Cours : Dynamique Non-Linéaire

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Rolls, stripes and their instabilities

Sources:

R. Hoyle, Pattern Formation: An Introduction to Methods, Cambridge 2006

M. Cross, http://www.its.caltech.edu/∼mcc

Pattern formation

No horizontal BCs (horizontally homogeneous domain) $\Longrightarrow$ eigenvectors are $e^{i\mathbf{q}\cdot\mathbf{x}}$ with $\mathbf{x} \equiv (x, y)$, $\mathbf{q} \equiv (q_x, q_y)$

Linear instability depends on magnitude $q \equiv |\mathbf{q}|$, not on orientation of $\mathbf{q}$

Near threshold, final solution = $\sum_{\mathbf{q}} u_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}$

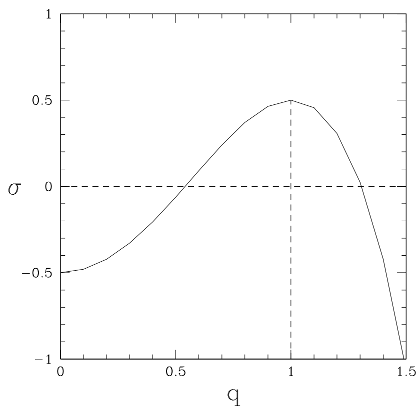
Nonlinear terms $\Longrightarrow$ relative magnitudes of different $u_{\mathbf{q}}$ $\Longrightarrow$ pattern

Mathematics not yet ready to choose between all possible patterns $\Longrightarrow$ Pose problem on fixed lattice and seek specified pattern, i.e. stripes, rectangles, squares, hexagons

Swift-Hohenberg equation

Instability of trivial state $u = 0$ to eigenvectors $e^{i\mathbf{q}\cdot\mathbf{x}}$ with growth rate $\sigma(q)$

$$\left.\begin{array}{l}\sigma \text{ depends on } q^2\\ \sigma > 0 \text{ for } q \sim q_c\\ \sigma < 0 \text{ for } |q| \gg 1\end{array}\right\} \Longrightarrow \begin{aligned}\sigma(q) &= a_0 + a_2 q^2 - q^4\\ &= \mu - (q_c^2 - q^2)^2\end{aligned}$$



Swift-Hohenberg equation

$$\begin{array}{lll} \sigma = \mu & -(q_c^2 - & q^2)^2 \\ \downarrow & & \downarrow \\ \partial_t & & \Delta \end{array}$$

Add saturating cubic term to halt exponential growth near q_c

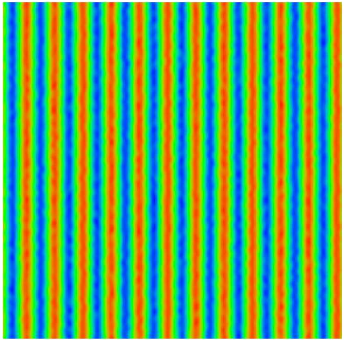
$$\partial_t u = \mu u - \left(q_c^2 + \Delta\right)^2 u - u^3$$

Add quadratic term to obtain hexagons

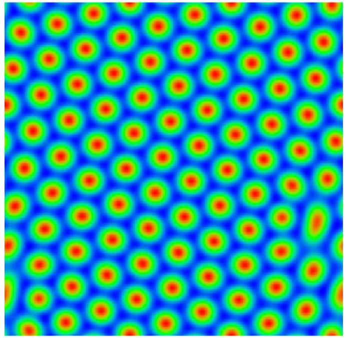
Include q_c and q_c' to obtain quasipatterns

Derived by J. Swift and P.C. Hohenberg (Phys. Rev. A 15, 319 (1977)) to describe pattern formation in convection

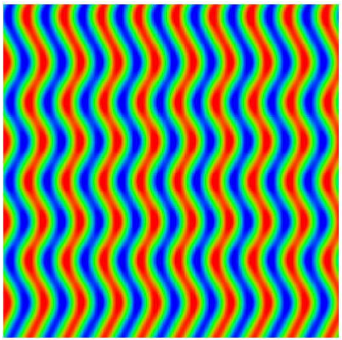
Patterns produced by Swift-Hohenberg equation



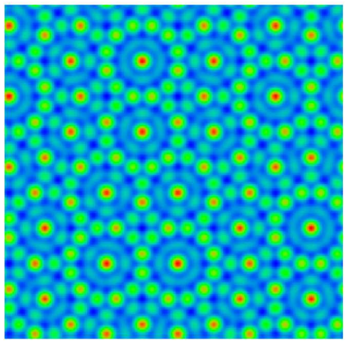
Stripes



Hexagons



Zigzag instability

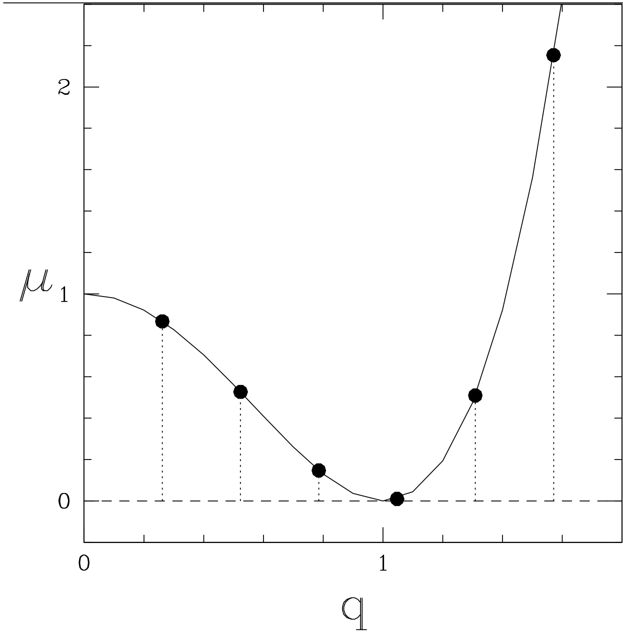
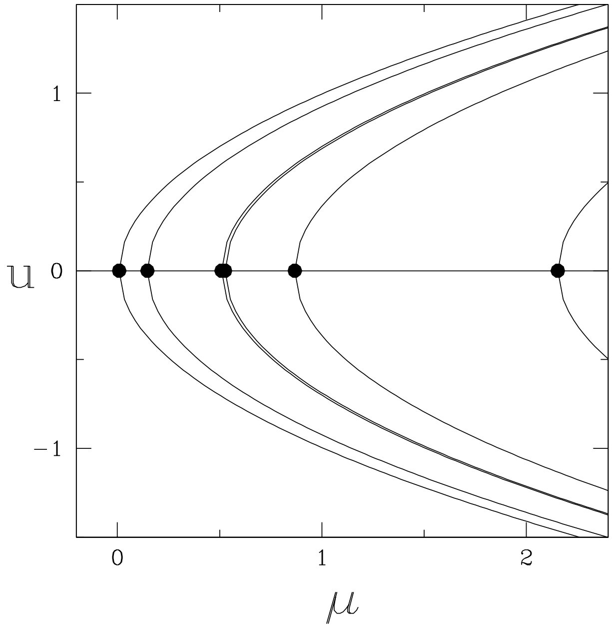


Quasicrystals

Stripes or Rolls

$$\partial_t u = \mu u - \left(q_c^2 + \Delta\right)^2 u - u^3$$

$q_c = 1$ **Domain of length** $L = 24$

Steady bifs at $\mu_q = (q_c^2 - q^2)^2$	Amplitude $u = \sqrt{\mu - (q_c^2 - q^2)^2}$
	

Periodic boundary conditions in x with wavelength $L \Longrightarrow q = n\frac{2\pi}{L}$

