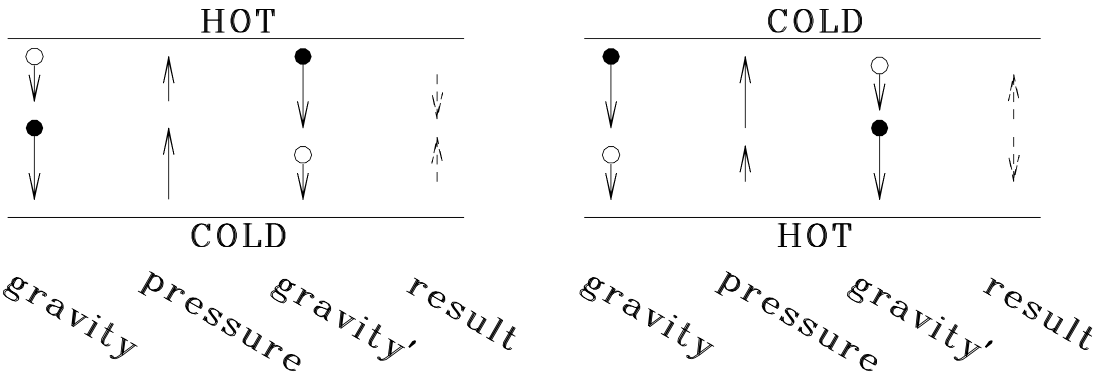
Hydrodynamics

Class 9

Laurette TUCKERMAN

laurette@pmmh.espci.fr

Rayleigh-Bénard Convection



Rayleigh-Bénard Convection

Boussinesq Approximation

Calculation and subtraction of the basic state

Non-dimensionalisation

Boundary Conditions

Linear stability analysis

Boussinesq Approximation

μ (viscosity $\sim$ diffusivity of momentum), κ (diffusivity of temperature),
ρ (density) constant except in buoyancy force.
Valid for $T_0 - T_1$ not too large.

$$\begin{aligned} \rho(T) &= \rho_0 \left[1 - \alpha(T - T_0)\right] \\ \boldsymbol{\nabla}\cdot\mathbf{U} &= 0 \end{aligned}$$

Governing equations:

$$\begin{aligned} \rho_0 \left[\partial_t + (\mathbf{U} \cdot \boldsymbol{\nabla})\right] \mathbf{U} &= \mu\Delta\mathbf{U} - \boldsymbol{\nabla} P - g\rho(T)\hat{\boldsymbol{e}}_z \\ \left[\partial_t + (\mathbf{U} \cdot \boldsymbol{\nabla})\right] T &= \kappa\Delta T \end{aligned}$$

$\uparrow$ advection $\quad$ $\uparrow$ diffusion $\quad$ $\uparrow$ buoyancy

Boundary conditions:

$$\begin{aligned} \mathbf{U} = 0 &\quad \text{at} \quad z = 0, d \\ T = T_{0,1} &\quad \text{at} \quad z = 0, d \end{aligned}$$

Calculation and subtraction of base state

Conductive solution: (U^*, T^*, P^*)

Motionless: $U^* = 0$

uniform temperature gradient: $T^* = T_0 - (T_0 - T_1)\dfrac{z}{d}$

density: $\rho(T^*) = \rho_0 \left[1 + \alpha(T_0 - T_1)\dfrac{z}{d}\right]$

Hydrostatic pressure counterbalances buoyancy force:

$$
\begin{aligned}
P^* &= -g \int dz\ \rho(T^*) \\
&= P_0 - g\rho_0 \left[z + \alpha(T_0 - T_1)\frac{z^2}{2d}\right]
\end{aligned}
$$

Write:

$$T = T^* + \hat{T} \qquad P = P^* + \hat{P}$$

Buoyancy:

$$\begin{aligned}
\rho(T^* + \hat{T}) &= \rho_0(1 - \alpha(T^* + \hat{T} - T_0)) \\
&= \rho_0(1 - \alpha(T^* - T_0)) - \rho_0 \alpha \hat{T} \\
&= \rho(T^*) - \rho_0 \alpha \hat{T} \\
-\boldsymbol{\nabla} P - g\rho(T)\hat{\boldsymbol{e}}_{\boldsymbol{z}} &= -\boldsymbol{\nabla} P^* - g\rho(T^*) - \boldsymbol{\nabla}\hat{P} + g\rho_0\alpha\hat{T}\hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
&= -\boldsymbol{\nabla}\hat{P} + g\rho_0\alpha\hat{T}\hat{\boldsymbol{e}}_{\boldsymbol{z}}
\end{aligned}$$

Advection of temperature:

$$\begin{aligned}
(U \cdot \boldsymbol{\nabla})T &= (U \cdot \boldsymbol{\nabla})T^* + (U \cdot \boldsymbol{\nabla})\hat{T} \\
&= (U \cdot \boldsymbol{\nabla})\left(T_0 - (T_0 - T_1)\frac{z}{d}\right) + (U \cdot \boldsymbol{\nabla})\hat{T} \\
&= -\frac{T_0 - T_1}{d} U \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}} + (U \cdot \boldsymbol{\nabla})\hat{T}
\end{aligned}$$

