Patterns and dynamics in transitional plane Couette flow

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Near transition, plane Couette flow takes the form of large-scale, oblique, and statistically steady alternating bands of turbulent and laminar flow. Properties of these flows are investigated using direct numerical simulation in a tilted computational domain. Four regimes—uniform, intermittent, periodic, and localized—are characterized. The Fourier spectrum along the direction of variation of the pattern is presented, and the component corresponding to the pattern wavenumber is investigated as an order parameter. The mean flow of a periodic pattern is characterized and shown to lead to a relation between the Reynolds number and the wavelength and angle of a pattern. © 2011 American Institute of Physics. [doi:10.1063/1.3580263]

I. INTRODUCTION

Transition to turbulence in wall-bounded shear flows has attracted the attention of a large number of researchers using a range of approaches. The three classic wall-bounded shear flows are plane Couette, plane Poiseuille, and pipe flow. This list can be broadened to include counter-rotating Taylor–Couette flow, boundary layer flow, and the flow between differentially rotating disks. The classic tool of linear stability analysis is not useful under these circumstances since the transition to turbulence takes place far from any linear instability.

Transitional flows contain turbulent regions, called spots, slugs, puffs, and bands, which coexist with laminar regions. This leads to questions concerning sizes or wavelengths and growth or decay. A particular coexistence regime consists of alternating turbulent and laminar bands. Both the wavelength and the angle between the bands and the streamwise direction are well-defined and depend on the Reynolds number. The discovery of regular turbulent-laminar patterns dates from about 2000 by Prigent and Dauchot\textsuperscript{1–4} in counter-rotating Taylor–Couette flow, although more irregular versions were observed as early as the 1960s by Coles\textsuperscript{5} and Van Atta\textsuperscript{6} and later by Andereck \textit{et al.}\textsuperscript{7}, Hegseth \textit{et al.}\textsuperscript{8} and Goharzadeh and Mutabazi.\textsuperscript{9} Turbulent-laminar banded patterns were subsequently observed in experiments on plane Couette flow\textsuperscript{1–4} and rotor-stator flow,\textsuperscript{10} and in numerical simulations of plane Couette flow,\textsuperscript{11–19} of plane Poiseuille flow,\textsuperscript{20,21} and of counter-rotating Taylor–Couette flow.\textsuperscript{22–24}

We have studied various aspects of turbulent-laminar banded patterns in plane Couette flow by simulating the Navier–Stokes equations in a rectangular computational box, aligned with the direction of the pattern wavevector. This article summarizes what we have learned about these remarkable flows.

II. GEOMETRY AND COMPUTATIONS

Figure 1 shows a visualization of one of our computed turbulent-laminar banded patterns at a Reynolds number of 350, more specifically the kinetic energy halfway between the two bounding plates. The small-scale structure seen in Fig. 1 within the turbulent bands is a manifestation of the streamwise vortices and streaks that comprise turbulence in wall-bounded shear flow.\textsuperscript{25–27} Figure 2, a spatiotemporal diagram of the variation of the streamwise velocity along the spanwise direction, shows both the dynamic nature of small-scale spatial and temporal features as well as the long-lived nature of the large-scale banded structure. In Fig. 1, the distance between successive turbulent bands is \( \lambda = 40 \), and the angle between the bands and the streamwise direction is \( \theta = 24^\circ \), approximately as found in experiment. This angle is actually imposed by our calculation, as we now describe.

