Cours : Dynamique Non-Linéaire

Laurette TUCKERMAN
laurette@pmmh.espci.fr

VII. Reaction-Diffusion Equations:

1. Excitability

2. Turing patterns

3. Lyapunov functionals

4. Spatial analysis and fronts

Reaction-Diffusion Systems

$$\partial_t u_i = \underbrace{f_i(u_1, u_2, \ldots)}_{\text{reaction}} + \underbrace{D_i \Delta u_i}_{\text{diffusion}}$$

Reactions f_i couple different species u_i at same location

Diffusivity D_i couples same species u_i at different locations

Describe oscillating chemical reactions, such as famous Belousov-Zhabotinskii reaction, discovered by two Soviet scientists in 1950s-1960s.

Also describe phenomena in

–biology (population biology, epidemiology, neurosciences)

–social sciences (economics, demography)

–physics

Two species

$$\partial_t u = f(u,v) + D_u \Delta u$$
$$\partial_t v = g(u,v) + D_v \Delta v$$

Spatially homogeneous

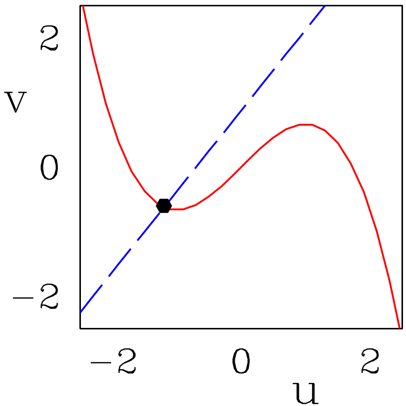
$$\partial_t u = f(u,v)$$
$$\partial_t v = g(u,v)$$

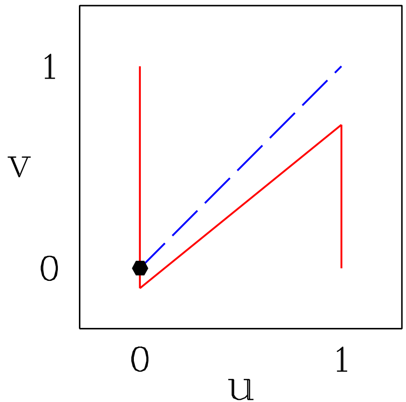
FitzHugh-Nagumo model

$$f(u,v) = u - u^3/3 - v + I$$
$$g(u,v) = 0.08\,(u + 0.7 - 0.8\,v)$$

Barkley model

$$f(u,v) = \frac{1}{\epsilon}\,u(1-u)\left(u - \frac{v+b}{a}\right)$$
$$g(u,v) = u - v$$



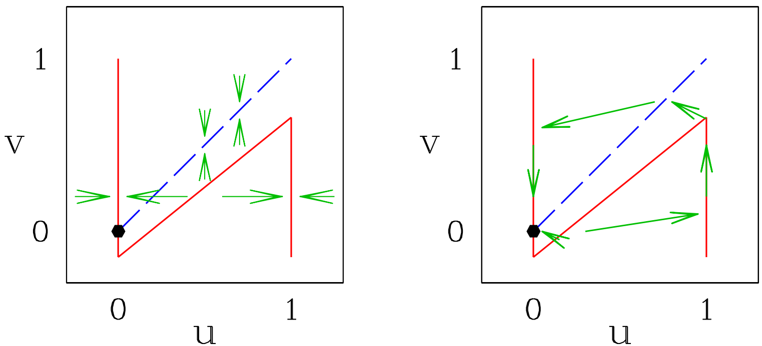


u-nullclines $f(u,v) = 0$, v-nullclines $g(u,v) = 0$, ● steady states

stable if eigenvalues of $\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ **have negative real parts**

Excitability

$$f(u,v) = \frac{1}{\epsilon}\, u(1-u)\left(u - \frac{v+b}{a}\right) \qquad g(u,v) = u - v$$

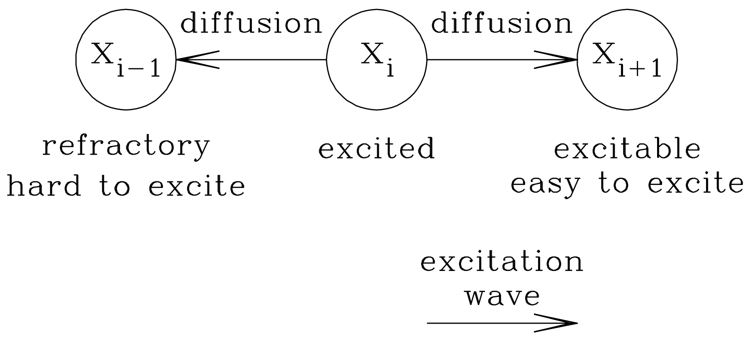


$\partial_t u = f = 0$ **separates** $\longleftarrow$ **and** $\longrightarrow$ $O(\epsilon^{-1})$

$\partial_t v = g = 0$ **separates** $\uparrow$ **and** $\downarrow$ $O(1)$

$u = 1$		**excited phase**
$u = 0$	$v \sim 1$	**refractory phase**
$u = 0$	$v \ll 1$	**excitable phase**
$u = (v+b)/a$		**excitation threshold**