Allowed q closest to $q_c = 1 \Longrightarrow$ smallest critical μ

n	$\lambda_n = \frac{L}{n}$	$q_n = \frac{2n\pi}{L}$	$\mu_n = (q_c^2 - q_n^2)^2$
4	6.0	1.05	0.01
3	8.0	0.79	0.15
5	4.8	1.31	0.51
2	12.0	0.52	0.53
1	24.0	0.26	0.87
6	4.0	1.57	2.15
7	3.4	1.83	5.56
8	3.0	2.09	11.47
9	2.7	2.36	20.72

$$L = 24 \approx 4\lambda_c = 8\pi$$

n	λ_n	q_n	μ_n
1	2π	1	$(1^2 - 1^2)^2 = 0$
2	π	2	$(1^2 - 2^2)^2 = 9$
3	$2\pi/3$	3	$(1^2 - 3^2)^2 = 64$

$$L = \lambda_c = 2\pi$$

	length scale for x	q_c	λ_c
RB convection free-slip BCs	depth	$\pi/\sqrt{2}$	$\sqrt{2}$
RB convection rigid BCs	depth	π	2
Swift-Hohenberg equation	2π	1	2π

Periodic BCs $\Longrightarrow$ circle pitchforks

Neumann BCs ($\partial_x u|_{0,\pi} = 0$) $\Longrightarrow$ ordinary pitchforks

$L \approx \lambda_c \Longrightarrow$ **thresholds separated $\Longrightarrow$ only first bifurcation important**

$L \gg 1 \Longrightarrow$ **thresholds close $\Longrightarrow$ discretization sometimes neglected**

Square patterns

Periodicity $L_x = L_y = \lambda_c = 2\pi \Longrightarrow$ **evecs** $e^{\pm i x_1}$ **and** $e^{\pm i x_2}$

$$u(x_1, x_2, t) = z_1(t)e^{ix_1} + \bar{z}_1(t)e^{-ix_1} + z_2(t)e^{ix_2} + \bar{z}_2(t)e^{-ix_2}$$

Symmetry group of square lattice generated by:

D_4: **rotation** $S_{\pi/2}$ **and reflection** κ **in** x_1

two-torus T^2: **translations by** $\mathrm{p} = (p_1, p_2)$ **in** x_1 **and** x_2

$$\begin{aligned}
S_{\pi/2}u(x_1, x_2, t) &\equiv u(x_2, -x_1, t) = z_1(t)e^{ix_2} + \bar{z}_1(t)e^{-ix_2} + z_2(t)e^{-ix_1} + \bar{z}_2(t)e^{ix_1} \\
\kappa u(x_1, x_2, t) &\equiv u(-x_1, x_2, t) = z_1(t)e^{-ix_1} + \bar{z}_1(t)e^{ix_1} + z_2(t)e^{ix_2} + \bar{z}_2(t)e^{-ix_2} \\
P_{p_1,p_2}u(x_1, x_2, t) &\equiv u(x_1 + p_1, x_2 + p_2, t) \\
&= z_1(t)e^{i(x_1+p_1)} + \bar{z}_1(t)e^{-i(x_1+p_1)} + z_2(t)e^{i(x_2+p_2)} + \bar{z}_2(t)e^{-i(x_2+p_2)}
\end{aligned}$$

$$\Longrightarrow \left\{ \begin{aligned}
S_{\pi/2}(z_1, z_2) &\equiv (\bar{z}_2, z_1) \\
\kappa(z_1, z_2) &\equiv (\bar{z}_1, z_2) \\
P_{p_1,p_2}(z_1, z_2) &\equiv (e^{ip_1}z_1, e^{ip_2}z_2)
\end{aligned} \right\}$$

Seek terms (up to cubic) which commute with $P_{p_1,p_2}(z_1, z_2)$

$\boxed{z_1 \to e^{ip_1} z_1}$	$\boxed{z_2 \to e^{ip_2} z_2}$	
$\bar{z}_1 \to e^{-ip_1} \bar{z}_1$	$\bar{z}_2 \to e^{-ip_2} \bar{z}_2$	
$z_1 z_1 \to e^{ip_1} z_1 e^{ip_1} z_1$	$z_1 \bar{z}_1 \to e^{ip_1} z_1 e^{-ip_1} \bar{z}_1$	$\bar{z}_1 \bar{z}_1 \to e^{-ip_1} \bar{z}_1 e^{-ip_1} \bar{z}_1$
$z_2 z_2 \to e^{ip_2} z_2 e^{ip_2} z_2$	$z_2 \bar{z}_2 \to e^{ip_2} z_2 e^{-ip_2} \bar{z}_2$	$\bar{z}_2 \bar{z}_2 \to e^{-ip_2} \bar{z}_2 e^{-ip_2} \bar{z}_2$
$z_1 z_2 \to e^{ip_1} z_1 e^{ip_2} z_2$	$\bar{z}_1 \bar{z}_2 \to e^{-ip_1} \bar{z}_1 e^{-ip_2} \bar{z}_2$	
$z_1 \bar{z}_2 \to e^{ip_1} z_1 e^{-ip_2} \bar{z}_2$	$\bar{z}_1 z_2 \to e^{-ip_1} \bar{z}_1 e^{ip_2} z_2$	
$\boxed{z_1 \bar{z}_1 z_1 \to e^{ip_1} z_1 \bar{z}_1 z_1}$	$\boxed{z_2 \bar{z}_2 z_1 \to e^{ip_1} z_2 \bar{z}_2 z_1}$	
$\boxed{z_1 \bar{z}_1 z_2 \to e^{ip_2} z_1 \bar{z}_1 z_2}$	$\boxed{z_2 \bar{z}_2 z_2 \to e^{ip_2} z_2 \bar{z}_2 z_2}$	

Commute with $\kappa \Longrightarrow$ coefficients real

Commute with $S_{\pi/2} \Longrightarrow$ equality of coefficients

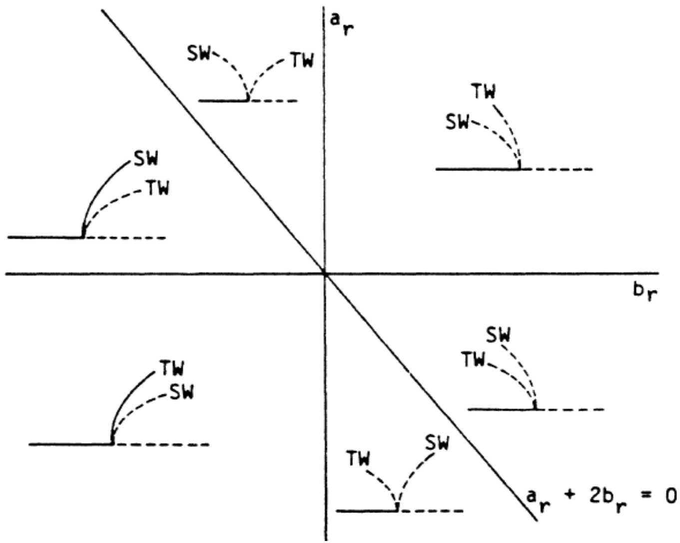
$$
\begin{aligned}
\dot{z}_1 &= \mu z_1 - (a_1 |z_1|^2 + a_2 |z_2|^2) z_1 \\
\dot{z}_2 &= \mu z_2 - (a_2 |z_1|^2 + a_1 |z_2|^2) z_2
\end{aligned}
$$

Same as equation for Hopf bifurcation with $O(2)$ symmetry!

rolls in x_1, x_2 direction $\to$ left, right travelling waves $\to r_1, r_2$

Solutions:

- **Rolls in x_1 direction ($r_1 \neq 0$, $r_2 = 0$)**
- **Rolls in x_2 direction ($r_1 = 0$, $r_2 \neq 0$)**
- **Squares with $r_1 = r_2$**