The distance between bands is very large compared to the small-scale variation within the bands, and so we choose our domain to be long in the direction of the pattern wavevector.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{(Color online) Computed turbulent-laminar pattern at Re=350. Shown is the kinetic energy at \( y=0 \), midway between bounding plates at \( y = \pm 1 \) which move to the right and left in the streamwise direction. Turbulent bands consist of streamwise streaks and vortices. The bands are oriented in the direction denoted by \( x \) at an angle of 24° from the streamwise direction, and are separated by a wavelength of 40 in the direction of the pattern wavevector, denoted by \( z \).}
\end{figure}
wavevector and short in the direction parallel to the bands. That is, our computational domain is rectangular, but tilted with respect to the streamwise and spanwise directions. Figure 1 has been constructed by filling a square oriented along the streamwise-spanwise axes with multiple copies of our tilted rectangular computational domain. Although our computational domain effectively imposes the angle, the wavevector and short in the direction parallel to the bands. Our numerical computations, like the physical experiments, are conducted by lowering the Reynolds number. Figures 3 and 4 are spatiotemporal diagrams, taken along the line \(x=y=0\), where the velocity is zero for laminar plane Couette flow (1). We show timeseries of the spanwise velocity at 32 equally spaced points in the \(z\) direction for a simulation in a domain of size \(L_z=120\), \(L_y=10\) as the Reynolds number is decreased. Left: modulus of Fourier transform in the \(z\) direction, averaged over time windows of length \(\Delta T=1000\), showing evolution of components corresponding to wavelengths of 40 (solid red), 60 (long-dashed blue), 120 (short-dashed green), and \(\infty\) (dotted black). Right: evolution of spanwise velocity within laminar (left) and turbulent (right) bands over interval of length \(\Delta T=1000\). Transition to an intermittent regime is seen at \(Re=410\), to three bands at \(Re=390\), to two bands at \(Re=310\), to a localized state at \(Re=300\), and to laminar Couette flow at \(Re=290\).

III. FOUR REGIMES

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number is lowered in discrete steps. The accompanying graph along the left shows the moderate-time ($\Delta T=1000$) average of the first Fourier components $m=0,1,2,3$ (i.e., wavelengths $\approx 120, 60, 40$) in the $z$-direction. The insets on the right show expanded versions of portions of the timeseries for $z$ locations in the laminar and the turbulent regions. The terms “turbulent” and “laminar” can be taken here to mean strongly and weakly chaotic, respectively. That is, the turbulent region does not display a Kolmogorov spectrum and the laminar region is not described by the linear plane Couette profile, as shown by the expanded timeseries in Fig. 3. Figure 2 illustrates the detailed behavior of the flow within the two regions.

We define four turbulent patterned regimes, which we call uniform, intermittent, periodic, and localized, listed in order of decreasing Reynolds number. (i) In the uniform regime, turbulence extends across the entire domain, whereas (ii) in the intermittent regime, laminar patches appear and disappear. (iii) In the periodic regime, laminar and turbulent regions are permanent. Although there is small-scale stochastic motion, large-scale motion of the bands is slow or absent. The spatial periodicity is well-defined. (iv) In the localized regime, a single turbulent region is surrounded by laminar flow. In this case, the turbulent region is exponentially localized in space and the laminar region is in fact described by the linear plane Couette profile.

We point out some particular events that can be seen in Fig. 3. At Re=310, the number of turbulent bands is reduced from three to two; simultaneously the Fourier component corresponding to wavelength 40 is succeeded by that of wavelength 60. At Re=300, the number of bands is further reduced to one. However, this is not a manifestation of a periodic pattern with wavelength 120, but a localized state, in which the turbulent region can be surrounded by a laminar region of any width. This is demonstrated$^{11}$ by producing a pattern containing a single turbulent band of the same width in domains varying from patterns with different wavelengths or orientations.

The formation process of turbulent-laminar banded patterns in a large-aspect-ratio domain with streamwise and spanwise extents of $L_{\text{str}}=800$ and $L_{\text{span}}=356$, in which the orientation of the turbulent bands arises out of initial conditions, is well documented by Duguet et al.$^{16}$ Competition between patterns with different wavelengths or orientations is characterized by Prigent et al.$^{1-4}$ and by Rolland and Manneville.$^{18}$

**IV. TRANSITION BETWEEN UNIFORM, INTERMITTENT, AND PERIODIC PATTERNS**

As shown in Figs. 3 and 4, the Fourier transform in the $z$ direction provides a good quantitative measure for distinguishing between the different turbulent-laminar patterns. In this section we present simulations carried out in a domain of length $L_z=40$, just large enough to accommodate a single wavelength, for fixed Reynolds numbers.

Figure 5 shows timeseries and spectra for simulations carried out over 20 000 time units at Reynolds numbers of 350, 410, and 500. The first row presents spatiotemporal diagrams like those in Figs. 3 and 4 of the spanwise velocity along a line in the $z$ direction in the midplane. The next row presents Fourier transforms in the $z$ direction of this spanwise velocity. These differ from those included in Figs. 3 and 4 as follows. First, we show the square modulus of the entire spectrum, rather than the first few components. Second, we average the instantaneous Fourier components over the entire simulation, rather than over intervals on the order of 100. The peak at $m=1$, i.e., at a wavelength of 40, corresponds to the turbulent-laminar pattern, most prominent at Re=350, reduced at 410 and barely present at 500. This component is plotted as a function of Re in Fig. 6.