Waves in Excitable Medium



Spatial variation + diffusion + excitability $\Longrightarrow$ propagating waves

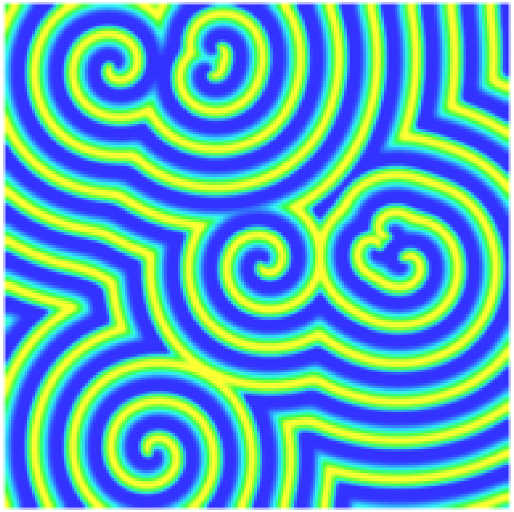
Excitable media in physiology:

–neurons

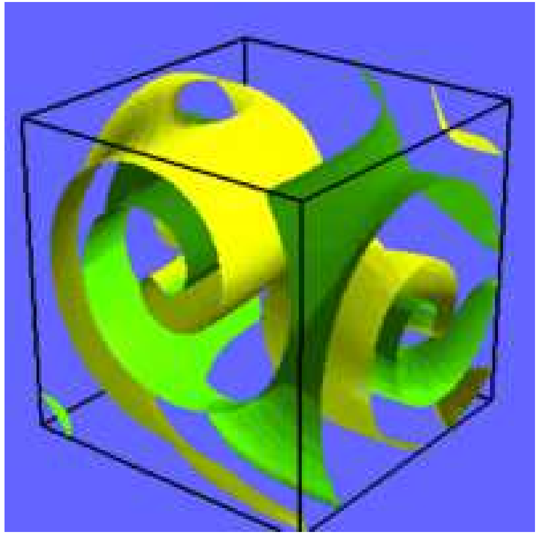
–cardiac tissue (the heart)

Pacemaker periodically emits electrical signals, propagated to rest of heart

Simulations from *Barkley model*, Scholarpedia



Spiral waves in 2D



Spiral waves in 3D

Turing patterns

Instability of homogeneous solutions $(\bar{u}, \bar{v})$ to reaction-diffusion systems

$$\left\{ \begin{array}{l} 0 = f(\bar{u}, \bar{v}) \\ 0 = g(\bar{u}, \bar{v}) \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} 0 = f(\bar{u}, \bar{v}) + D_u \Delta \bar{u} \\ 0 = g(\bar{u}, \bar{v}) + D_v \Delta \bar{v} \end{array} \right\}$$

What about stability? Does diffusion damp spatial variations?

Linear stability analysis:

$$\left\{ \begin{array}{l} u(x,t) = \bar{u} + \tilde{u} e^{\sigma t + i\mathrm{k}\cdot\mathrm{x}} \\ v(x,t) = \bar{v} + \tilde{v} e^{\sigma t + i\mathrm{k}\cdot\mathrm{x}} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} \sigma \tilde{u} = f_u \tilde{u} + f_v \tilde{v} - D_u k^2 \tilde{u} \\ \sigma \tilde{v} = g_u \tilde{u} + g_v \tilde{v} - D_v k^2 \tilde{v} \end{array} \right\}$$

$$M_k \equiv \begin{pmatrix} f_u - D_u k^2 & f_v \\ g_u & g_v - D_v k^2 \end{pmatrix} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} - k^2 \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}$$

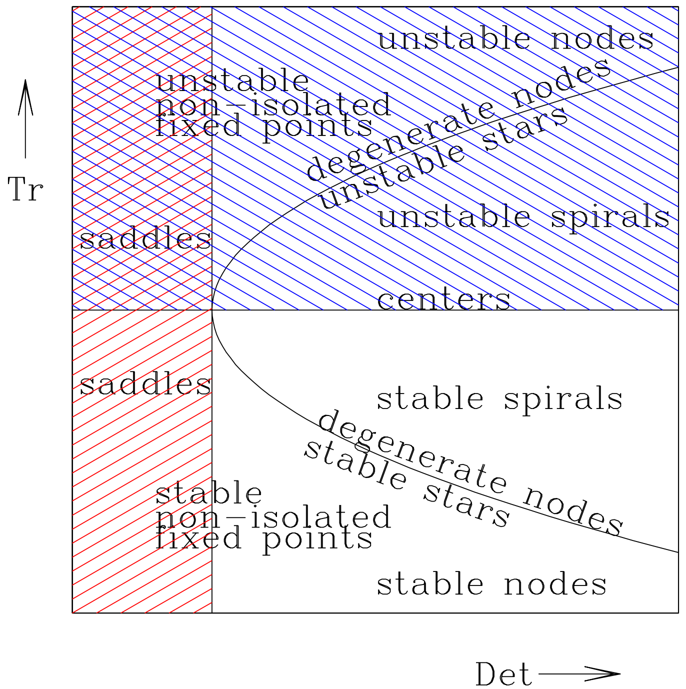
If $D_u = D_v \equiv D$, then

$$\sigma_{k\pm} = \sigma_{0\pm} - k^2 D \leq \sigma_{0\pm}$$

$(\bar{u}, \bar{v})$ stable to homogeneous perturbations $\Longrightarrow$
$(\bar{u}, \bar{v})$ stable to inhomogeneous perturbations. Diffusion is stabilizing.