Governing equations:

$$
\begin{aligned}
\rho_0 \left[\partial_t + (U \cdot \boldsymbol{\nabla})\right] U &= -\boldsymbol{\nabla} \hat{P} + g\rho_0\alpha\hat{T}\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \mu\Delta U \\
\boldsymbol{\nabla}\cdot U &= 0 \\
\left[\partial_t + (U \cdot \boldsymbol{\nabla})\right] \hat{T} &= \frac{T_0 - T_1}{d} U \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}} + \kappa\Delta\hat{T}
\end{aligned}
$$

Homogeneous boundary conditions:

$$
\begin{aligned}
U &= 0 \ \text{ at } \ z = 0, d \\
\hat{T} &= 0 \ \text{ at } \ z = 0, d
\end{aligned}
$$

Non-dimensionalization

Scales:

$$z = d\bar{z}, \quad t = \frac{d^2}{\kappa}\bar{t}, \quad U = \frac{\kappa}{d}\bar{U}, \quad \hat{T} = \frac{\mu\kappa}{d^3 g\rho_0\alpha}\bar{T}, \quad \hat{P} = \frac{\mu\kappa}{d^2}\bar{P}$$

Equations :

$$\begin{aligned}
\frac{\kappa^2\rho_0}{d^3}\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\boldsymbol{\nabla}})\right]\bar{U} &= -\frac{\mu\kappa}{d^3}\bar{\boldsymbol{\nabla}}\bar{P} + \frac{\mu\kappa}{d^3}\bar{T}\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \frac{\mu\kappa}{d^3}\bar{\Delta}\bar{U} \\
\frac{\kappa}{d^2}\bar{\boldsymbol{\nabla}}\cdot\bar{U} &= 0 \\
\frac{\mu\kappa^2}{d^5 g\rho_0\alpha}\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\boldsymbol{\nabla}})\right]\bar{T} &= \frac{\kappa}{d}\frac{T_0 - T_1}{d}\bar{U}\cdot\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \frac{\mu\kappa^2}{d^5 g\rho_0\alpha}\bar{\Delta}\bar{T}
\end{aligned}$$

Dividing through, we obtain:

$$\begin{aligned}
\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\boldsymbol{\nabla}})\right]\bar{U} &= \frac{\mu}{\rho_0\kappa}\left[-\bar{\boldsymbol{\nabla}}\bar{P} + \bar{T}\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \bar{\Delta}\bar{U}\right] \\
\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\boldsymbol{\nabla}})\right]\bar{T} &= \frac{(T_0 - T_1)d^3 g\rho_0\alpha}{\kappa\mu}\bar{U}\cdot\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \bar{\Delta}\bar{T}
\end{aligned}$$

Non-dimensional parameters:

the Prandtl number: $Pr \equiv \dfrac{\mu}{\rho_0 \kappa}$

momentum diffusivity / thermal diffusivity

the Rayleigh number: $Ra \equiv \dfrac{(T_0 - T_1) d^3 g \rho_0 \alpha}{\kappa \mu}$

non-dimensional measure of thermal gradient

Boundary conditions

Horizontal direction: periodicity $2\pi/q$

Vertical direction: at $z = 0, 1$

$$T = 0|_{z=0,1} \quad \text{perfectly conducting plates}$$

$$w = 0|_{z=0,1} \quad \text{impenetrable plates}$$

Rigid boundaries at $z = 0, 1$:

$$u|_{z=0,1} = v|_{z=0,1} = 0 \quad \text{zero tangential velocity}$$

Incompressibility

$$\partial_x u + \partial_y v + \partial_z w = 0$$

$$\Longrightarrow \partial_z w = -(\partial_x u + \partial_y v)$$

$$u|_{z=0,1} = v|_{z=0,1} = 0 \Longrightarrow \quad \partial_x u|_{z=0,1} = \partial_y v|_{z=0,1} = 0$$

$$\Longrightarrow \quad \partial_z w|_{z=0,1} = 0$$

Free surfaces at $z = 0, 1$ to simplify calculations:

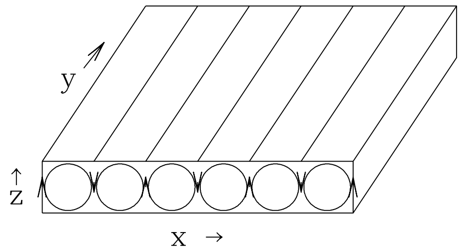
$$[\partial_z u + \partial_x w]_{z=0,1} \quad = [\partial_z v + \partial_y w]_{z=0,1} = 0$$

zero tangential stress

$$\begin{aligned} w|_{z=0,1} = 0 \quad &\Longrightarrow \partial_x w|_{z=0,1} = \partial_y w|_{z=0,1} = 0 \\ &\Longrightarrow \partial_z u|_{z=0,1} = \partial_z v|_{z=0,1} = 0 \\ &\Longrightarrow \partial_x \partial_z u|_{z=0,1} = \partial_y \partial_z v|_{z=0,1} = 0 \\ &\Longrightarrow \partial_{zz} w|_{z=0,1} = - \partial_z (\partial_x u + \partial_y v)|_{z=0,1} = 0 \end{aligned}$$