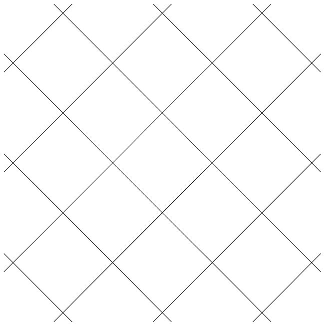


Square patterns

$$\begin{aligned} u(x_1, x_2, t) &= z_1(t)e^{ix_1} + \bar{z}_1(t)e^{-ix_1} + z_2(t)e^{ix_2} + \bar{z}_2(t)e^{-ix_2} \\ &= r(t)e^{ix_1} + r(t)e^{-ix_1} + r(t)e^{ix_2} + r(t)e^{-ix_2} \\ &= 2r(t)(\cos(x_1) + \cos(x_2)) \\ &= 4r(t)\cos\left(\frac{x_1 + x_2}{2}\right)\cos\left(\frac{x_1 - x_2}{2}\right) \end{aligned}$$

Nodal lines $u = 0$ are diagonals with slopes ± 1:

$$x_1 + x_2 = \pi + 2n\pi, \qquad x_1 - x_2 = \pi + 2n\pi$$



Container with square symmetry

Eigenvector ψ_1 with two rolls along $x_1 \Longrightarrow$

Rotated eigenvector ψ_2 with two rolls along $x_2 \Longrightarrow$

Four equivalent branches resembling $\pm\psi_1$, $\pm\psi_2$ generated at bifurcation

$\psi_1 + \psi_2$ is also an eigenvector, oriented along diagonal

Four other branches resembling $\pm(\psi_1 + \psi_2)$, $\pm(\psi_1 - \psi_2)$

Equivalent to each other but not to branches oriented along x_1 or x_2

Different stability, secondary bifurcations

Two sets of four branches generated at pitchfork

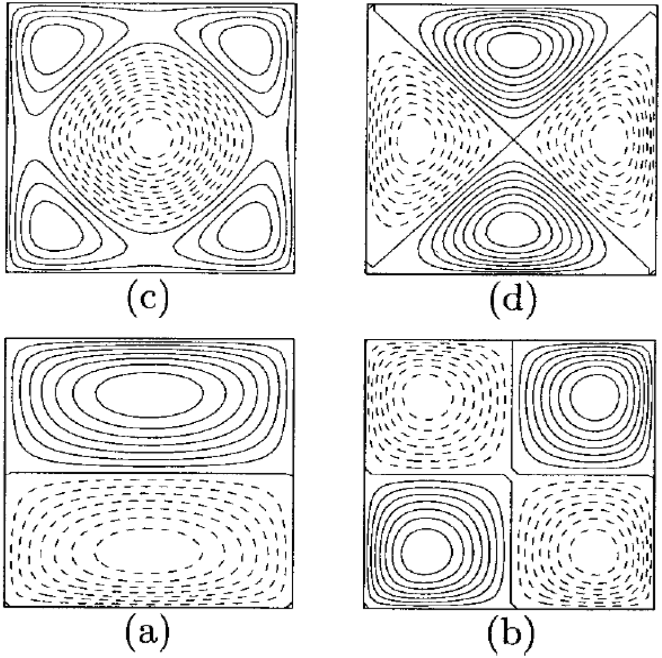
Marangoni convection:
results from temperature dependence of surface tension at free surface

Fluid dragged along surface $\Longrightarrow$ vertical motion (conservation of mass)

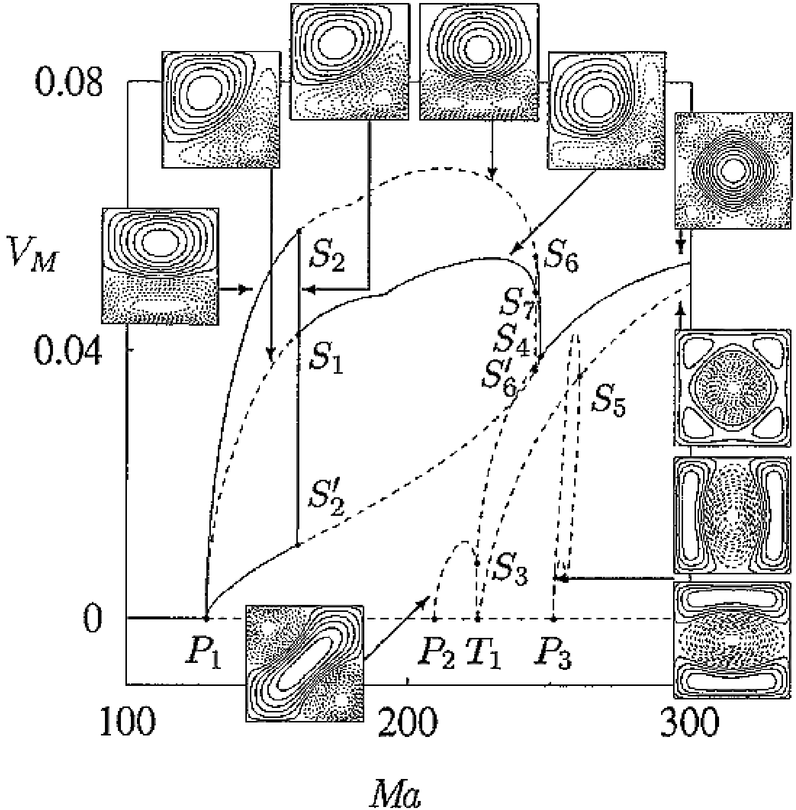
Eigenvector with full D_4 symmetry $\Longrightarrow$ transcritical bifurcation

Eigenvectors of Marangoni convection in container with square horizontal cross section

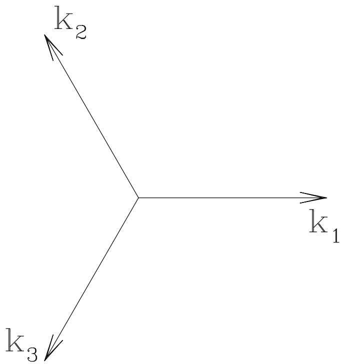
From Bergeon, Henry, Knobloch., Phys. Fluids 13, 92 (2001)



Bifurcation diagram of Marangoni convection in container with square cross section



Hexagons



$$u(x,y,t) = z_1(t)e^{i\mathrm{k}_1\cdot\mathrm{x}} + z_2(t)e^{i\mathrm{k}_2\cdot\mathrm{x}} + z_3(t)e^{i\mathrm{k}_3\cdot\mathrm{x}} + c.c.$$

Groups D_6 and T^2 generated by

$$\begin{aligned} S_{2\pi/3}(z_1,z_2,z_3) &\equiv (z_3,z_1,z_2) \\ \kappa(z_1,z_2,z_3) &\equiv (\bar{z}_1,\bar{z}_2,\bar{z}_3) \\ P_{\mathrm{p}}(z_1,z_2,z_3) &\equiv (e^{i\mathrm{k}_1\cdot\mathrm{p}}z_1, e^{i\mathrm{k}_2\cdot\mathrm{p}}z_2, e^{i\mathrm{k}_3\cdot\mathrm{p}}z_3) \end{aligned}$$

$\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3 = 0 \Longrightarrow$ **quadratic terms allowed!**

$$\bar{z}_2\bar{z}_3 \to e^{-i\mathrm{k}_2\cdot\mathrm{p}}\bar{z}_2 e^{-i\mathrm{k}_3\cdot\mathrm{p}}\bar{z}_3 = e^{-i(\mathrm{k}_2+\mathrm{k}_3)\cdot\mathrm{p}}\bar{z}_2\bar{z}_3 = e^{i\mathrm{k}_1\cdot\mathrm{p}}\bar{z}_2\bar{z}_3$$

