Université Pierre et Marie Curie

Master Sciences et Technologie (M2)

Spécialité : Concepts fondamentaux de la physique

Parcours : Physique des Liquides et Matière Molle

Cours : Dynamique Non-Linéaire

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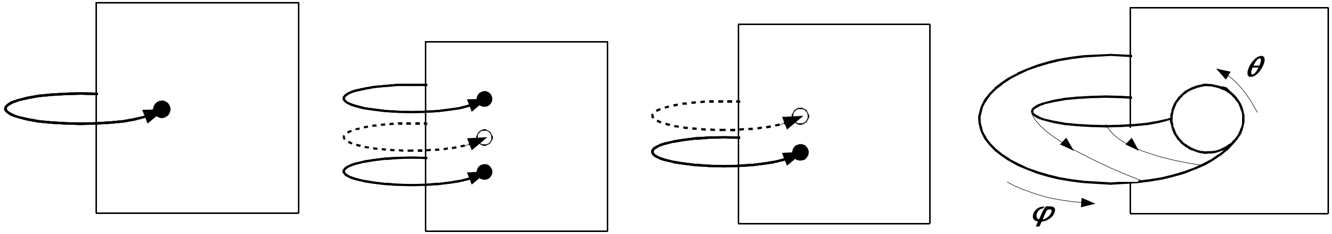
Quasiperiodicity

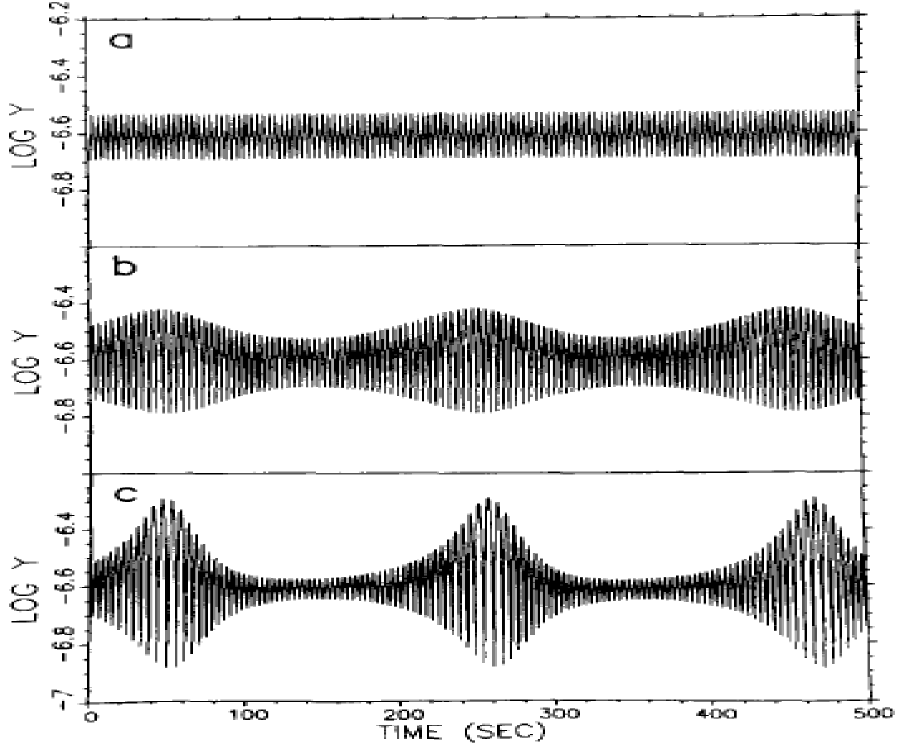
Bifurcations undergone by limit cycles



before bifurcation	after pitchfork	after saddle-node	after Hopf

	Continuous system	Poincaré map
Before bifurcation	Limit cycle	Fixed point
After bifurcation	Torus	Circle
Name of bifurcation	{ Secondary Hopf / Neimark-Sacker }	Hopf

Periodic and quasiperiodic behavior in oscillating chemical reaction model



From D. Barkley, J. Ringland & J.S. Turner, J. Chem. Phys. **87**, *3812 (1987)*

Circle maps

$$x_{n+1} = f(x) \mod 1$$

Sine circle map

V. Arnold in the 1960s (Moscow, later also Paris IX, Dauphine, died 2010):

$$x_{n+1} = f_{\Omega,K}(x_n) \equiv \left[x_n + \Omega - \frac{K}{2\pi} \sin(2\pi x_n) \right] \mod 1$$