The last row of Fig. 5 shows the spectrum of the streamwise velocity. The $m=1$ component is still very prominent at Re=350 and 410, but there are additional peaks at $7 \leq m \leq 11$ for all three Reynolds numbers. These components reflect the small-scale structure seen in Figs. 1 and 2. At these Reynolds numbers, a streamwise vortex occupies approximately the entire gap ($L_z=2$) with an equal spanwise extent, leading to a pair of vortices whose spanwise width is 4, with an extension in the $z$ direction of $2 \times L_z/\cos(\theta) = 4/\cos(2\theta) = 4.38$, or a Fourier component of 40/4.38 $= 9.1$. The streamwise vortices in turn lead to streaks, meaning that they advect high and low streamwise velocity toward the midplane from near the bounding plates. The absence of peaks corresponding to small-scale structure in the spanwise spectrum confirms this heuristic picture: while a streamwise vortex should have substantial spanwise velocity, the corresponding velocity in the midplane should be primarily in the $y$ direction.

We return to the $m=1$ Fourier component of the spanwise velocity and now consider the probability distribution of its instantaneous values. The Fourier component has both an amplitude $a$ and a phase $\phi$, but, by translational symmetry, its probability distribution $p$ must be independent of phase, i.e., of the location along the $z$ axis. We write

$$1 = \int_0^\infty a^2 d\phi \int_0^{2\pi} d\theta \frac{d\rho(a,\phi)}{\rho(a)}$$

We estimate $p(a)$ via

$$p(a) = \frac{1}{a_i \Delta a} \int_{a_i}^{a_i+\Delta a} \int_0^{2\pi} d\phi d\theta$$

where the integral in Eq. (4) is estimated by counting the proportion of values of $a$ falling within each of 20 bins of width $\Delta a$ centered on $a_i$.

Figure 7 shows $p(a)$ obtained from timeseries for Re =350, 410, and 500 on a logarithmic scale, along with fits of
In $p(a)$ to even polynomials. At $Re=500$, when the turbulence is uniform, $p(a)$ has a clear maximum at $a=0$; it is in fact extremely well fit by a Gaussian

$$\ln p(a) = c_0 + c_2a^2.$$  \hspace{1cm} (5)

The probability distribution function is almost identical for $500 \leq Re \leq 600$.\footnote{The most probable value shifts from 0 to positive $a$ as Re is lowered, attaining values near 1 (on an arbitrary scale) for Re=350. We generalize Eq. (5) to a quartic polynomial

$$\ln p(a) = c_0 + c_2a^2 + c_4a^4.$$  \hspace{1cm} (6)}

In the usual scenario for phase transitions, $c_4$ would vary little with Re while $c_2$ would change sign at the transition.
However, a quartic polynomial does not provide a good fit for the patterned flows, as exemplified by the curve for Re = 350. Weighting the points by their probability changes the fit, but does not improve it. Figure 8 shows the coefficients of the fit as a function of Re. The coefficients change very little for Re \( \lesssim 500 \) and within the range 350 \( \leq \) Re \( \leq 370 \). The coefficients change dramatically within 410 \( \leq \) Re \( \leq 430 \): \( c_4 \) decreases to near zero and \( c_0 \) and \( c_2 \) change signs. The most probable value \( a_{\text{max}} \) is difficult to determine at intermediate values because \( p(a) \) is flat and contains noise. Therefore, we take \( a_{\text{max}} \) to be the maximum value of the quartic function (6), or 0 if \( |c_4| < 0.1 \), i.e.,

\[
a_{\text{max}} = \begin{cases} 
0 & \text{if } |c_4| < 0.1 \\
-c_2/2c_4 & \text{otherwise}.
\end{cases}
\] 

Any of these—the amplitude of \( a_{\text{max}} \) or \( c_4 \) or the sign of \( c_0 \), \( c_2 \)—can be used as an order parameter for the existence of a turbulent-laminar banded pattern.

