Net pressure gradient and net flux in periodic directions

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In a periodic domain, the pressure gradient (rather than the pressure) must be periodic in each periodic direction. This allows for a term which is linear in each periodic direction. For channel flows (whether Poiseuille or Couette), the assumption of a periodic pressure gradient leads to:

$$\begin{aligned} -p(x,y,z) &= ax + bz + f(x,y,z) \\ \Longrightarrow -p_x(x,y,z) &= a + f_x(x,y,z) \\ \Longrightarrow -p_y(x,y,z) &= f_y(x,y,z) \\ \Longrightarrow -p_z(x,y,z) &= b + f_z(x,y,z) \end{aligned}$$

The constant terms above can be exactly counter-balanced by viscous forces if parabolic profiles are added as follows:

$$u = Re\, a(1-y^2)/2 \Longrightarrow \frac{1}{Re}\nabla^2 u = -a$$
$$w = Re\, b(1-y^2)/2 \Longrightarrow \frac{1}{Re}\nabla^2 w = -b$$

The parabolic profiles satisfy homogeneous boundary conditions at $y = \pm 1$ and are associated with net fluxes in the x and z directions:

$$\int_{-1}^{1} u(y)dy = -Re\, a \left[y^3/3\right]_{-1}^{1} = -2Re\, a/3$$
$$\int_{-1}^{1} w(y)dy = -Re\, b \left[y^3/3\right]_{-1}^{1} = -2Re\, b/3$$

Hence, a parabolic profile can be added to a velocity field in any periodic direction if the pressure is changed by a compensating linear term, the net pressure gradient, in that periodic direction. The resulting velocity field will still satisfy both the Navier-Stokes equations and the boundary conditions.

This indeterminacy is lifted by imposing the value of either the net pressure gradient or else the net flux (or, less commonly, some linear combination of the two). The necessity for making this choice is clear and well known when the net pressure gradient or the net flux must be set to a finite value, i.e. in the streamwise direction for Poiseuille flow. However, imposing the net pressure gradient or the net flux is necessary in *every* periodic direction – i.e. the spanwise direction for Poiseuille flow, and both the streamwise and spanwise directions for Couette flow – in order to lift the indeterminacy mentioned above. Imposing a net flux is somewhat more difficult computationally than imposing a net pressure gradient; it can be done with Green's functions, i.e. calculating the net pressure gradient which will lead to the desired value of the net flux.

If the flow is reflection symmetric in a periodic direction, then the two requirements coincide: zero net pressure in that direction corresponds to zero flux in that direction. However, if reflection symmetry is broken, then the two requirements lead to different flows.

A striking example of this was given in 1993 by Edwards et al.

• W.S. Edwards, R.P. Tagg, B.C. Dornblaser, L.S. Tuckerman, H.L. Swinney, *Periodic traveling waves with nonoperiodic pressure*, Eur. J. Mech. B/Fluids **10** 205–210 (1991); Erratum in Eur. J. Mech. B/Fluids **10** 575 (1991).

for the case of the axial direction in Taylor-Couette flow. The excellent agreement between theory and experiment for obtained in 1923 by G.I. Taylor for the formation of axisymmetric vortices comprised the final argument for the acceptance of the Navier-Stokes equations. However, the agreement between theory and experiment for spiral vortices in the case in which the cylinders counter-rotate remained imperfect for many decades. In 1993, Edwards et al. carried out numerical simulations of Taylor-Couette flow in which zero net flux was imposed by allowing a net axial pressure gradient, in contrast with previous calculations which had assumed an axially periodic pressure. This new calculation rectified the long-standing disagreement that had existed between experimental and calculated wavespeeds.

A few additional notes:
The difference between setting the overall pressure gradient and the flux is not manifested at the linear level: the eigenmodes leading to spiral vortex flow have both zero net axial flow and zero overall pressure gradient.
The difference between setting the overall pressure gradient and the flux is also not manifested when the flow is reflection-symmetric in that periodic direction. Spiral vortex flow breaks the reflection symmetry in the axial direction.

In a genuinely periodic direction, for example the azimuthal direction in Taylor-Couette flow, the pressure itself must be the same at $\theta = 2\pi$ and at $\theta = 0$; there can be no overall pressure gradient in θ, and so an azimuthally periodic pressure must be imposed. However, the periodicity assumed for the axial direction in Taylor-Couette flow is different. The points $z = 0$ and $z = L_z$ are not the same physical locations; instead, a periodic flow is observed and this is used as a modelling assumption. The assumption of a periodic flow is formally compatible with either an overall pressure gradient or an overall flux. However, since the actual Taylor-Couette apparatus is long but finite, and the net axial flux must be constant at every axial location, imposing zero axial flux corresponds more closely to what would happen in an axial section of a Taylor-Couette apparatus.

The assumption of periodicity in plane channel flows is made for the same reason as that in the axial direction in Taylor-Couette flow. Experimental plane channel flow configurations have very long streamwise and spanwise extents. These directions are modeled as infinite, composed of periodically repeating sections. To correspond to a finite experimental apparatus, it seems preferable to impose zero flux in the spanwise direction, along with whichever conditions are chosen in the streamwise direction.

Papers that mention this point are:

• W.S. Edwards, R.P. Tagg, B.C. Dornblaser, L.S. Tuckerman, H.L. Swinney, *Periodic traveling waves with nonoperiodic pressure*, Eur. J. Mech. B/Fluids **10** 205–210 (1991); Erratum in Eur. J. Mech. B/Fluids **10** 575 (1991).
• J. Antonijoan, F. Marquès, J. Sánchez, *Non-linear spirals in the Taylor-Couette problem*, Phys. Fluids **10**, 829 (1998)
• P. Ashwin, G.P. King, *A study of particle paths in non-axisymmetric Taylor-Couette flows*, J. Fluid Mech **338**, 341 (1997).
• J.G. Heywood, R. Rannacher, S. Turek, *Artificial boundaries and flux and pressure conditions for the incompresible Navier-Stokes equations*, Int. J. Num. Meth. Fluids **22**, 325 (1996).
• R. Raffai, P. Laure, *The influence of an axial mean flow on the Couette-Taylor Problem*, Eur. J. Mech. B/Fluids **12**, 277 (1993).

Scanned copies of all of these papers can be found at:

http://www.pmmh.espci.fr/ laurette/papers/Edwards_corrected.pdf
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