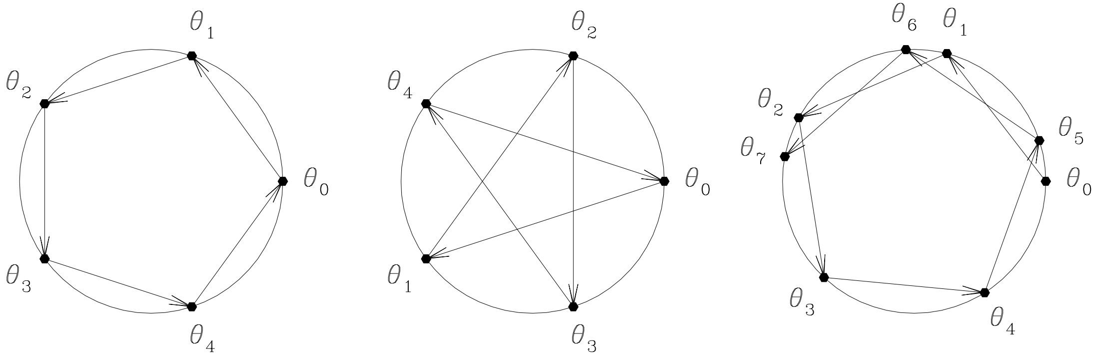
K **measures nonlinearity, Ω is basic frequency.**

$$K = 0 \quad \Longrightarrow \quad x_{n+1} = [x_n + \Omega] \bmod 1$$

$\Omega = p \quad \Longrightarrow$ all x are fixed points of f

$\Omega = p/q \quad \Longrightarrow$ all x are members of q-cycles of f

$$f^q(x) = \Big[x + \underbrace{\frac{p}{q} + \frac{p}{q} + \ldots \frac{p}{q}}_{q \text{ times}}\Big] \bmod 1 = [x + p] \bmod 1 = x$$



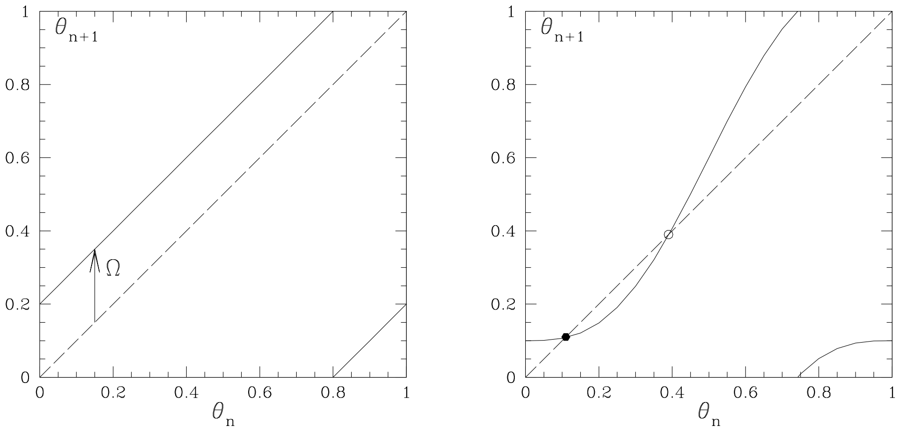
$\Omega = 1/5$ $\quad$ $\Omega = 3/5$ $\quad$ $\Omega \gtrsim 1/5$

Ω irrational $\Longrightarrow$ *no* fixed points or q-cycles.

All x are members of *quasiperiodic orbits*. Each orbit is dense on the circle.

$K > 0$: Frequency locking

$$x_{n+1} = f_{\Omega,K}(x_n) \equiv \left[x_n + \Omega - \frac{K}{2\pi} \sin(2\pi x_n) \right] \mod 1$$



$K = 0$

$\Omega = 0.2$

$f(x) = x + \Omega$

$K = 1$

$\Omega = 0.1$

saddle-node bifurcation

creates pair of fixed points

$$x_{n+1} = f_{\Omega,K}(x_n) \equiv \left[x_n + \Omega - \frac{K}{2\pi}\sin(2\pi x_n)\right] \mod 1$$

Condition for bifurcation	Condition for fixed point
$f'(x) = 1$	$f(x) = x$
$1 - K\cos(2\pi x) = 1$	$x + \Omega - \frac{K}{2\pi}\sin(2\pi x) = x + n$
$\cos(2\pi x) = 0$	$\sin(2\pi x) = \frac{2\pi}{K}(\Omega - n)$