Alan Turing (famous WW II UK cryptologist, founder of computer science)

1952: homogeneous state can be unstable if $D_u \neq D_v$



For instability, need $\mathbf{Tr}_k > 0$ or $\mathbf{Det}_k < 0$

For instability, need $\mathbf{Tr}_k > 0$ or $\mathbf{Det}_k < 0$

$$\text{Homogeneous stability} \iff \left\{ \begin{array}{l} \mathrm{Tr}_0 = f_u + g_v < 0 \quad \text{and} \\ \mathrm{Det}_0 = f_u g_v - f_v g_u > 0 \end{array} \right\}$$

$$\mathrm{Tr}_k = f_u + g_v - (D_u + D_v)k^2 = \mathrm{Tr}_0 - (D_u + D_v)k^2 < \mathrm{Tr}_0 < 0$$

So for instability, need $\mathbf{Det}_k < 0$

$$\begin{aligned} \mathrm{Det}_k &= f_u g_v - f_v g_u + D_u D_v k^4 - (D_v f_u + D_u g_v)k^2 \\ &= \underbrace{\mathrm{Det}_0}_{>0} + \underbrace{D_u D_v k^4}_{>0,\ \text{dominates for } k \gg 1} - (D_v f_u + D_u g_v)k^2 \end{aligned}$$

Find negative minimum for intermediate k^2:

$$\begin{aligned} 0 &= \left. \frac{\mathrm{d}\,\mathrm{Det}_k}{\mathrm{d}k^2} \right|_{k_*} = 2D_u D_v k_*^2 - (D_v f_u + D_u g_v) \\ k_*^2 &= \frac{D_v f_u + D_u g_v}{2 D_u D_v} \implies \text{need } D_v f_u + D_u g_v > 0 \end{aligned}$$

Need $\mathrm{Det}_k < 0$ at $k_*^2 = (D_v f_u + D_u g_v)/(2D_u D_v)$:

$$\begin{aligned}
0 > \mathrm{Det}_k|_{k_*} &= \mathrm{Det}_0 + D_u D_v k_*^4 - (D_v f_u + D_u g_v) k_*^2 \\
&= \mathrm{Det}_0 + \frac{(D_v f_u + D_u g_v)^2}{4D_u D_v} - \frac{2(D_v f_u + D_u g_v)^2}{4D_u D_v} \\
&= \mathrm{Det}_0 - \frac{(D_v f_u + D_u g_v)^2}{4D_u D_v} \\
0 &> 4D_u D_v (f_u g_v - f_v g_u) - (D_v f_u + D_u g_v)^2
\end{aligned}$$

Collecting the four conditions:

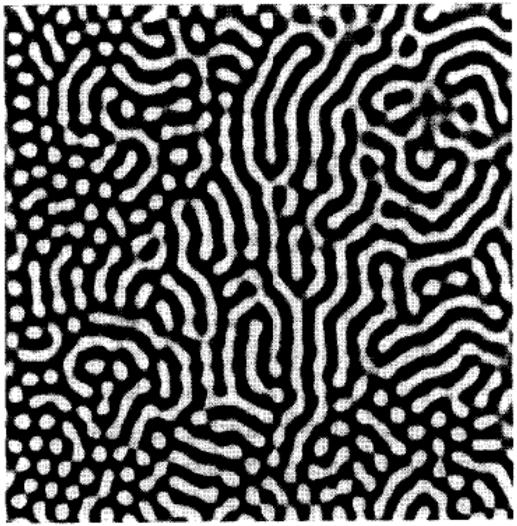
$$\mathrm{Tr}_0 = f_u + g_v < 0$$

$$\mathrm{Det}_0 = f_u g_v - f_v g_u > 0$$

$$2D_u D_v k_*^2 = D_v f_u + D_u g_v > 0$$

$$4D_u D_v\ \mathrm{Det}_k|_{k_*} = 4D_u D_v (f_u g_v - f_v g_u) - (D_v f_u + D_u g_v)^2 < 0$$

Turing patterns were first produced experimentally:
–in 1990 by de Kepper et al. at Univ. of Bordeaux
–in 1992 by Swinney et al. at Univ. of Texas at Austin



Turing pattern in a chlorite-iodide-malonic acid chemical laboratory experiment. From R.D. Vigil, Q. Ouyang & H.L. Swinney, *Turing patterns in a simple gel reactor*, **Physica A 188, 17 (1992)**

Might be mechanism for:
–differentiation within embryos
–formation of patterns on animal coats, e.g. zebras and leopards

Lyapunov functionals

1D systems: no limit cycles, usually just convergence to fixed point
Generalize to multidimensional variational, potential, or gradient flows:

$$\frac{d\mathbf{u}}{dt} = -\nabla\Phi \qquad \Longleftrightarrow \qquad \frac{du_i}{dt} = -\frac{\partial\Phi}{\partial u_i}$$