Can include $\bar{z}_2\bar{z}_3$ in evolution equation for z_1

$$
\begin{aligned}
\dot{z}_1 &= \left(\mu - b|z_1|^2 - c(|z_2|^2 + |z_3|^2)\right) z_1 + a\bar{z}_2\bar{z}_3 \\
\dot{z}_2 &= \left(\mu - b|z_2|^2 - c(|z_3|^2 + |z_1|^2)\right) z_2 + a\bar{z}_3\bar{z}_1 \\
\dot{z}_3 &= \left(\mu - b|z_3|^2 - c(|z_1|^2 + |z_2|^2)\right) z_3 + a\bar{z}_1\bar{z}_2 \\
z_j &= r_j e^{i\phi_j}
\end{aligned}
$$

Hexagons: $r_1 = r_2 = r_3 = r$

$$
\left(\dot{r} + ri\dot{\phi}_1\right) = (\mu - (b+2c)r^2)r + ar^2 e^{-i(\phi_1+\phi_2+\phi_3)}
$$

$$
\begin{aligned}
\dot{r} &= (\mu - (b+2c)r^2)r + ar^2\cos(\phi_1+\phi_2+\phi_3) \\
r\dot{\phi}_1 &= -ar^2\sin(\phi_1+\phi_2+\phi_3)
\end{aligned}
$$

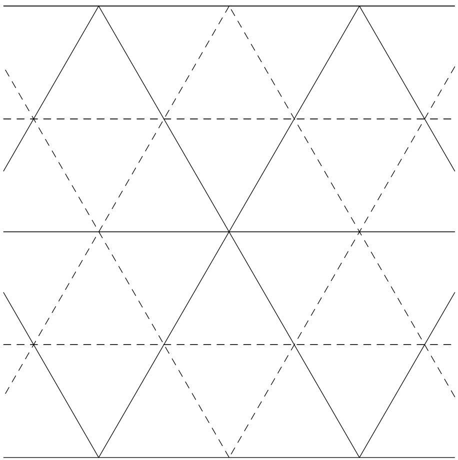
Steady states:

$$
\Phi \equiv \phi_1 + \phi_2 + \phi_3 = 0, \pi \implies \cos(\Phi) = \pm 1
$$

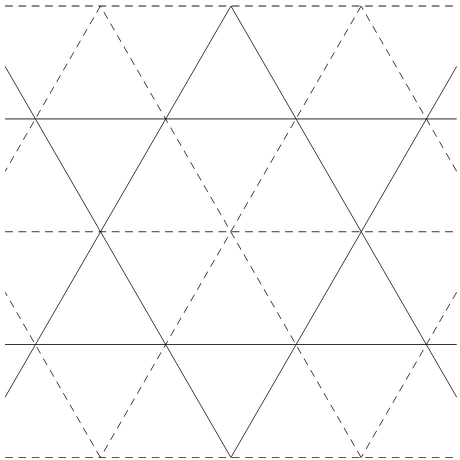
$$
0 = \mu - (b+2c)r^2 \pm ar \implies r = \begin{cases} \frac{1}{b+2c}\left[-a \pm \sqrt{a^2 + 4\mu(b+2c)}\right] \\ \frac{1}{b+2c}\left[+a \pm \sqrt{a^2 + 4\mu(b+2c)}\right] \end{cases}
$$

Hexagons

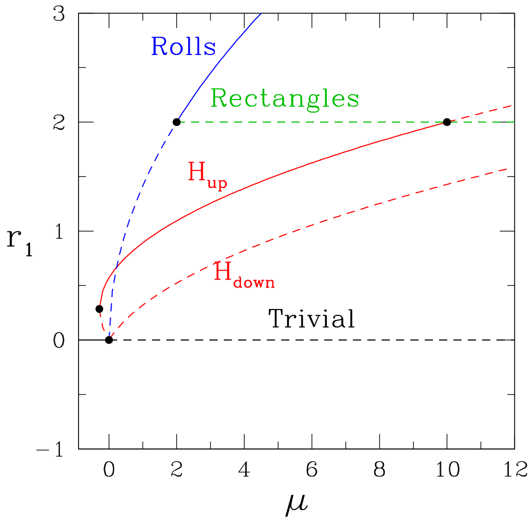
Up hexagons: $\Phi = 0$



Down hexagons: $\Phi = \pi$



Bifurcation Diagram



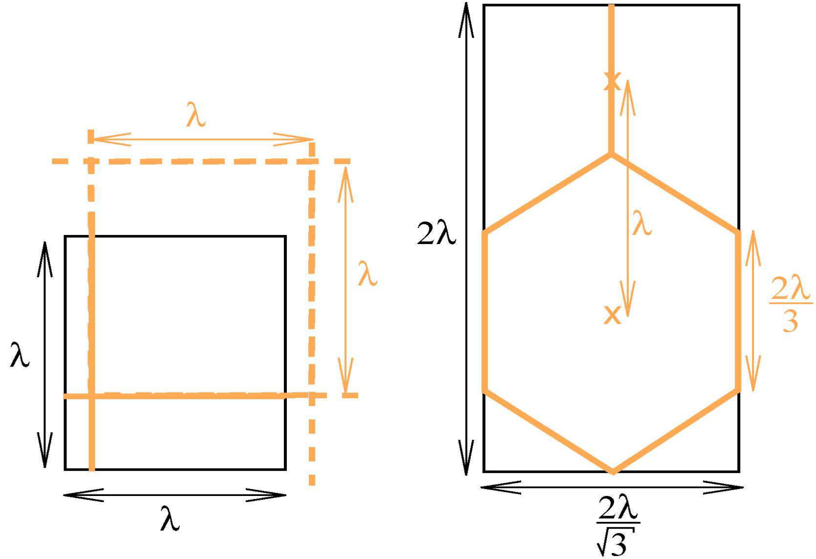
Up- and down-hexagons bifurcate transcritically from 0 at $\mu = 0$

Saddle-node bifurcation at $\mu = -a^2/(4(b+2c))$, where $r = a$

Rolls created at pitchfork bifurcation

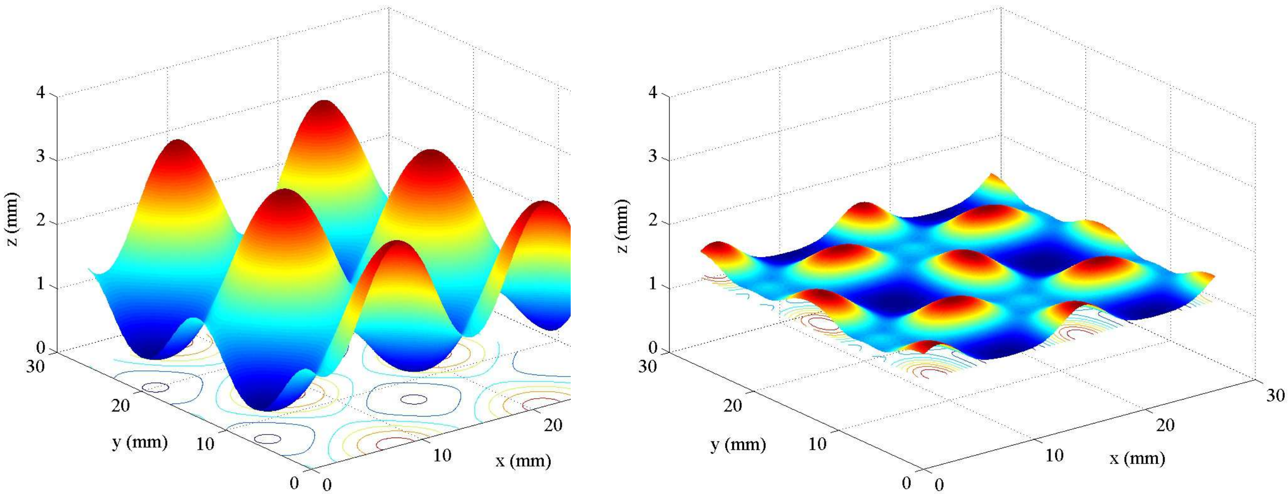
Rectangles created in secondary bifurcation from roll branch

Squares and hexagons in simulation of Faraday experiment



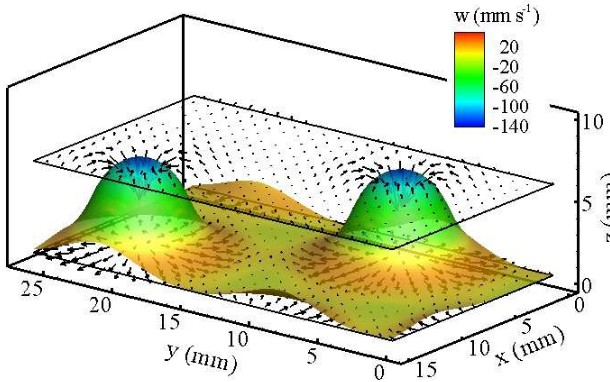
Boxes supporting the periodic patterns in the square and hexagonal cases.