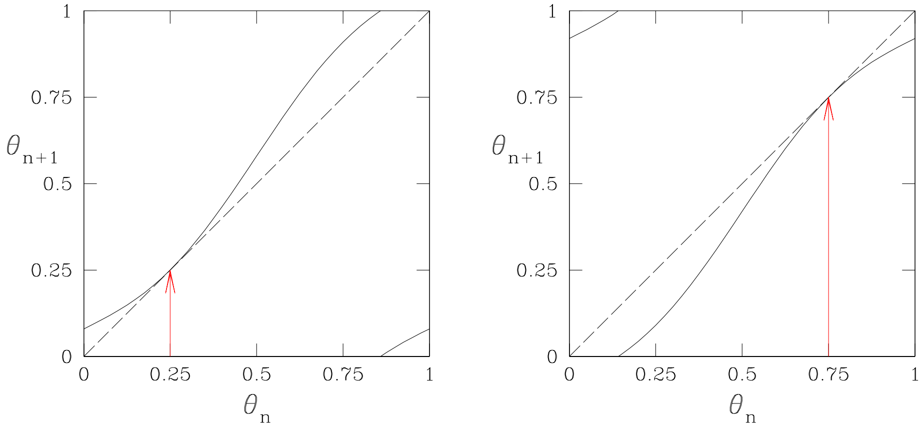
$$x = \frac{1}{4} \implies \sin\left(2\pi\frac{1}{4}\right) = 1 = \frac{2\pi}{K}(\Omega - n)$$

$$\implies K = 2\pi\Omega$$

$$x = \frac{3}{4} \implies \sin\left(2\pi\frac{3}{4}\right) = -1 = \frac{2\pi}{K}(\Omega - 1)$$

$$\implies K = 2\pi(1 - \Omega)$$

Saddle-node bifurcations of sine circle map create stable-unstable pairs of fixed points

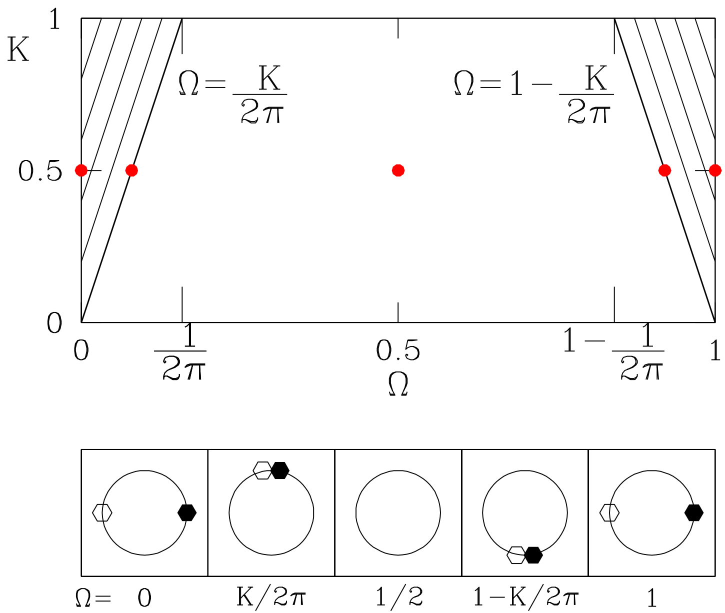


$x = \frac{1}{4}$ $\qquad$ $x = \frac{3}{4}$

$$K = 0.5$$

$\Omega = \frac{K}{2\pi} = 0.08$ $\qquad$ $\Omega = 1 - \frac{K}{2\pi} = 0.92$

First frequency-locking tongue in (Ω, K) plane



Fixed points (one-cycles) exist inside tongue,

for $0 \leq \Omega < \frac{K}{2\pi}$ and $1 - \frac{K}{2\pi} < \Omega \leq 1$

Two-cycles: fixed points of $f^2(x) \equiv f(f(x))$

Since $f^2_{(\Omega=1/2,K=0)}(x) = x \;\forall x$ **, set** $\Omega_\pm(K) = \frac{1}{2} \pm \epsilon(K)$ **with** $K, \epsilon \ll 1$

$$
\begin{aligned}
f^2(x) &= f(x) + \Omega_\pm - \frac{K}{2\pi}\sin(2\pi f(x)) \\
&= x + \frac{1}{2} \pm \epsilon - \frac{K}{2\pi}\sin(2\pi x) + \frac{1}{2} \pm \epsilon - \frac{K}{2\pi}\sin(2\pi f(x)) \\
&= x + 1 \pm 2\epsilon - \frac{K}{2\pi}\sin(2\pi x) - \frac{K}{2\pi}\sin(2\pi f(x))
\end{aligned}
$$