For gradient flow, Jacobian is Hessian matrix:

$$\mathcal{H} = -\begin{pmatrix} \partial^2\Phi/(\partial u_1 \partial u_1) & \partial^2\Phi/(\partial u_1 \partial u_2) & \ldots \\ \partial^2\Phi/(\partial u_2 \partial u_1) & \partial^2\Phi/(\partial u_2 \partial u_2) & \ldots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$\mathcal{H}$ **symmetric** $\Longrightarrow$ **no complex eigenvalues** $\Longrightarrow$ **no Hopf bifurcations**

$$\frac{d\Phi}{dt} = \sum_i \frac{\partial\Phi}{\partial u_i}\frac{du_i}{dt} = -\sum_i \frac{\partial\Phi}{\partial u_i}\frac{\partial\Phi}{\partial u_i} = -|\nabla\Phi|^2$$

Φ **decreases monotonically, either to** $-\infty$ **or to point where**
$d\mathbf{u}/dt = -\nabla\Phi = 0 \Longrightarrow$ **no limit cycles**

Generalize to reaction-diffusion systems involving potential $\Phi(u)$:

$$\frac{\partial \mathrm{u}}{\partial t} = -\nabla\Phi + \frac{\partial^2 \mathrm{u}}{\partial x^2} \qquad \text{on } x_{\mathrm{lo}} \le x \le x_{\mathrm{hi}}$$

Boundary conditions:

	Dirichlet	$\mathrm{u}(x_{\mathrm{lo}}) = \mathrm{u}_{\mathrm{lo}}$	$\mathrm{u}(x_{\mathrm{hi}}) = \mathrm{u}_{\mathrm{hi}}$
or	**Neumann (homogeneous)**	$\frac{\partial \mathrm{u}}{\partial x}(x_{\mathrm{lo}}) = 0$	$\frac{\partial \mathrm{u}}{\partial x}(x_{\mathrm{hi}}) = 0$

Define free energy or Lyapunov functional:

$$\mathcal{F}(\mathrm{u}) \equiv \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\underbrace{\Phi(\mathrm{u}(x,t))}_{\text{potential energy}} + \underbrace{\frac{1}{2}\left|\frac{\partial \mathrm{u}(x,t))}{\partial x}\right|^2}_{\text{kinetic energy}} \right]$$

Seek quantity analogous to gradient:

$$F(\mathrm{x} + \mathrm{dx}) = \mathrm{F}(\mathrm{x}) + \nabla\mathrm{F}(\mathrm{x}) \cdot \mathrm{dx} + \mathrm{O}(|\mathrm{dx}|)^2 \text{ for all dx}$$

The functional derivative $\delta\mathcal{F}/\delta\mathrm{u}$ is defined to be such that

$$\mathcal{F}(\mathrm{u} + \delta\mathrm{u}) = \mathcal{F}(\mathrm{u}) + \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \, \frac{\delta\mathcal{F}}{\delta\mathrm{u}} \cdot \delta\mathrm{u} + O(\delta\mathrm{u})^2 \text{ for every } \delta\mathrm{u}$$

Expand:

$$\begin{aligned}
\mathcal{F}(\mathrm{u}+\delta\mathrm{u}) &= \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\Phi(\mathrm{u}+\delta\mathrm{u}) + \frac{1}{2}\left|\frac{\partial(\mathrm{u}+\delta\mathrm{u})}{\partial x}\right|^2\right] \\
&= \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\Phi(\mathrm{u}) + \nabla\Phi(\mathrm{u})\cdot\delta\mathrm{u} + \ldots + \frac{1}{2}\left|\frac{\partial\mathrm{u}}{\partial x} + \frac{\partial\delta\mathrm{u}}{\partial x} + \ldots\right|^2\right] \\
&= \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\Phi(\mathrm{u}) + \frac{1}{2}\left|\frac{\partial\mathrm{u}}{\partial x}\right|^2\right] \\
&\quad + \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\nabla\Phi(\mathrm{u})\cdot\delta\mathrm{u} + \frac{\partial\mathrm{u}}{\partial x}\cdot\frac{\partial\delta\mathrm{u}}{\partial x}\right] + O(\delta\mathrm{u})^2
\end{aligned}$$

Integrate by parts:

$$\int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx\, \frac{\partial\mathrm{u}}{\partial x}\cdot\frac{\partial\delta\mathrm{u}}{\partial x} = \left[\frac{\partial\mathrm{u}}{\partial x}\cdot\delta\mathrm{u}\right]_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} - \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx\, \frac{\partial^2\mathrm{u}}{\partial x^2}\cdot\delta\mathrm{u}$$

Surface term vanishes since $\begin{cases} \frac{\partial\mathrm{u}}{\partial x}(x_{\mathrm{lo}}) = \frac{\partial\mathrm{u}}{\partial x}(x_{\mathrm{hi}}) = 0 & \textbf{for Neumann BCs} \\ \delta\mathrm{u}(x_{\mathrm{lo}}) = \delta\mathrm{u}(x_{\mathrm{hi}}) = 0 & \textbf{for Dirichlet BCs} \end{cases}$

$$\mathcal{F}(\mathrm{u}+\delta\mathrm{u}) = \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\Phi(\mathrm{u}) + \left| \frac{\partial \mathrm{u}}{\partial x} \right|^2 \right] + \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\nabla\Phi(\mathrm{u}) - \frac{\partial^2 \mathrm{u}}{\partial x^2} \right] \cdot \delta\mathrm{u} + O(\delta\mathrm{u})^2$$