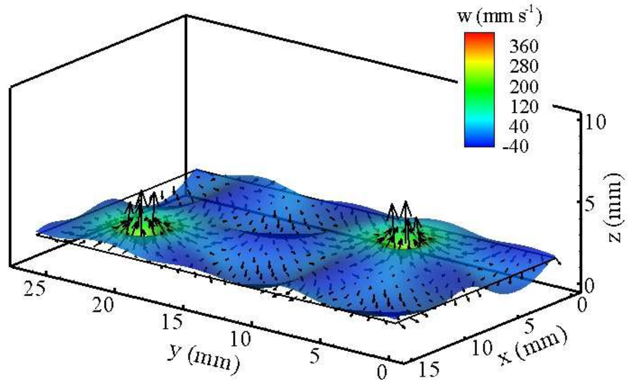
Square Pattern in Faraday Simulation

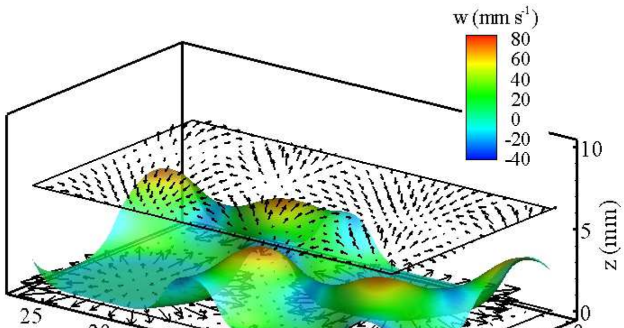


From Perinet, Juric, Tuckerman, J. Fluid Mech. 635, 1 (2009)

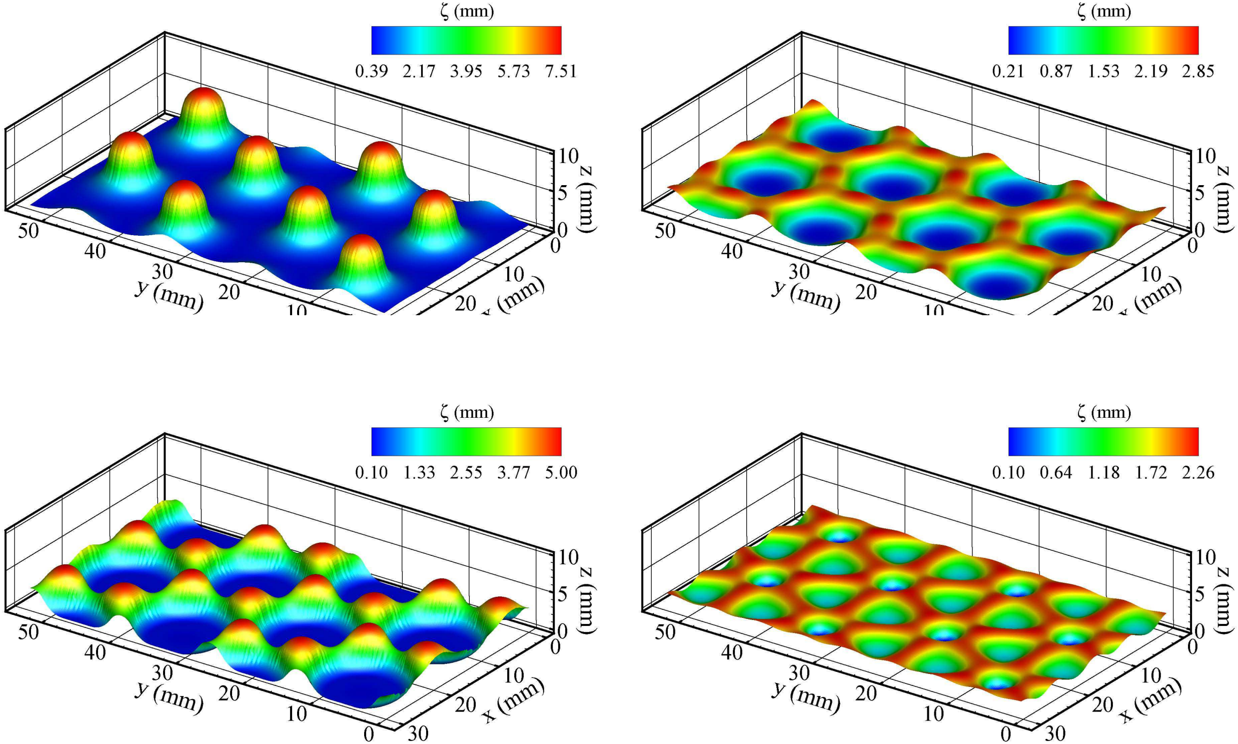
Hexagonal Patterns: interface and velocity fields