$$
\begin{aligned}
-\sin(2\pi f(x)) &= -\sin\left(2\pi\left(x + \frac{1}{2} \pm \epsilon - \frac{K}{2\pi}\sin(2\pi x)\right)\right) \\
&= -\sin\left(2\pi x + \pi \pm 2\pi\epsilon - K\sin(2\pi x)\right) \\
&= \underbrace{-\sin\left(2\pi x + \pi\right)}_{=\sin(2\pi x)} \underbrace{\cos\left(\pm 2\pi\epsilon - K\sin(2\pi x)\right)}_{\approx 1} \\
&\quad \underbrace{-\cos\left(2\pi x + \pi\right)}_{=\cos(2\pi x)} \underbrace{\sin\left(\pm 2\pi\epsilon - K\sin(2\pi x)\right)}_{\approx \pm 2\pi\epsilon - K\sin(2\pi x)} \\
&\approx \sin\left(2\pi x\right) + \cos\left(2\pi x\right)\left(\pm 2\pi\epsilon - K\sin(2\pi x)\right) \\
&= \sin\left(2\pi x\right) \pm 2\pi\epsilon\cos\left(2\pi x\right) - \frac{K}{2}\sin(4\pi x)
\end{aligned}
$$

$$f^2(x) \approx x + 1 \pm 2\epsilon \pm \underbrace{\epsilon K}_{\ll \epsilon} \cos(2\pi x) - \frac{K^2}{4\pi} \sin(4\pi x)$$

Fixed points of $f^2(x) \bmod 1$:

$$x = f^2(x) \bmod 1 \approx x \pm 2\epsilon - \frac{K^2}{4\pi} \sin(4\pi x)$$

$$\epsilon \approx \pm\frac{K^2}{8\pi} \sin(4\pi x)$$

At saddle-node bifurcation point of f^2:

$$1 = \frac{d}{dx} f^2(x) \approx 1 - K^2 \cos(4\pi x) \implies \cos(4\pi x) = 0$$

$$\implies x \approx \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8} \implies \sin(4\pi x) = \pm 1$$

Therefore:

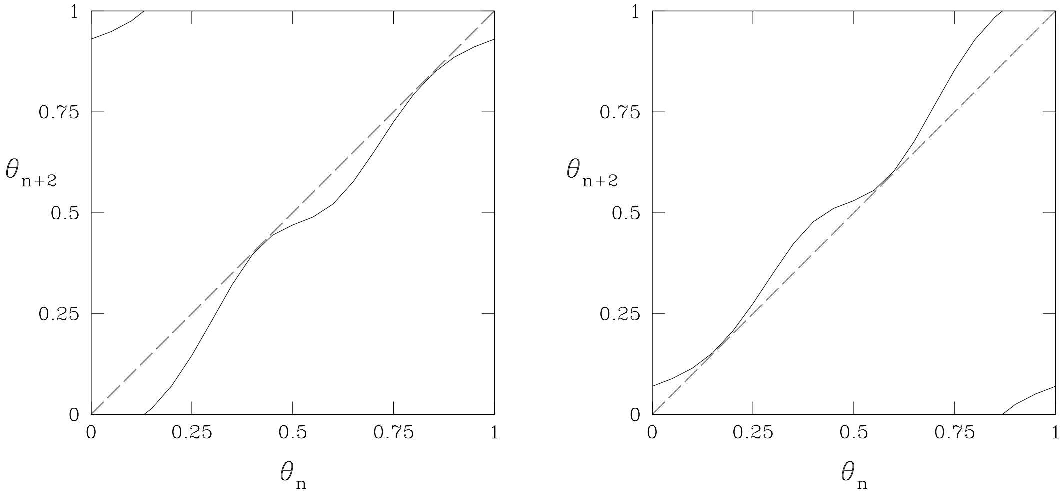
$$\epsilon(K) \approx \pm\frac{K^2}{8\pi}$$

$$\Omega_\pm(K) = \tfrac{1}{2} \pm \epsilon(K) \approx \tfrac{1}{2} \pm \frac{K^2}{8\pi}$$

