Time integration

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We consider implicit schemes for the linear diffusive terms of the Navier-Stokes equations. A necessary condition for numerical methods for diffusive operators is that high-wavenumber modes of the solution should decay with the wavenumber. The nonlinear term in the Navier-Stokes equation can be seen as a generator and amplifier of high wavenumber coefficients while the viscous term damps them. Time-step evolution operators differ in their capability of damping the high spatial frequencies in an evolving function. The intensity of this damping depends on the particular time integration scheme and on the way the Laplacian is evaluated. For systems naturally generating higher frequencies,the damping must be strong enough to oppose the effect of the nonlinear term.

A-stable methods are those which, when applied to the differential equation $du/dt = -qt$ whose exact solution is $u(t) = e^{-qt}u(0)$, yield numerical solutions which approach zero as $t \to \infty$. A-stable methods must be implicit and, moreover, can be at most second-order accurate in time. We consider six A-stable methods: first-order backward Euler, Crank-Nicolson (trapezoidal), backward differentiation (retarded Euler), Adams type, Lees type and two-step trapezoidal.

We present a brief and simplified analysis for a one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \nu \Delta u$$

For simplicity we represent u in the Fourier basis (*von Neumann stability analysis*). This analysis will therefore have only qualitative meaning since in this approach the effect of boundaries is neglected. This is, however, sufficient to identify general properties of the time discretization schemes discussed. We then have

$$u(x,t) = \sum_k \hat{u}(k,t)e^{ikx}$$

The evolution equation for the Fourier coefficients is

$$\frac{\partial \hat{u}}{\partial t}(k,t) = -\nu k^2 \hat{u}(k,0)$$

whose exact solution is

$$\hat{u}(k,t) = e^{-\nu k^2 t}\hat{u}(k,0)$$

Time-stepping schemes can be viewed as methods of representing the exponential $e^{-\nu k^2 t}$

A general two-step time integration scheme can be written after Beam & Warming [2] as follows:

$$(1+\xi)\hat{u}^{n+1} - (1+2\xi)\hat{u}^{n} + \xi\hat{u}^{n-1} = \underbrace{\Delta t \nu}_{\epsilon}(-k^2)\left[\theta\hat{u}^{n+1} + (1-\theta+\eta)\hat{u}^{n} - \eta\hat{u}^{n-1}\right] \quad (1)$$

where we have substituted the eigenvalues $spec(\Delta) = -k^2$ in place of the Laplace operator. For unconditional stability, ξ, θ, η must satisfy

$$\theta \geq \eta + \frac{1}{2} \qquad \xi \geq -\frac{1}{2} \qquad \xi \leq \theta + \eta - \frac{1}{2}$$

If $\eta = \xi - \theta + 1/2$ then the scheme is second-order accurate.

We wish to write a two-step iterative process (1) in a one-step form of type

$$G_1\bar{u}^{n+1} = G_0\bar{u}^{n} \quad \Rightarrow \quad \bar{u}^{n+1} = \underbrace{G_1^{-1}G_0}_{G}\,\bar{u}^{n} \quad (2)$$

where G is called the *amplification factor*. To do so we introduce an auxiliary variable $\hat{z}$ so that (1) can be written in matrix notation (2) by defining

$$\bar{u}^{n} = \begin{bmatrix} \hat{u}^{n} \\ \hat{z}^{n} \end{bmatrix} \qquad \hat{z}^{n} = \hat{u}^{n-1}$$

This makes it possible to write (1) using only two time steps $n+1$ and n:

$$\begin{aligned}(1+\xi+k^2\epsilon\theta)\hat{u}^{n+1} &= [1+2\xi-k^2\epsilon(1-\theta+\eta)]\hat{u}^{n} - (\xi+k^2\epsilon\eta)\hat{z}^{n} \\ \hat{z}^{n+1} &= \hat{u}^{n}\end{aligned} \quad (3)$$

so that (2) corresponding to (3) can be written as:

$$\underbrace{\begin{bmatrix} 1+\xi+k^2\epsilon\theta & 0 \\ 0 & 1 \end{bmatrix}}_{G_1} \underbrace{\begin{bmatrix} \hat{u}^{n+1} \\ \hat{z}^{n+1} \end{bmatrix}}_{\bar{u}^{n+1}} = \underbrace{\begin{bmatrix} 1+2\xi-k^2\epsilon(1-\theta+\eta) & k^2\epsilon\eta-\xi \\ 1 & 0 \end{bmatrix}}_{G_0} \underbrace{\begin{bmatrix} \hat{u}^{n} \\ \hat{z}^{n} \end{bmatrix}}_{\bar{u}^{n}}$$

The *amplification matrix* $G \equiv G_1^{-1}G_0$ is characterized by its eigenvalues μ_1, μ_2 which can be found from the characteristic polynomial $P(\mu)$ of G:

$$P(\mu) = (1+\xi+k^2\epsilon\theta)\mu^2 - (1+2\xi-k^2\epsilon(1-\theta+\eta))\mu - k^2\epsilon\eta + \xi$$

The values of θ, ξ, η corresponding to the time integration schemes considered are presented in table 1.

