LINEAR AND NONLINEAR STABILITY ANALYSIS OF PERTURBED PLANE COUETTE FLOW

DWIGHT BARKLEY[1] AND LAURETTE S. TUCKERMAN[2]
[1] *Mathematics Institute, University of Warwick*
Coventry CV4 7AL, U.K. *barkley@maths.warwick.ac.uk*
[2] *LIMSI, BP 133, 91403 Orsay Cedex, France, laurette@limsi.fr*

Despite being linearly stable for all Reynolds numbers, plane Couette flow becomes turbulent in laboratory experiments and in numerical simulations for $Re \approx 300$. In order to induce and to study the transition to three-dimensionality and then to turbulence, systematic perturbations are sometimes applied to the flow. In particular, plane Couette flow perturbed by a wire is currently the subject of laboratory experiments at CEA-Saclay [1, 2, 3].

We have performed a numerical linear and nonlinear stability analysis of plane Couette flow perturbed by the presence of a thin narrow ribbon midway between bounding plates. Both in the experiment and in the numerical study, the streamwise (x) dimension L is increased until length-independent results are obtained. In practice, this requires L between 20 and 40 times the cross-channel (y) dimension. In our numerical study the ribbon had formally infinite extent in the spanwise (z) direction. In experiments the perturbing wire has a large spanwise extent.

In our numerical approach we first solve the steady 2D Navier-Stokes equations subject to the boundary conditions:

$$\begin{aligned}
&\mathbf{u}(x-L,y) = \mathbf{u}(x+L,y)\\
&\mathbf{u}(x,y=\pm 1) = \pm\hat{\mathbf{x}}\\
&\mathbf{u}(x=0,y) = 0, \qquad \text{for } -\rho \le y \le \rho,
\end{aligned}$$

where ρ is the half-height of the ribbon ($\rho = 0.086$ for all results reported). We use the spectral element code `Prism` [4], which permits grid refinement near the ribbon. Each velocity component is represented by $O(10^4)$ gridpoints or basis functions. Figure 1 shows an example steady 2D flow.

Next we calculate the linear stability of the steady flows to three-dimensional z-periodic perturbations (eigenfunctions) of the form $(\mathbf{u}, p) =$

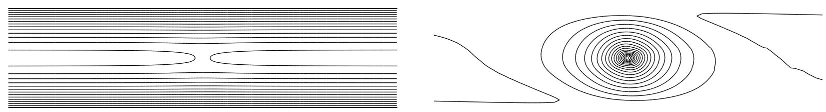

Figure 1. (a) Streamfunction near ribbon ($-3 < x < 3$) of steady 2D flow $\mathbf{U}(x, y)$ at $Re = 250$. (b) Effect of ribbon is highlighted by subtracting unperturbed plane Couette flow $y\mathbf{e_x}$ from flow of (a). This flow displays centro-symmetry but not reflection symmetry in x and y. The effect of the ribbon is to set up a circulation opposing that of the plane Couette flow.

$(\hat{u}\cos\beta z, \hat{v}\cos\beta z, \hat{w}\sin\beta z, \hat{p}\cos\beta z)$. For fixed β, this is essentially a 2D calculation [5]. The vector $\hat{\mathbf{u}}(x, y, t) = (\hat{u}, \hat{v}, \hat{w})$ of Fourier coefficients evolves according to:

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} = -(\hat{\mathbf{u}} \cdot \nabla)\mathbf{U} - (\mathbf{U} \cdot \nabla)\hat{\mathbf{u}} - (\nabla - \beta\hat{\mathbf{z}})\hat{p} + \frac{1}{Re}(\nabla^2 - \beta^2)\hat{\mathbf{u}}$$
$$(\nabla + \beta\hat{\mathbf{z}}) \cdot \hat{\mathbf{u}} = 0$$

where ∇, etc. are two-dimensional differential operators. The boundary conditions on $\hat{\mathbf{u}}$ are the same as for the steady flow except that $\hat{\mathbf{u}}(x, y) = 0$ at $y = \pm 1$. The leading eigenvalues are calculated by the Arnoldi method [5]. Figure 2 shows the maximal growth rate of 3D eigenfunctions as a function of Re and β. At $Re_c \approx 230$ there is a linear instability with $\beta_c \approx 1.3$. The corresponding wavelength $\lambda_c \approx 4.8$ agrees well with experimental observation.

Finally, to determine the nonlinear character of the instability, we carry out a single nonlinear 3D simulation starting from the initial condition $\mathbf{U} + \epsilon\mathbf{u}$, where $\mathbf{U}$ and $\mathbf{u}$ are the steady flow and eigenfunction at $Re = 250$ and ϵ is small. The evolution of the amplitude $A(t)$ with time is fit to the normal form $\dot{A} = \sigma A + \alpha A^3$. A positive (negative) value of α implies that the bifurcation is subcritical (supercritical). Figure 3 demonstrates that the growth is faster than exponential, and hence that the bifurcation is subcritical. We have further verified the subcritical nature of the instability by computing a steady 3D flow at $Re = 200$, below Re_c.