The functional derivative $\delta\mathcal{F}/\delta\mathrm{u}$ is defined to be such that

$$\mathcal{F}(\mathrm{u}+\delta\mathrm{u}) = \mathcal{F}(\mathrm{u}) + \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \, \frac{\delta\mathcal{F}}{\delta\mathrm{u}} \cdot \delta\mathrm{u} + O(\delta\mathrm{u})^2 \text{ for every } \delta\mathrm{u} \Longrightarrow$$

$$\int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \, \frac{\delta\mathcal{F}}{\delta\mathrm{u}} \cdot \delta\mathrm{u} = \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left[\nabla\Phi(\mathrm{u}) - \frac{\partial^2 \mathrm{u}}{\partial x^2} \right] \cdot \delta\mathrm{u}$$

Choosing $\delta\mathrm{u}$ to be delta function centered on any x and pointing in any vector direction leads to pointwise equality:

$$\frac{\delta\mathcal{F}}{\delta\mathrm{u}} = \nabla\Phi(\mathrm{u}) - \frac{\partial^2 \mathrm{u}}{\partial x^2} = -\frac{\partial \mathrm{u}}{\partial t}$$

$$
\begin{aligned}
\frac{d\mathcal{F}}{dt} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathcal{F}(t+\delta t) - \mathcal{F}(t)\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathcal{F}(\mathrm{u}(t+\Delta t)) - \mathcal{F}(\mathrm{u}(t))\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathcal{F}\left(\mathrm{u}(t) + \frac{\partial \mathrm{u}}{\partial t}\Delta t + \ldots\right) - \mathcal{F}(\mathrm{u}(t))\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\mathcal{F}(\mathrm{u}(t)) + \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \frac{\delta \mathcal{F}}{\delta \mathrm{u}} \cdot \frac{\partial \mathrm{u}}{\partial t}\Delta t + \ldots - \mathcal{F}(\mathrm{u}(t))\right] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \frac{\delta \mathcal{F}}{\delta \mathrm{u}} \cdot \frac{\partial \mathrm{u}}{\partial t}\Delta t + \ldots\right] \\
&= \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \frac{\delta \mathcal{F}}{\delta \mathrm{u}} \cdot \frac{\partial \mathrm{u}}{\partial t} = \int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left(-\frac{\partial \mathrm{u}}{\partial t}\right) \cdot \frac{\partial \mathrm{u}}{\partial t} \\
&= -\int_{x_{\mathrm{lo}}}^{x_{\mathrm{hi}}} dx \left|\frac{\partial \mathrm{u}}{\partial t}\right|^2 \leq 0
\end{aligned}
$$

$\mathcal{F}$ decreases so limit cycles cannot occur. Can be applied in higher spatial dimensions via volume integration and Gauss's Divergence Theorem.

Spatial Analysis and Fronts

$$\frac{\partial u}{\partial t} = -\frac{d\Phi}{du} + \frac{\partial^2 u}{\partial x^2}$$

Travelling wave solutions:

$$\begin{aligned} u(x,t) &= U(x-ct) \quad \text{with } c = 0 \text{ for steady states} \\ \xi &\equiv x - ct \\ \frac{\partial u}{\partial t}(x,t) &= -c\,\frac{dU}{d\xi}(\xi) \\ \frac{\partial^2 u}{\partial x^2}(x,t) &= \frac{d^2U}{d\xi^2}(\xi) \end{aligned}$$

Equation obeyed by steady states and travelling waves becomes

$$-c\,\frac{du}{d\xi} = -\frac{d\Phi}{du} + \frac{d^2u}{d\xi^2} \qquad \Longrightarrow \qquad \frac{d^2u}{d\xi^2} = \frac{d\Phi}{du} - c\,\frac{du}{d\xi}$$

Analogy between space and time $\Longrightarrow$ x must be 1D

Spatial analysis or Mechanical analogy

$$\underbrace{\frac{d^2u}{d\xi^2}}_{\text{“acceleration”}} = - \underbrace{\frac{d(-\Phi)}{du}}_{\text{“potential gradient”}} \underbrace{-c\,\frac{du}{d\xi}}_{\text{“friction”}}$$

u	position	ξ	time	$E(\xi) \equiv -\Phi + \frac{1}{2}\left(\frac{du}{d\xi}\right)^2$ energy
$\frac{du}{d\xi}$	velocity	$-\Phi$	potential	
$\frac{d^2u}{d\xi^2}$	acceleration	$-c\frac{du}{d\xi}$	friction	

$$\dot{E} = \frac{dE}{d\xi} = \frac{d}{d\xi}\left[-\Phi + \frac{1}{2}\left(\frac{du}{d\xi}\right)^2\right]$$

$$= -\frac{d\Phi}{du}\frac{du}{d\xi} + \frac{du}{d\xi}\frac{d^2u}{d\xi^2}$$

$$= \left[-\frac{d\Phi}{du} + \frac{d^2u}{d\xi^2}\right]\frac{du}{d\xi} = -c\left(\frac{du}{d\xi}\right)^2 \begin{cases} < 0 & \text{if } c > 0 \\ = 0 & \text{if } c = 0 \\ > 0 & \text{if } c < 0 \end{cases}$$

$$c < 0 \iff \left\{ \begin{matrix} \text{“Increase in energy”} \\ \text{“Negative friction”} \end{matrix} \right\} \iff \text{just leftwards motion}$$