Schemes that attenuate high wavenumbers satisfy the criterion

$$\lim_{k\to\infty} \max\{|\mu_1(k)|, |\mu_2(k)|\} < 1 \quad (4)$$

Method	θ	ξ	η	Order	μ_1, μ_2	$\lim_{k\to\infty} \lvert\max\{\mu_1,\mu_2\}\rvert$
Backward Euler	1	0	0	1	$0, \frac{1}{1+k^2\epsilon}$	0
Crank-Nicolson	1/2	0	0	2	$0, \frac{2-k^2\epsilon}{2+k^2\epsilon}$	1
Backward differentiation	1	1/2	0	2	$\frac{2\pm\sqrt{1-2k^2\epsilon}}{3+2k^2\epsilon}$	0
Adams type	3/4	0	−1/4	2	$\frac{2\pm\sqrt{4-4k^2\epsilon-3k^4\epsilon^2}}{4+3k^2\epsilon}$	$\sqrt{3}/3$
Lees type	1/3	−1/2	−1/3	2	$-\frac{k^2\epsilon\pm\sqrt{-3k^4\epsilon^2+9}}{3+2k^2\epsilon}$	1
Two-step trapezoidal	1/2	−1/2	−1/2	2	$\pm\frac{\sqrt{1-k^4\epsilon^2}}{1+k^2\epsilon}$	1

Table 1: List of A-stable one- and two-step schemes (based on Beam & Warming [3])

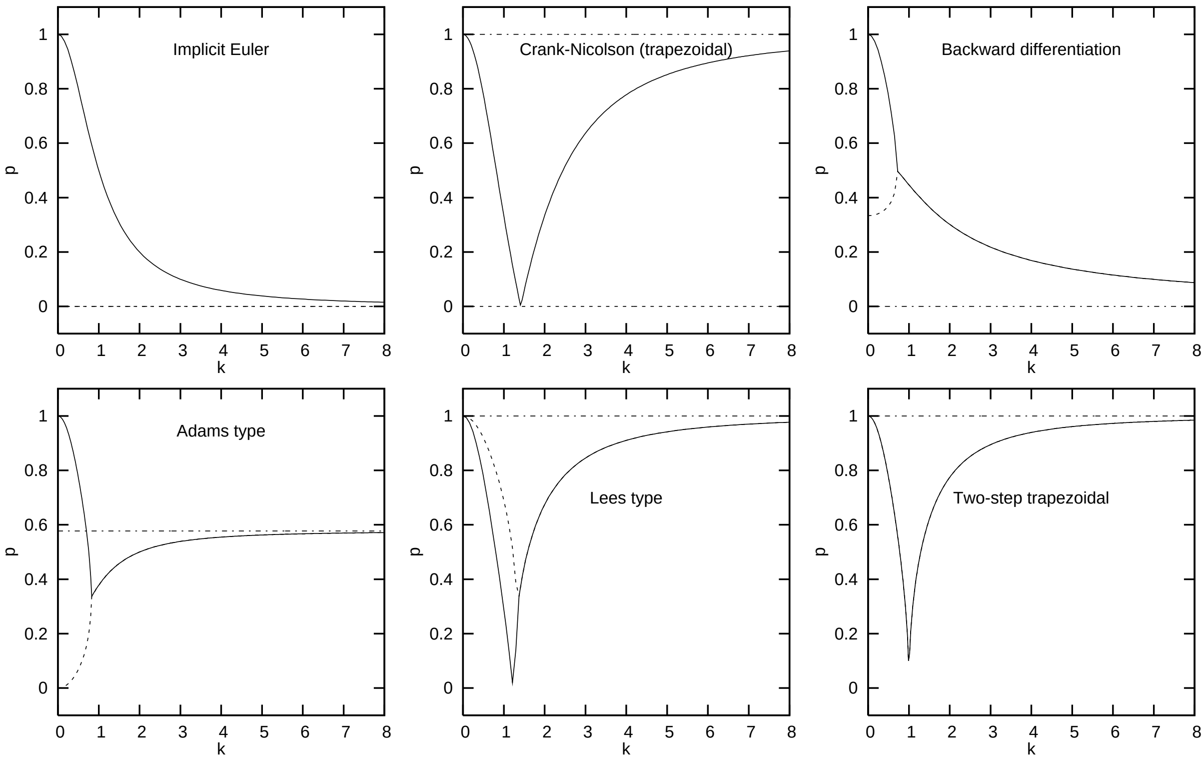

Figure 1: Plots of p = { $|\mu_1(k)|$ – solid line, $|\mu_2(k)|$ – dashed line, $\lim_{k\to\infty} \max\{|\mu_1(k)|, |\mu_2(k)|\}$ – dot-dash line } for A-stable schemes.

Asymptotic values for all the schemes as well as an expression for μ_1, μ_2 are presented in table 1. Only three methods have the property (4): backward Euler, backward differentiation and Adams type. The behavior of μ_1, μ_2 as of the functions of k for fixed $\epsilon = 1$ is presented in figure 1 for all schemes considered. The first-order backward Euler scheme attenuates high wavenumbers faster than all other methods. If second order accuracy is needed then the backward differentiation represent a natural choice and the Adams-type scheme can possibly be considered for situations where weaker damping of high-wavenumber modes is sufficient.

The nonlinear term can be treated by a second order explicit Adams-Bashforth scheme:

$$\mathbf{s}^{n+1} = \frac{1}{2}\left(3\mathbf{s}^{n} - \mathbf{s}^{n-1}\right)$$

Tests performed with the Crank-Nicolson method for von Kármán flow confirm that for this scheme, nonlinear simulation was unstable even for quite small Reynolds numbers $Re \approx 300$. This behavior was also observed for the von Kármán flow by Speetjens [7] and Lopez *et al.* [5] and also by Marcus [6] in the Taylor-Couette configuration. Choosing smaller values of the time-step helps very little in this situation since $\Delta t \sim k^{-2}$ for constant ϵ and the maximal time-step quickly becomes very small for higher spatial resolutions. The backward Euler/Adams-Bashforth time integration scheme for the Navier-Stokes equation can be written as

$$(\mathbb{1} - \epsilon\Delta)\mathbf{u}^{n+1} = \mathbf{u}^{n} + \frac{\Delta t}{2}\left(3\mathbf{s}^{n} - \mathbf{s}^{n-1}\right)$$

References

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