Figure 4 depicts the final steady 3D flow. Small streamwise vortices can be seen in each of the four corners of the $x = 0$ plane containing the ribbon. The lower ($y < 0$) pair evolves with x into the strong pair of vortices at $x = 1$. The vortices at $x = 3$ are tilted, attesting to the nonlinear generation of the second spanwise harmonic 2β. Far from the ribbon, the 3D flow returns to plane Couette flow. This flow is similar to that observed in the Saclay experiments.

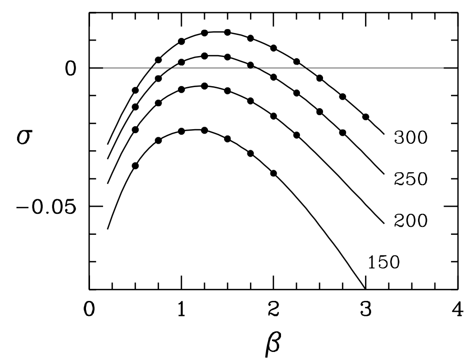

Figure 2. Growth rate σ of most unstable 3D eigenfunction as a function of spanwise wavenumber β for $Re = 150, 200, 250, 300$. Critical values for instability are $Re_c \approx 230$ and $\beta_c \approx 1.3$. Recall that in the absence of a ribbon or wire, plane Couette flow is linearly stable for all Re, β.

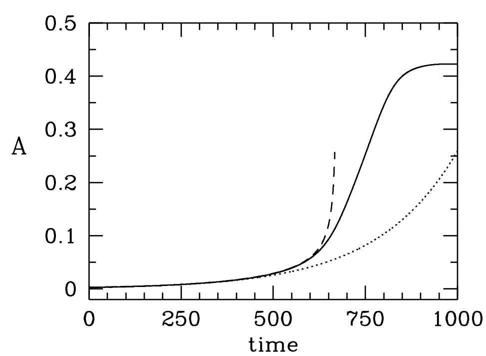

Figure 3. Nonlinear growth of the amplitude A of the 3D flow from simulation (solid) at $Re = 250$. First-order (dotted) and third-order (dashed) dynamics are shown with $\sigma = 0.0046$ and $\alpha = 1.4$.

In conclusion, we have accurately determined the extent to which the basic steady 2D profile is modified by the presence of a small spanwise-oriented ribbon. We have determined that a ribbon, comparable in size to the cylinders used in the Saclay experiments, is large enough to induce linear instability of the basic profile at Reynolds numbers of order a few hundred. We have found that the spanwise wavelength of the most unstable mode is in good agreement with the value seen experimentally. Finally, we have shown that the linear instability is subcritical and leads to a flow with streamwise vortices.

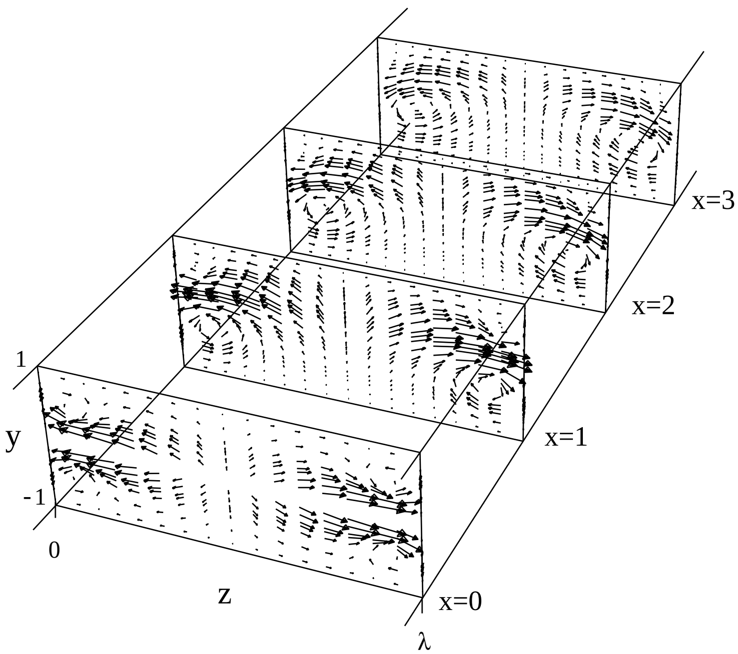

Figure 4. Steady 3D velocity field at $Re = 250$. Shown are (v, w) velocity plots at four streamwise locations containing streamwise vortices.

References

1. O. Dauchot and F. Daviaud (1995) Streamwise vortices in plane Couette flow, *Phys. Fluids* **7**, 901–903.
2. S. Bottin, O. Dauchot and F. Daviaud (1997) Intermittency in a locally forced plane Couette flow, *Phys. Rev. Lett.* **79**, 4377–4380.
3. S. Bottin, O. Dauchot, F. Daviaud and P. Manneville (1998) Experimental evidence of streamwise vortices as finite amplitude solutions in transitional plane Couette flow, *Phys. Fluids*, submitted.
4. R.D. Henderson (1994), Unstructured spectral element methods; parallel algorithms and simulations, Ph.D Thesis, Princeton University.
5. D. Barkley and R. Henderson (1996) Floquet stability analysis of the periodic wake of a circular cylinder, *J. Fluid Mech.* **322**, 215–241.