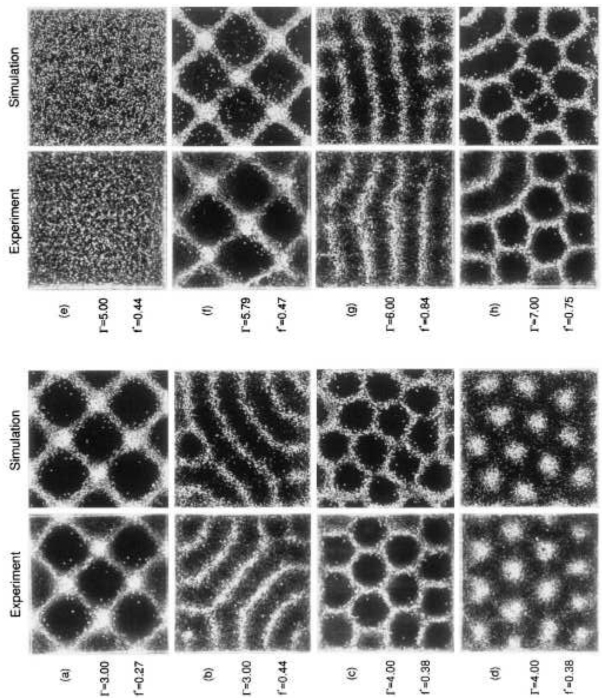




Hexagonal pattern in Faraday simulation

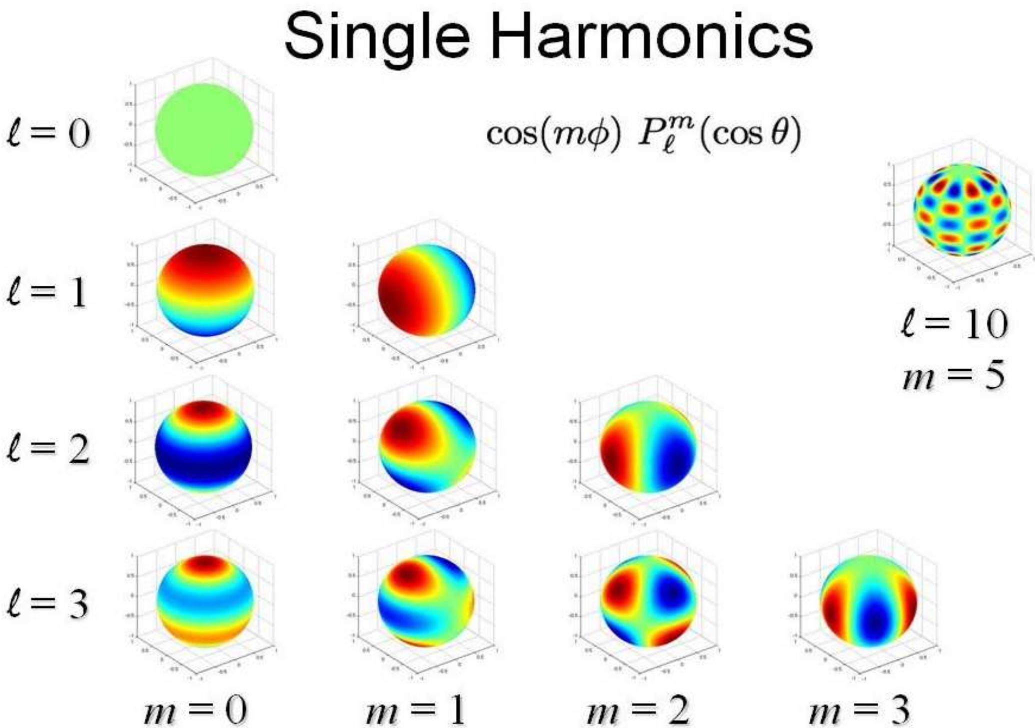


Squares, stripes, hexagons in a granular layer



From Bizon, Shattuck, Swift, McCormick & Swinney, *Patterns in 3D vertically oscillated granular layers: simulation and experiment*, **Phys. Rev. Lett. 80, 57 (1998).**

Spherical Symmetry



Bifurcation diagram for $\ell = 6$:

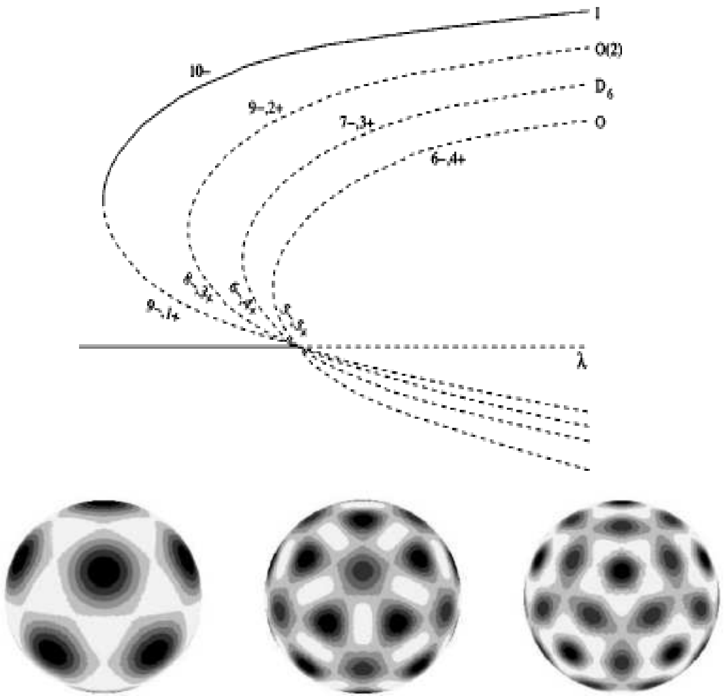


FIG. 2. Stationary solutions of Eq. (2) with icosahedral symmetry for $l=6$, $l=10$, and $l=12$.

Matthews, Phys. Rev. E, 2003 and Nonlinearity, 2003

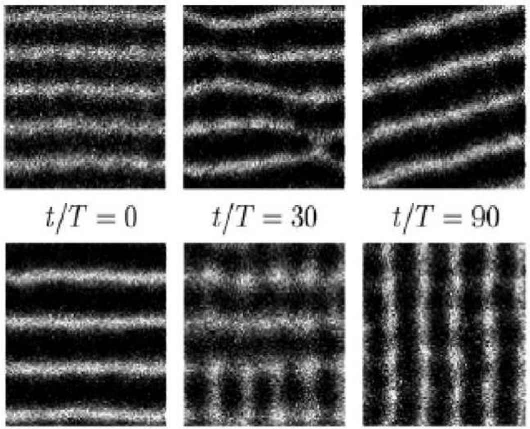
Instabilities of roll patterns

Swift-Hohenberg equation reproduces instabilities of striped (roll) patterns:

(E)	**Eckhaus:**	**change in wavelength**
(Z)	**zigzag:**	**sinusoidal in-phase oscillations along roll axes**
(SV)	**skew-varicose:**	**sinusoidal out-of-phase oscillations along roll axes**
(CR)	**cross-roll:**	**appearance of perpendicular rolls**
(OS)	**oscillatory:**	**time-dependent oscillations along roll axes**

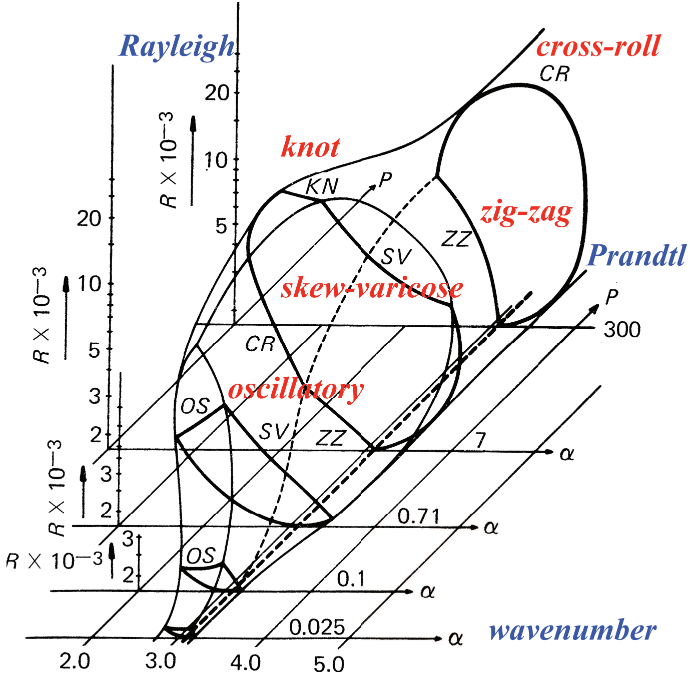
skew-varicose

cross-roll



Vertically vibrated granular layer. From de Bruyn, Bizon, Shattuck, Goldman, Swift, Swinney, Phys. Rev. Lett. 81, 1421 (1998)

Busse Balloon for RB Convection (F. Busse & R. Clever, 1967–79)



From F.H. Busse, Transition to turbulence in Rayleigh-Bénard convection, in Hydrodynamic Instabilities and the Transition to Turbulence, ed. by H.L. Swinney and J.P. Gollub, Springer, 1981.