V. OTHER DOMAINS

Although we have primarily studied the case \( \theta = 24^\circ \), we have also studied other angles. A summary of our survey in Re and \( \theta \) is given in Fig. 9. Figure 10 shows stationary patterned states, some periodic and some localized, with extremal angles and wavelengths at Re = 350. By fixing \( \theta = 24^\circ \)
and $L_z$, and varying $L_x$, we produced turbulent-banded patterns with wavelengths between 35 and 65. By fixing $L_z = 120$ and varying $\lambda$ and $L_x$ according to $L_x = 4 / \sin \lambda$, we were able to produce patterns at angles between 15° and 66°. The pattern at 66° is a localized state. We expect most of these states to be unstable in a less restricted domain.

Figures 11 and 12 show simulations at the extremal angles of 0° and 90°, i.e., in more classic rectangular domains aligned with the streamwise and spanwise directions. At $\theta=0^\circ$, the spanwise extent is $L_z = 120$ and we have fixed the streamwise extent to $L_x = 10$. We see that turbulent patches subsist down to $Re=220$, far lower than in the $\theta=24^\circ$ case. This is not a well-defined threshold; when simulations are continued at fixed intermediate Reynolds numbers from the intermediate fields in Fig. 11, turbulence persists in some cases and not in others. The Reynolds-number-behavior of the threshold for turbulence is statistical and strongly dependent on the initial condition and the procedure. It is also extremely dependent on the dimensions of the domain. In computations of plane Couette flow in a domain similar to that in Fig. 11, Duguet also finds growth of turbulent patches at Reynolds numbers below 300, and suggests that this behavior is a manifestation of the unstable states located on homoclinic snaking branches computed by Schneider et al. In contrast, for $\theta=90^\circ$, i.e., in a domain with a long streamwise extent of 120 and short spanwise length of 4, the flow becomes laminar throughout when $Re \leq 385$ without passing through any intermediate pattern.

VI. MEAN FLOW OF A PERIODIC TURBULENT-LAMINAR PATTERN

We now analyze in detail the mean flow corresponding to a well-established turbulent-laminar pattern at parameter values $Re=350$, $\theta=24^\circ$, and $L_z = 40$ using the data from the timeseries shown in Fig. 5 between $t=6000$ and $t=8000$. In this section, we calculate the time-average of the velocity field over the entire domain, rather than just the spanwise velocity at sampled points across a line at the midplane. This time-averaged velocity varies little in $x$, the direction parallel to the turbulent bands; it is therefore meaningful to average over $x$ as well, defining

$$\langle u(y,z) \rangle = \frac{1}{TL_x} \int_{t_0}^{t_0+T} dt \int_{x=0}^{L_x} dx u(x,y,z,t).$$

The distinctive features of $\langle u \rangle = (\langle u \rangle, \langle v \rangle, \langle w \rangle)$ are best viewed by subtracting from it the basic Couette profile (1) and by expressing the flow in the $(y,z)$ plane via a stream-function, since $\partial_y \langle v \rangle + \partial_z \langle w \rangle = 0$. 

FIG. 10. (Color online) Extremal turbulent-laminar banded patterns at $Re=350$. Top row: $\theta=24^\circ$ with $\lambda = 35$ (left) and $\lambda = 65$ (right). Bottom row: $\theta=15^\circ$ and $\theta=66^\circ$.

FIG. 11. (Color online) Evolution in a domain with streamwise extent $L_x = 10$ and spanwise extent $L_z = 120$. Timeseries taken at points along long spanwise direction indicated by red line. Turbulent regions subsist far below $Re=300$.

FIG. 12. (Color online) Evolution in a domain with streamwise extent of 120 and spanwise extent of 4. Timeseries taken at points along long streamwise direction indicated by red line. Turbulence disappears throughout the domain for $Re \approx 385$. 

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\begin{equation}
\langle \mathbf{u} \rangle = \mathbf{U}_{\text{Cou}} + U_e \mathbf{e}_y + \nabla \times \psi \mathbf{e}_z.
\end{equation}

Figure 13 shows $U$ and $\psi$ in the $(y,z)$ plane as well as the turbulent kinetic energy $\langle \mathbf{u} \cdot \mathbf{u} \rangle / 2$ and the mean pressure $\langle p \rangle$. The fields in Fig. 13 all display centro-symmetry,

\begin{equation}
U(-y,-z) = U(y,z),
\end{equation}

where $z=0$ is defined to be at the center of the turbulent region. As was demonstrated in Secs. III and IV, the $z$ dependence is extremely well approximated by a single Fourier mode. This leads to representations of the form

\begin{equation}
U(y,z) \approx U_0(y) + U_z(y)\cos(2\pi y/\lambda) + U_y(y)\sin(2\pi y/\lambda),
\end{equation}

involving only three scalar functions $U_0$, $U_z$, and $U_y$, where $U_0$, $U_z$, and $U_y$ are odd in $y$ and $U_y$ is even.