Note: Period-doubling bifurcation (= pitchfork for f^2) requires $K \geq 2$:

$$f'(x) = 1 - K\cos(2\pi x) \geq 1 - K$$

Saddle-node bifurcations of f^2 create stable-unstable pairs of 2-cycles

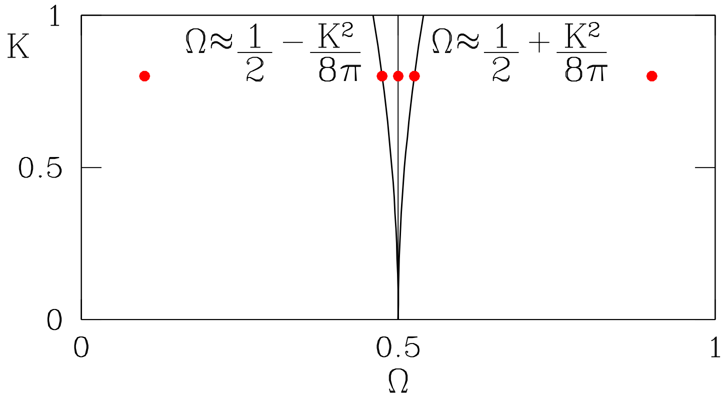


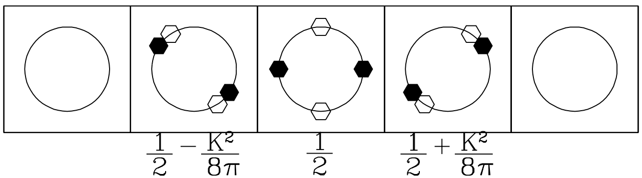
$$K = 0.8$$

$$\Omega \approx \frac{1}{2} - \frac{K^2}{8\pi} \approx 0.475 \qquad \Omega \approx \frac{1}{2} + \frac{K^2}{8\pi} \approx 0.525$$

$$x \approx \frac{3}{8}, \frac{7}{8} \qquad x \approx \frac{1}{8}, \frac{5}{8}$$

Second frequency-locking tongue in (Ω, K) plane





Two-cycles exist inside tongue

$$\frac{1}{2} - \frac{K^2}{8\pi} \lesssim \Omega \lesssim \frac{1}{2} + \frac{K^2}{8\pi}$$

Three-cycles: fixed points of $f^3(x)$

Emerge from $K = 0, \Omega = 1/3$ and $\Omega = 2/3$ via saddle-nodes of f^3

Exist within Ω-intervals of width $O(K^3)$

For any (p, q), seek Ω, K, x such that

$$f^q_{\Omega,K}(x) = x + p$$

q-cycles are produced via saddle-node bifurcations of f^q

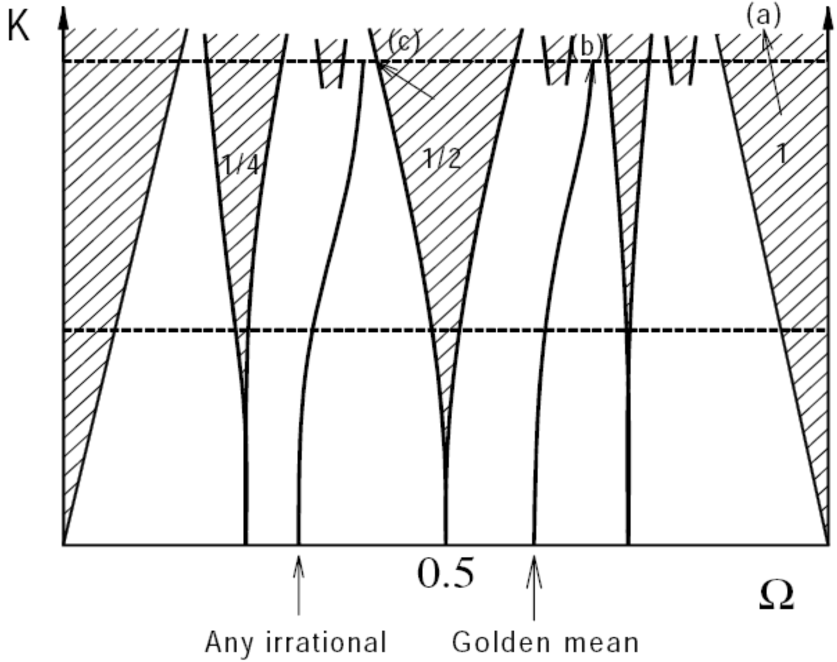
Exist within Ω-intervals $I_{p,q}(K)$ surrounding p/q of width $O(K^q)$ called

frequency-locking **or** *Arnold tongues*

$$K = 0 \implies \bigcup_{p,q} I_{p,q} = \text{rationals} \qquad \text{measure}\left(\bigcup_{p,q} I_{p,q}\right) = 0$$

