\text{Coriolis}} \Longrightarrow \partial_t(ru_{\theta'}) = -2r\Omega u_r\end{aligned} \tag{7b}$$

Using a rotating frame introduces the classic *fictitious forces*, the centrifugal and Coriolis forces, in addition to the terms that result from transforming to polar coordinates.