Newell-Whitehead-Segur equation (1969)

Describes stability of rolls with wavenumber close to critical q_c

Amplitude $A(X, Y, T)$ with X, Y, T slow variables

$$u(x, y, t) = A(X, Y, T)e^{iq_c x} + c.c.$$

$$\text{Swift-Hohenberg} \quad \underset{\text{multiple scales}}{\overset{\text{method of}}{\Longrightarrow}} \quad \text{NWS}$$

$$\partial_T A = \mu A - |A|^2 A + \left(\partial_X - \frac{i}{2}\partial_{YY}\right)^2 A$$

Revert $X, Y, Z \to x, y, z$

$u \sim e^{iq_c x}$	A constant
$u \sim e^{i(q_c+q)x}$	$A \sim e^{iqx}$

Newell-Whitehead-Segur equation

$$\partial_t A = \mu A - |A|^2 A + \left(\partial_x - \frac{i}{2}\partial_{yy}\right)^2 A$$

Linear stability of $A = 0$ (uniform basic state):

$$u \sim e^{i(q_c+q)x} \Longrightarrow A_q \sim e^{iqx}$$

$$\lambda = \mu - q^2 \Longrightarrow \mu_c = q^2$$

$$0 = \mu - |A_q|^2 - q^2 \Longrightarrow A_q = \sqrt{\mu - q^2}e^{i\phi}e^{iqx}$$

Linear stability analysis of A_q (rolls with wavenumber $q + q_c$):
Substitute $A_q + e^{\sigma t}a(x, y)$ into NWS, drop nonlinear terms:

$$\sigma a = \mu a - 2|A_q|^2 a - A_q^2 a^* + \left(\partial_x - \frac{i}{2}\partial_{yy}\right)^2 a$$

Eckhaus Instability

Eigenvectors varying in x only:

$$a_0(x) \equiv \alpha_0 e^{iqx}$$

$$a_k(x) \equiv \alpha_k e^{i(q+k)x} + \beta_k e^{i(q-k)x}, \quad k > 0$$

$$\sigma a_0 = -(\mu - q^2)\alpha_0 e^{iqx} - (\mu - q^2)\alpha_0^* e^{iqx}$$

or

$$\sigma_0 \begin{pmatrix} \alpha_0^R \\ \alpha_0^I \end{pmatrix} = \begin{pmatrix} -2(\mu - q^2) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0^R \\ \alpha_0^I \end{pmatrix}$$

Eigenvalues/vectors of circle pitchfork:

$\sigma_0 = -2(\mu - q^2) \rightarrow$ **contraction along radius**

$\sigma_0 = 0 \rightarrow$ **marginal direction around circle**

For $k > 0$, $a_k(x) \equiv \alpha_k e^{i(q+k)x} + \beta_k e^{i(q-k)x}$

$$\begin{aligned}
(\mu \quad & -2|A_q|^2 + \partial_{xx})a_k \\
&= \left(\mu - 2(\mu - q^2) - (q+k)^2\right)\alpha_k e^{i(q+k)x} + \left(\mu - 2(\mu - q^2) - (q-k)^2\right)\beta_k e^{i(q-k)x} \\
&= -\left(\mu - q^2 + k^2 + 2qk\right)\alpha_k e^{i(q+k)x} - \left(\mu - q^2 + k^2 - 2qk)\right)\beta_k e^{i(q-k)x}
\end{aligned}$$

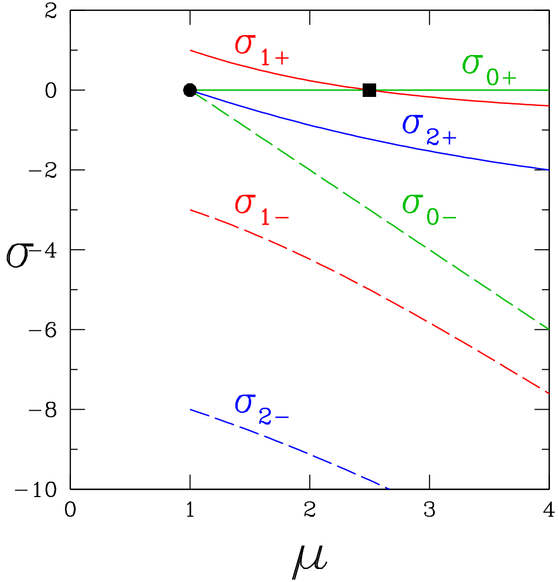
$$A_q^2 a_k^* = (\mu - q^2)e^{i2qx}\left(\alpha_k^* e^{-i(q+k)x} + \beta_k^* e^{-i(q-k)x}\right) = (\mu - q^2)\left(\alpha_k^* e^{i(q-k)x} + \beta_k^* e^{i(q+k)x}\right)$$

$$\begin{aligned}
\sigma_k a_k &= \left(\mu - 2|A_q|^2 + \partial_{xx}\right)a - A_q^2 a^* \\
&= -\left(\mu - q^2 + k^2 + 2qk\right)\alpha_k e^{i(q+k)x} - \left(\mu - q^2 + k^2 - 2qk\right)\beta_k e^{i(q-k)x} \\
&\quad -(\mu - q^2)\left(\alpha_k^* e^{i(q-k)x} + \beta_k^* e^{i(q+k)x}\right)
\end{aligned}$$

$$\sigma_k \begin{pmatrix} \alpha_k^R \\ \beta_k^R \end{pmatrix} = \begin{pmatrix} -(\mu - q^2 + k^2) - 2qk & -(\mu - q^2) \\ -(\mu - q^2) & -(\mu - q^2 + k^2) + 2qk \end{pmatrix} \begin{pmatrix} \alpha_k^R \\ \beta_k^R \end{pmatrix}$$

$$\sigma_k \begin{pmatrix} \alpha_k^I \\ \beta_k^I \end{pmatrix} = \begin{pmatrix} -(\mu - q^2 + k^2) - 2qk & (\mu - q^2) \\ (\mu - q^2) & -(\mu - q^2 + k^2) + 2qk \end{pmatrix} \begin{pmatrix} \alpha_k^R \\ \beta_k^R \end{pmatrix}$$

$$\sigma_{k\pm} = -(\mu - q^2 + k^2) \pm \sqrt{(2qk)^2 + (\mu - q^2)^2}$$



Eigenvalues $\sigma_{k\pm}$ of A_q for $q = 1$, $k = 0, 1, 2$. Branch A_q created from trivial state at primary circle pitchfork bifurcation ● and stabilized at secondary Eckhaus bifurcation ■

$$\sigma_{k\pm} = -(\mu - q^2 + k^2) \pm \sqrt{(2qk)^2 + (\mu - q^2)^2}$$

At threshold $\mu = q$, A_q is unstable if $|q| > 1/2$ since

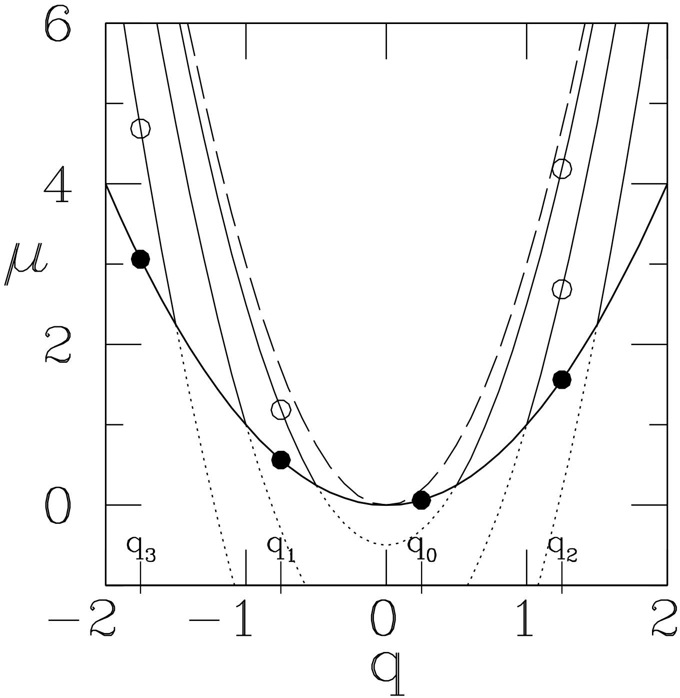
$$\sigma_{k+} = -k^2 + |2qk| = k(2|q| - k) > 0 \text{ for } k < 2|q|$$

Eigenvalues cross zero and become negative at:

$$\begin{aligned}
(\mu - q^2 + k^2)^2 &= (2qk)^2 + (\mu - q^2)^2 \\
k^4 + 2(\mu - q^2)k^2 &= (2qk)^2 \\
(\mu - q^2) &= 2q^2 - \frac{k^2}{2} \\
\mu &= 3q^2 - \frac{k^2}{2}
\end{aligned}$$