$$K = 1 \implies \qquad \text{measure}\left(\bigcup_{p,q} I_{p,q}\right) = 1$$

Schematic representation of frequency-locking tongues



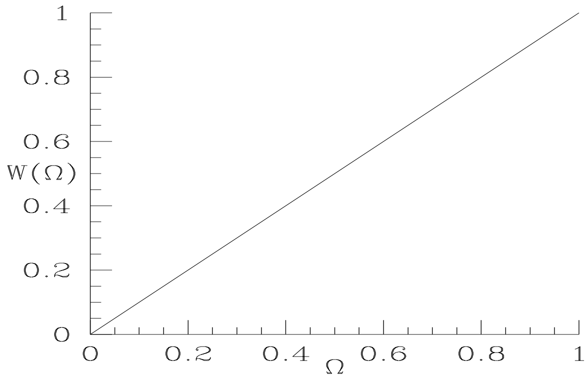
Winding number of circle map f

$$W(f) \equiv \lim_{n\to\infty} \frac{f^n(x_0) - x_0}{n} \qquad f^n \text{ not truncated to } [0, 1]$$

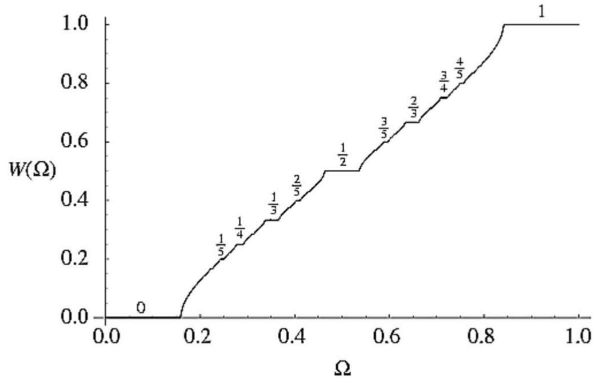
Poincaré: f monotonic & continuous $\Longrightarrow$ limit exists & independent of x_0.

For sine circle map:

$K = 0$



$K = 1$



$W = \lim_{n\to\infty} \frac{x_0 + n\Omega - x_0}{n} = \Omega$

diagonal line

Devil's staircase
continuous,
constant on set of measure one,
jumps at each irrational number

The golden mean: “most irrational” number

Stays furthest away from frequency-locking tongues

$$
\begin{aligned}
w_0 &\equiv 0 \\
w_{n+1} &\equiv \frac{1}{1+w_n} \\
w_1 &= \frac{1}{1+0} = \frac{1}{1} = 1 \\
w_2 &= \frac{1}{1+\frac{1}{1+0}} = \frac{1}{1+w_1} = \frac{1}{1+1} = \frac{1}{2} \\
w_3 &= \frac{1}{1+\frac{1}{1+\frac{1}{1+0}}} = \frac{1}{1+w_2} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}
\end{aligned}
$$

Golden mean: $w_* \equiv \lim_{n\to\infty} w_n$

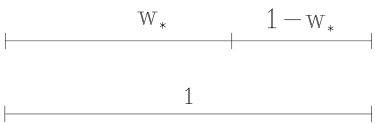
$$\textbf{Golden mean:} \quad w_* \equiv \lim_{n\to\infty} w_n \quad \text{with} \quad w_{n+1} = \frac{1}{1+w_n}$$

$$w_* = \frac{1}{1+w_*}$$

$$w_*(1+w_*) = 1$$

$$w_*^2 + w_* - 1 = 0$$

$$w_* = \frac{-1+\sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2} = 0.618\ldots$$



Parthenon, plants, shells, Greeks, Renaissance, . . .

$$(1-w_*) : w_* = w_* : 1$$

Fibonacci sequence:

$$F_0 = F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \Longrightarrow 1, 1, 2, 3, 5, 8, 13, \ldots$$

leads to equivalent definition of w_n:

$$w_{n+1} \equiv \frac{F_n}{F_{n+1}} = \frac{F_n}{F_n + F_{n-1}} = \frac{1}{\frac{F_n + F_{n-1}}{F_n}} = \frac{1}{1 + w_n}$$