If $c = 0$, then E constant with

$$
\begin{aligned}
E &= -\Phi(u(\xi)) + \frac{1}{2}\left(\frac{du}{d\xi}\right)^2 \\
E + \Phi(u(\xi)) &= \frac{1}{2}\left(\frac{du}{d\xi}\right)^2 \\
\sqrt{2(E + \Phi(u(\xi)))} &= \frac{du}{d\xi} \\
\int d\xi &= \int \frac{du}{\sqrt{2(E + \Phi(u))}} \\
[\xi]_{\xi_{\text{lo}}}^{\xi} &= \int_{u_{\text{lo}}}^{u(\xi)} \frac{du}{\sqrt{2(E + \Phi(u))}} \\
&= \text{elliptic integral if } \Phi(u) = u^3 \\
&\implies \xi(u) \implies u(\xi)
\end{aligned}
$$

yields results but no intuition

Dynamical systems approach with ξ as time

$$v \equiv \frac{du}{d\xi} \Longrightarrow \begin{cases} \dot{u} = v \\ \dot{v} = \frac{d\Phi}{du} - cv \end{cases}$$

If $c = 0$, then system is Hamiltonian:

$$\mathcal{H} = -\Phi + \frac{1}{2}v^2 \Longrightarrow \begin{cases} \dot{u} = \frac{\partial \mathcal{H}}{\partial v} \\ \dot{v} = -\frac{\partial \mathcal{H}}{\partial u} \end{cases}$$

Add diffusion to supercritical pitchfork $\Longrightarrow$ Ginzburg-Landau equation:

$$\frac{\partial u}{\partial t} = \mu u - u^3 + \frac{\partial^2 u}{\partial x^2}$$

Steady states

$$0 = \mu u - u^3 + \frac{d^2 u}{dx^2}$$

Integrate to obtain the potential:

$$-\frac{d\Phi}{du} = \mu u - u^3 \qquad \Longrightarrow \qquad -\Phi = \frac{\mu}{2}u^2 - \frac{1}{4}u^4$$

Steady states:

$$\frac{d^2u}{dx^2} = \frac{d\Phi}{du} \Longrightarrow \begin{cases} \dot{u} = v \\ \dot{v} = \frac{d\Phi}{du} \end{cases}$$

Fixed points of new dynamical system:

$$\begin{aligned} 0 &= v \\ 0 &= \frac{d\Phi}{du} = -\mu\bar{u} + \bar{u}^3 \quad \Longrightarrow \quad \bar{u} = 0 \quad or \quad \bar{u} = \pm\sqrt{\mu} \end{aligned}$$

Same $\bar{u}$ as without diffusion, but stability under new dynamics is different:

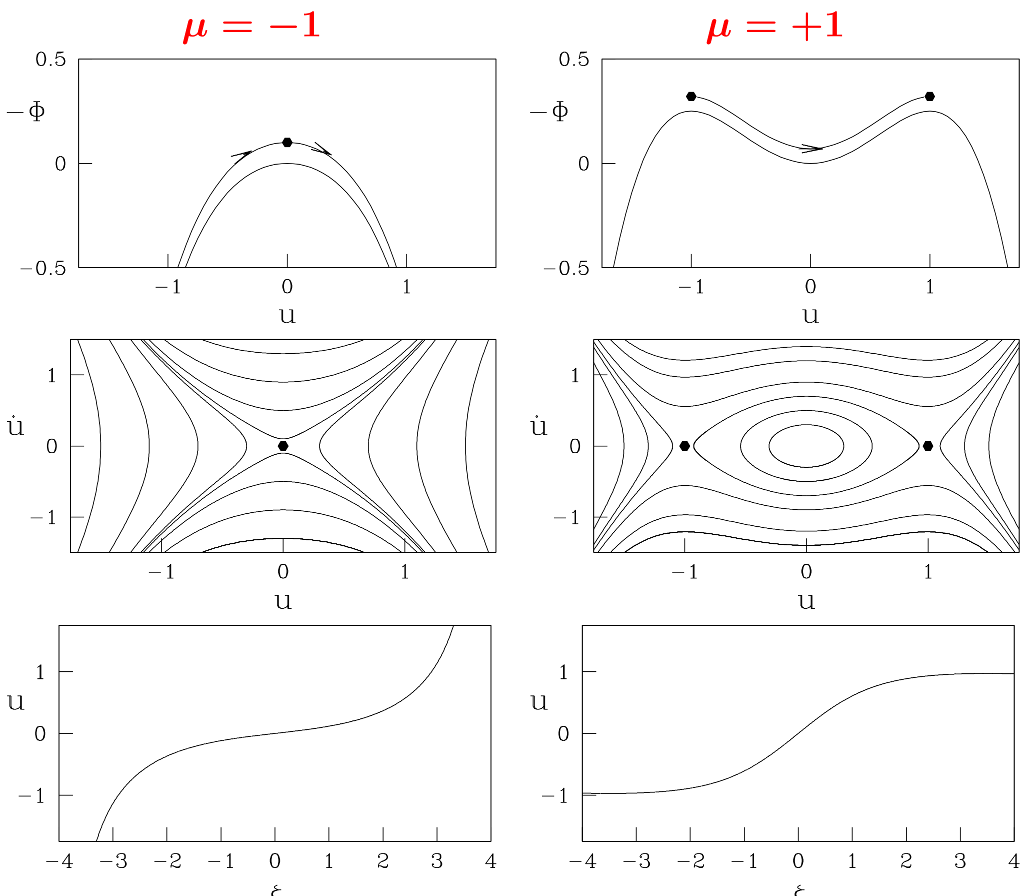
$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ \Phi'' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3\bar{u}^2 - \mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mu & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 2\mu & 0 \end{pmatrix}$$