Last bifurcation has $k = 1$ and is Eckhaus instability:

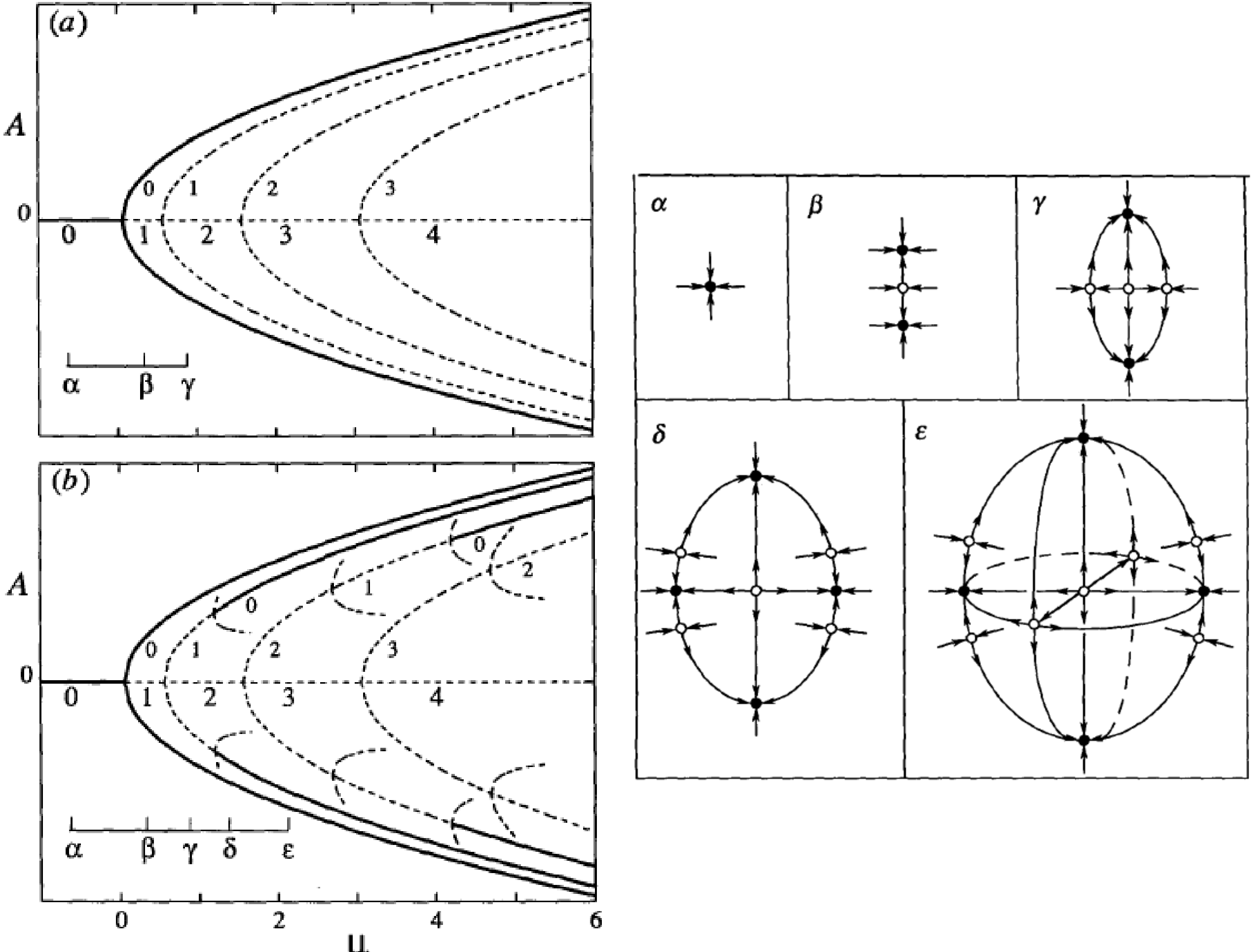
$$\mu_E = 3q^2 - \frac{1}{2}$$



● **primary bifurcation points along** $\mu = q^2$

○ **secondary bifurcation points along** $\mu = 3q^2 - k^2/2$

Eckhaus bifurcation points along $\mu = 3q^2 - 1/2$


(a)
A
0
0 1 2 3 4
0 1 2 3
α β γ
(b)
A
0
0 1 2 3 4
0 1 2 3
0 1 0 2
α β γ δ ε
0 2 4 6
μ
α
β
γ
δ
ε

Zig-zag instability

Eigenvectors depend on x and y:

$$a_m(x) \equiv \alpha_m e^{i(qx+my)} + \beta_m e^{i(qx-my)}, \quad m > 0$$

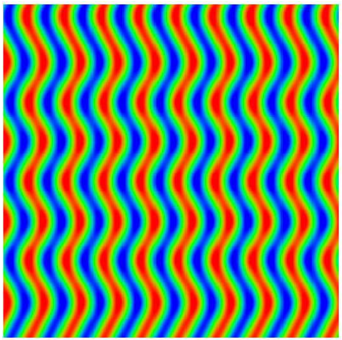
$$\begin{aligned}
\left(\partial_x - \frac{i}{2}\partial_{yy}\right)^2 a_m &= \left(iq - \frac{i}{2}(im)^2\right)^2 a_m = -\left(q^2 + m^2\left(q + \frac{m^2}{4}\right)\right) a_m \\
(\mu - 2|A_q|^2) a_m &= (\mu - 2(\mu - q^2)) a_m = (-\mu + 2q^2) a_m \\
A_q^2 a_m^* &= (\mu - q^2) e^{i2qx}\left(\alpha_m^* e^{-i(qx+my)} + \beta_m^* e^{-i(qx-my)}\right) \\
&= (\mu - q^2)\left(\alpha_m^* e^{i(qx-my)} + \beta_m^* e^{i(qx+my)}\right)
\end{aligned}$$

$$\begin{aligned}
&\sigma_m\left(\alpha_m e^{i(qx+my)} + \beta_m e^{i(qx-my)}\right) \\
&= \left(-(\mu - q^2) - m^2\left(q + \frac{m^2}{4}\right)\right)\left(\alpha_m e^{i(qx+my)} + \beta_m e^{i(qx-my)}\right) \\
&-(\mu - q^2)\left(\alpha_m^* e^{i(qx-my)} + \beta_m^* e^{i(qx+my)}\right)
\end{aligned}$$

$$\sigma_m \begin{pmatrix} \alpha_m^R \\ \beta_m^R \end{pmatrix} = \begin{pmatrix} -(\mu - q^2) - m^2\left(q + \frac{m^2}{4}\right) & -(\mu - q^2) \\ -(\mu - q^2) & -(\mu - q^2) - m^2\left(q + \frac{m^2}{4}\right) \end{pmatrix} \begin{pmatrix} \alpha_m^R \\ \beta_m^R \end{pmatrix}$$

Eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ **are** $\sigma = \frac{a+a}{2} \pm \sqrt{\left(\frac{(a-a)}{2}\right)^2 + b^2} = a \pm b$

$$\begin{aligned} \sigma_{m\pm} &= -(\mu - q^2) - m^2\left(q + \frac{m^2}{4}\right) \pm (\mu - q^2) \\ &= \begin{cases} -m^2\left(q + \frac{m^2}{4}\right) \\ -2(\mu - q^2) - m^2\left(q + \frac{m^2}{4}\right) \end{cases} \\ \sigma_{m+} &> 0 \text{ for all } \mu \text{ if } q + \frac{m^2}{4} < 0 \iff q < -\frac{m^2}{4} \end{aligned}$$



Instability:

Recall pattern has wavenumber $q + q_c$

$|q| \uparrow \Longrightarrow$ **more unstable** σ_{m+}**'s**

$q < 0 \iff q + q_c < q_c \iff \lambda + \lambda_c > \lambda_c$

Rolls bend $\Longrightarrow$ **wavelengths decrease**