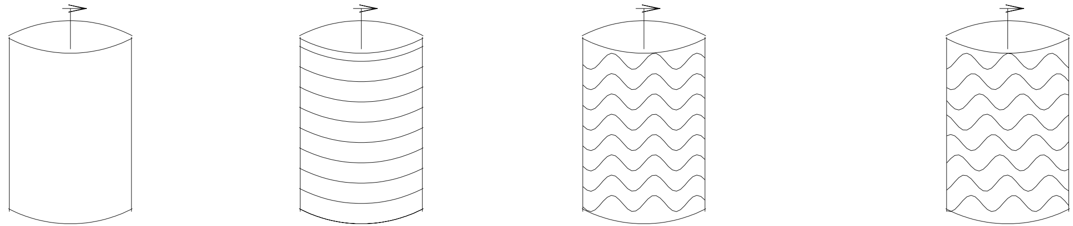
Closest rational approximation obtained by truncating continued fraction:

$$a = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

w_* is irrational least well approximated by rational: $a_1 = a_2 = \ldots = 1$

Following path in (Ω, K) space with $W_{\Omega,K} = w_*$ will keep furthest away from frequency-locking tongues

Taylor-Couette flow

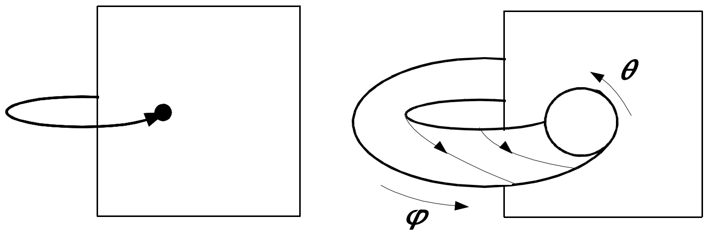


Laminar Couette $\mathrm{U}_C(r)$

Taylor Vortex $\mathrm{U}_{TV}(r, z)$

Wavy Vortex $\mathrm{U}_{WV}(r, \theta, z, t)$

Modulated Wavy Vortex $\mathrm{U}_{MWV}(r, \theta, z, t)$



No frequency-locking in modulated wavy vortex flow! Why not?

Rand (1981): Symmetry! In rotating frame,

wavy vortex flow is steady and modulated wavy vortex flow is periodic.

Points on circle (phases in θ) dynamically equivalent $\Longrightarrow$ no saddle-nodes.

Lyapunov exponents

Steady state $\bar{x}$: eigenvalues of Jacobian matrix

Limit cycle $\bar{x}(t \bmod T)$: Floquet exponents

Any attractor: Lyapunov exponents

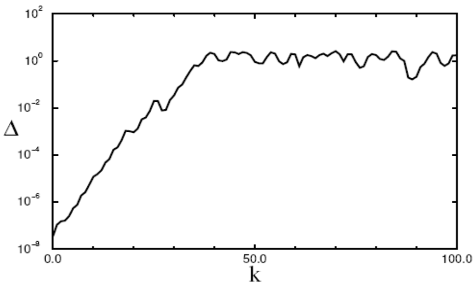
Let $\bar{x}(t)$ evolve according to full nonlinear system: $\dot{\bar{x}} = f(\bar{x}(t))$

Let $\epsilon(t)$ evolve according to linearized system: $\dot{\epsilon} = \left[Df_{\bar{x}(t)}\right] \epsilon$

Largest Lyapunov exponent:

$$\lambda^{(1)} \equiv \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\epsilon(t)}{\epsilon(0)} \right|$$

Independent of initial condition if within same attractor



Integrate perturbed non-linear system:
Initial slope is largest Lyapunov exponent
Stop when trajectory reaches attractor boundary.

Winding number: average rotation per iteration

Lyapunov exponent: average growth or decay per iteration

Rate of growth of area: $\lambda^{(1)} + \lambda^{(2)}$

Rate of growth of volume: $\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)}$**, etc.**

Map:

$$\begin{aligned} \epsilon_1 &= f'(\bar{x}_0)\epsilon_0 \\ \epsilon_n &= \prod_{k=0}^{n-1} f'(\bar{x}_k)\epsilon_0 \\ \lambda &= \lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{k=0}^{n-1} f'(\bar{x}_k) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(\bar{x}_k)| \end{aligned}$$