Hamiltonian $\Longleftrightarrow \mathrm{Tr}(\mathcal{J}) = \frac{\partial^2\mathcal{H}}{\partial u \partial v} - \frac{\partial^2\mathcal{H}}{\partial v \partial u} = 0 \Longleftrightarrow$ **eigs are** $\pm\lambda$

$$\lambda(-\lambda) = -\Phi'' \Longrightarrow \lambda_{\pm} = \pm\sqrt{\Phi''} = \begin{cases} \pm\sqrt{-\mu} & \text{for} \quad \bar{u} = 0 \\ \pm\sqrt{2\mu} & \text{for} \quad \bar{u} = \pm\sqrt{\mu} \end{cases}$$

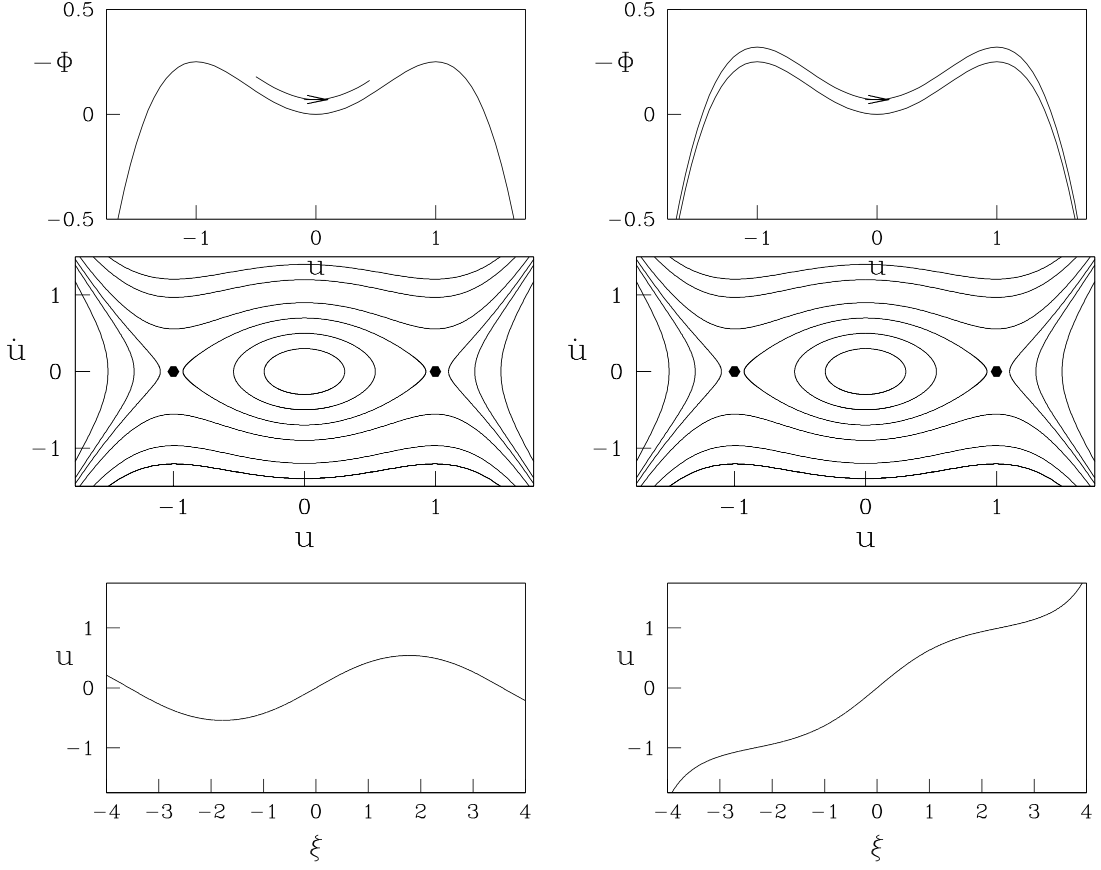
$\lambda = \pm i\omega \Longrightarrow$ **center = elliptic fixed point**

$\lambda = \pm\sigma \Longrightarrow$ **saddle = hyperbolic fixed point**



$\mu = -1$
$\mu = +1$
0.5
$-\Phi$
0
−0.5
−1
0
1
u
$\dot{u}$
u
−4
−3
−2
−1
0
1
2
3
4
ξ

$\mu = +1$



Types of Trajectories

	$\mu = -1$	$\mu = +1$
unbounded, crossing between left to right	✓	✓
unbounded, staying on left or on right	✓	✓
periodic		✓
front (limiting case of periodic)		✓

Periodic:
Trajectories in the $(u, \dot{u})$ phase plane are elliptical.
Particle oscillates back and forth in potential well.