A view of $\langle \mathbf{u} \rangle$ in two streamwise-spanwise planes at $y = \pm 0.725$ is shown in Fig. 14. Computations of the mean flow and turbulent kinetic energy for turbulent-laminar banded patterns are presented by Tsukahara et al.\textsuperscript{21} for plane Poiseuille flow and by Dong\textsuperscript{22,23} for Taylor–Couette flow.

The field $\langle \mathbf{u} \rangle$ obeys the $(x,t)$-averaged Navier–Stokes equations whose $x$ component is

\begin{equation}
0 = -\frac{1}{\text{Re}} \nabla^2 \langle \mathbf{u} \rangle - \langle \mathbf{u} \cdot \nabla \rangle \langle \mathbf{u} \rangle - \langle \mathbf{u} \cdot \mathbf{u} \rangle + \frac{1}{\text{Re}} \nabla^2 \langle \mathbf{u} \rangle = O\left( \frac{\pi^2}{\text{Re}} \right).
\end{equation}

The nonlinear term is dominated by advection in the $z$ direction by $\mathbf{U}_{\text{Cou}}$. Evaluating it at a typical value $y=1/2$, we obtain as an estimate for the nonlinear term

\begin{equation}
\langle (\mathbf{u} \cdot \nabla) \rangle \approx \mathbf{e}_z \cdot \mathbf{U}_{\text{Cou}} \frac{\partial}{\partial z} \sin \theta \frac{2\pi}{\lambda} \mathbf{e}_z = \sin \theta \frac{\pi}{\lambda}.
\end{equation}

A more complete justification\textsuperscript{13} of Eqs. (13) and (14) relies on the full computed fields and the functional form (11). The balance between Eqs. (13) and (14) leads to

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig13}
\caption{(Color online) $U(y,z)$: transverse component of mean flow; $\Psi(y,z)$: streamfunction of in-plane mean flow. A long cell extends from one laminar-turbulent boundary to the other. Gradients of $\Psi$ are much larger in $y$ than in $z$, i.e., $|W|/|V|$. In the laminar region at the center, $W,V=0$. $E_{\text{turb}}(y,z)$: mean turbulent kinetic energy $\langle \mathbf{u} \cdot \mathbf{u} \rangle / 2$. There is a phase difference of $\lambda_2/4=10$ between extrema of $E_{\text{turb}}$ and $U$. $P(y,z)$: mean pressure field. Pressure gradients are primarily in the $y$ direction and within the turbulent region. Color ranges for each field from blue to red: $U [-0.4, 0.4]$, $\Psi [0, 0.9]$, $E_{\text{turb}} [0, 0.4]$, $P [0, 0.007]$.
}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig14}
\caption{Mean velocity components seen in three planes with standard orientation for Couette flow. The turbulent regions are shaded. Top: velocity components in the streamwise-spanwise plane at $y=0.725$ (upper part of the channel). Middle: same except $y=-0.725$ (lower part of the channel). Bottom: flow in a constant spanwise cut. The mean velocity is shown in the enlarged region.
}
\end{figure}
Turbulent-laminar patterns are a fascinating feature of many wall-bounded shear flows near transition. We have carried out detailed studies of turbulent-laminar patterns in plane Couette flow. Our main findings are as follows. First, turbulent-laminar banded patterns can be further divided into different regimes—intermittent, periodic, or localized. Second, the Fourier component corresponding to the pattern wavevector (the direction we have called $\mathbf{z}$) leads to the appropriate order parameter for describing such patterns. The transition from uniform turbulence to a turbulent-laminar pattern is described by a bifurcation in its probability distribution function. Third, the mean flow associated with a periodic turbulent-laminar pattern consists primarily of flow along the turbulent-laminar boundaries (the direction we have called $\mathbf{x}$), maintained by a weaker circulation around the turbulent regions. The mean balance of forces determines the relation between the angle, wavelength, and Reynolds number of the patterns.

It seems plausible that studies of such patterns will lead to insights concerning the cause or nature of the fundamental problem of transition to turbulence. But even in the absence of such results, turbulent-laminar patterns are a perplexing and exotic object of study in their own right.

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