Chaotic attractors: nearby initial conditions eventually diverge

$\Longrightarrow$ **at least one Lyapunov exponent is positive**

One of the definitions of chaos

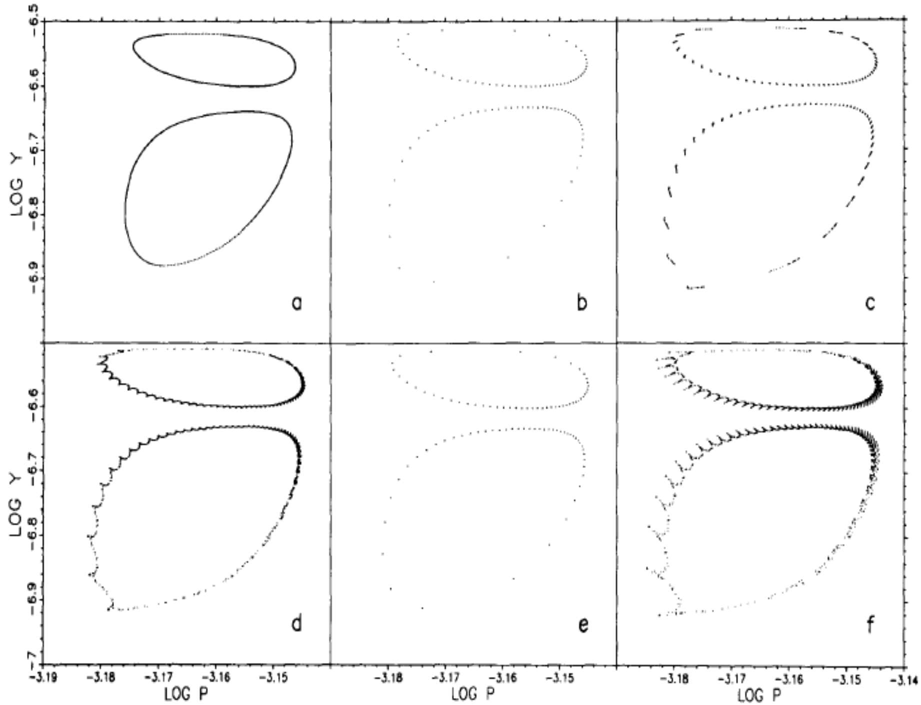
Wrinkling of a torus

When $K > 1$, sine circle map becomes non-invertible

$\Longrightarrow$ **it cannot be the Poincaré mapping of a flow**

$\Longrightarrow$ **it can become chaotic (an invertible map cannot become chaotic)**

Attractor can no longer be mapped onto a circle and may become *wrinkled*



(a) Torus (quasiperiodic) **(b) Frequency locking (1:49)** **(c) Bands on wrinkled torus**
(d) Wrinkled torus **(e) Frequency locking (1:48)** **(c) Wrinkled torus**

From D. Barkley, J. Ringland & J.S. Turner, J. Chem. Phys. **87**, *3812 (1987).*

Route to chaos from a torus

$<$ **1970s Landau:** **Hopf$_1$ (Ω_1), Hopf$_2$ (Ω_2), Hopf$_3$ (Ω_3), $\ldots \Longrightarrow$ Turbulence**
$\approx$ **1980s Lorenz, May, Feigenbaum, etc.:**
Small number (3) of ODEs can display chaos

Ruelle & Takens (1971); Newhouse, Ruelle & Takens (1978):
Theorem concerning quasiperiodic motion (motion on torus) of dimension $n \geq 3$. Perturbations can lead to chaos:

"Let v be a constant vector field on the torus $T^n = R^n/Z^n$. If $n \geq 3$, every C^2 neighborhood of v contains a vector field v' with a strange Axiom A attractor. If $n \geq 4$, we may take C^∞ instead of C^2."

Curry & Yorke (1978); Grebogi, Ott & Yorke (1985): **Numerical investigation of probability of random perturbations leading to chaos**

Flow	Poincaré map
two-torus	circle map
three-torus	pair of coupled circle maps

$$\begin{aligned}\theta_{n+1} &= \theta_n + \omega_1 + KP_1(\theta_n, \phi_n) \bmod 1 \\ \phi_{n+1} &= \phi_n + \omega_2 + KP_2(\theta_n, \phi_n) \bmod 1\end{aligned}$$

Solutions: quasiperiodic with three frequencies, quasiperiodic with two frequencies, periodic, or chaotic. Map is non-invertible for $K \geq K_c$.

Attractor	Lyapunov exp	$\frac{K}{K_c} = \frac{3}{8}$	$\frac{K}{K_c} = \frac{3}{4}$	$\frac{K}{K_c} = \frac{9}{8}$
Three-frequency quasiperiodic	$0, 0$	82%	44%	0%
Two-frequency quasiperiodic	$0, -$	16%	38%	33%
Periodic	$-, -$	2%	11%	31%
Chaotic	$+, ?$	0%	7%	36%