Fronts:
Trajectory leaves $\bar{u} = -\sqrt{\mu}$ at zero velocity, arrives exactly at $\bar{u} = \sqrt{\mu}$ with zero velocity, since there is no friction.
Profile has $u = -\sqrt{\mu}$ on left, narrow transition region, $u = \sqrt{\mu}$ on right.

Type of trajectory is determined by the initial conditions (temporal point of view) or the boundary conditions (spatial point of view). Periodic boundary conditions on a domain of fixed wavelength select the periodic profile. Boundary conditions $u(\pm\infty) = \pm\sqrt{\mu}$ lead to front solution.

Front solutions connect two maxima of $-\Phi$, i.e. hyperbolic unstable fixed points of the transformed dynamical system.
These correspond to stable spatially homogeneous solutions to the original reaction-diffusion system:

$$\frac{du}{dt} = -\frac{d\Phi}{du}$$

Stability determined by

$$-\frac{d^2\Phi}{du^2}(\bar{u}) \left\{ \begin{array}{c} < 0 \\ > 0 \end{array} \right\} \quad \Longrightarrow \quad \bar{u} \left\{ \begin{array}{c} \text{stable} \\ \text{unstable} \end{array} \right\}$$

Thus, homogeneous stable steady states are maxima of $-\Phi$.

Nonzero c

Front between $u_{-\infty}$ and $u_{+\infty}$, which are maxima of $-\Phi(u)$ and hence stable solutions to spatially homogeneous equations,

Dirichlet BCs $u(\xi = \pm\infty) = u_{\pm\infty} \Longrightarrow$ Neumann BCs $\dfrac{du}{d\xi}(\xi = \pm\infty) = 0$

Travelling wave solutions:

$$0 = c\frac{du}{d\xi} - \frac{d\Phi}{du} + \frac{d^2u}{d\xi^2}$$

Multiply by $du/d\xi$:

$$0 = c\left(\frac{du}{d\xi}\right)^2 - \frac{d\Phi(u(\xi))}{d\xi} + \frac{1}{2}\frac{d}{d\xi}\left(\frac{du}{d\xi}\right)^2$$

Integrate over ξ interval:

$$\begin{aligned} 0 &= c\int_{-\infty}^{+\infty} d\xi \left(\frac{du}{d\xi}\right)^2 - \int_{-\infty}^{+\infty} d\xi\, \frac{d\Phi(u(\xi))}{d\xi} + \int_{-\infty}^{+\infty} d\xi\, \frac{1}{2}\frac{d}{d\xi}\left(\frac{du}{d\xi}\right)^2 \\ &= c\int_{-\infty}^{+\infty} d\xi \left(\frac{du}{d\xi}\right)^2 - [\Phi]_{-\infty}^{+\infty} + \frac{1}{2}\left[\left(\frac{du}{d\xi}\right)^2\right]_{-\infty}^{+\infty} \Leftarrow \end{aligned}$$

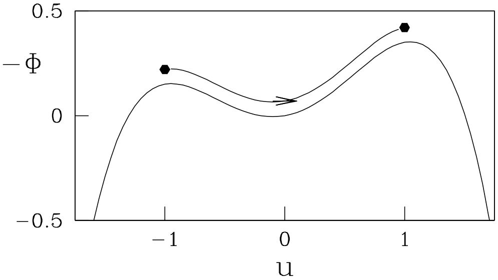
vanishes because of Neumann BCs

$$c = \frac{\Phi_{+\infty} - \Phi_{-\infty}}{\int_{-\infty}^{+\infty} d\xi \, \left(\frac{du}{d\xi}\right)^2} \qquad \text{where } \Phi_{\pm\infty} \equiv \Phi(u_{\pm\infty})$$

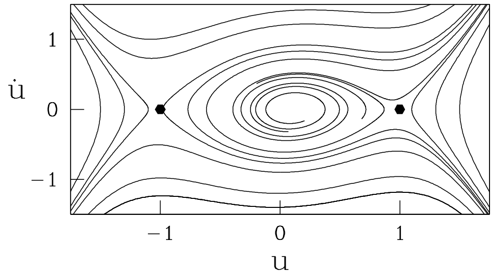
Front velocity $c > 0$ if $\Phi_{-\infty} < \Phi_{+\infty}$, i.e. if $-\Phi_{-\infty} > -\Phi_{+\infty}$.
Front moves from left to right $\Longrightarrow$
$u_{-\infty}, -\Phi_{-\infty}$ domain invades $u_{+\infty}, -\Phi_{+\infty}$ domain
Front motion increases size of domain with greater $-\Phi$.

Mechanical analogy:
Trajectory goes from $u_{-\infty}, -\Phi_{-\infty}$ to $u_{+\infty}$, with lower potential $-\Phi_{+\infty}$.
For "velocity" $du/d\xi$ and "kinetic energy" to vanish at both endpoints, energy must be lost via friction. Hence c is positive.

"Negative friction" is possible since $c < 0$ just means that the front moves towards the left.



Trajectory
from lower left hill to higher right hill
uses "negative friction" to increase its energy



Phase portrait
Former center has become focus
surrounded by spiralling trajectories

Perturbed Ginzburg-Landau equation

$$0 = c\frac{du}{d\xi} + \mu u - u^3 - 0.1 + \frac{d^2u}{d\xi^2}$$

Potential

$$-\Phi = \frac{1}{2}\mu u^2 - \frac{1}{4}u^4 - 0.1u$$

has two maxima of different heights