Not realistic, but allows trigonometric functions $\sin(k\pi z)$

Two-dimensional case



$$U = \boldsymbol{\nabla} \times \psi\ \hat{\boldsymbol{e}}_{\boldsymbol{y}} \Longrightarrow \left\{ \begin{array}{c} u = -\partial_z \psi \\ w = \partial_x \psi \end{array} \right\} \Longrightarrow \boldsymbol{\nabla} \cdot U = 0$$

No-penetration boundary condition:

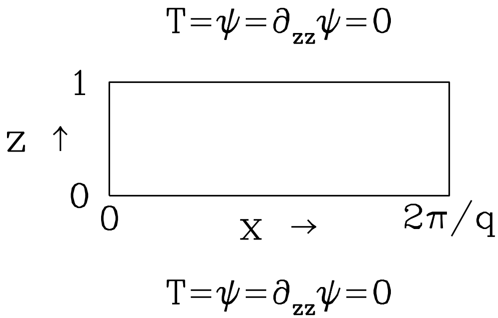
Horizontal flux: $0 = w = \partial_x \psi \Longrightarrow \begin{cases} \psi = \psi_1 & \text{at } z = 1 \\ \psi = \psi_0 & \text{at } z = 0 \end{cases}$

$$\int_{z=0}^{1} dz\ u(x,z) = -\int_{z=0}^{1} dz\ \partial_z \psi(x,z) = -\ \psi(x,z)]_{z=0}^{1} = \psi_0 - \psi_1$$

Arbitrary constant $\Longrightarrow \psi_0 = 0$ Zero flux $\Longrightarrow \psi_1 = 0$

Stress-free: $0 = \partial_z u = -\partial_{zz}^2 \psi$ Rigid: $0 = u = \partial_z \psi$ at $z = 0, 1$

Two-dimensional case



Temperature equation:

$$\partial_t T + \boldsymbol{U} \cdot \boldsymbol{\nabla} T = Ra\, U \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}} + \Delta T$$

$$\begin{aligned} U \cdot \boldsymbol{\nabla} T &= u\, \partial_x T + w\, \partial_z T \\ &= -\partial_z \psi\, \partial_x T + \partial_x \psi\, \partial_z T \equiv J[\psi, T] \end{aligned}$$

$$\partial_t T + J[\psi, T] = Ra\, \partial_x \psi + \Delta T$$

Velocity equation

$$\partial_t U + (U \cdot \boldsymbol{\nabla})U = Pr\left[-\boldsymbol{\nabla} P + T\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \Delta U\right]$$

Take $\hat{e}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times$:

$$\begin{aligned}
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times\partial_t U &= \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\partial_t\psi\hat{\boldsymbol{e}}_{\boldsymbol{y}} = -\partial_t\Delta\psi \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times\boldsymbol{\nabla} P &= 0 \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times T\hat{\boldsymbol{e}}_{\boldsymbol{z}} &= -\partial_x T \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times\Delta U &= \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times\Delta\boldsymbol{\nabla}\times\psi\hat{\boldsymbol{e}}_{\boldsymbol{y}} = -\Delta^2\psi
\end{aligned}$$

$$\partial_t\Delta\psi - \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times(U \cdot \boldsymbol{\nabla})U = Pr[\partial_x T + \Delta^2\psi]$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times f = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot f - \Delta f$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla} \times (U \cdot \boldsymbol{\nabla}) U \;=\; \partial_z (U \cdot \boldsymbol{\nabla}) u - \partial_x (U \cdot \boldsymbol{\nabla}) w$$

$$\begin{aligned}
&= \partial_z (u \partial_x u + w \partial_z u) - \partial_x (u \partial_x w + w \partial_z w) \\
&= \partial_z u \; \partial_x u + \partial_z w \; \partial_z u - \partial_x u \; \partial_x w - \partial_x w \; \partial_z w \\
&+ u \; \partial_{xz} u + w \; \partial_{zz} u - u \; \partial_{xx} w - w \; \partial_{xz} w \\
&= \partial_z u \; (\partial_x u + \partial_z w) - \partial_x w \; (\partial_x u + \partial_z w) \\
&+ u \; \partial_x (\partial_z u - \partial_x w) + w \partial_z (\; \partial_z u - \partial_x w) \\
&= (-\partial_z \psi) \partial_x (-\partial_{zz} \psi - \partial_{xx} \psi) \\
&+ (\partial_x \psi) \partial_z (-\partial_{zz} \psi - \partial_{xx} \psi) \\
&= (\partial_z \psi) \partial_x (\Delta \psi) - (\partial_x \psi) \partial_z (\Delta \psi)
\end{aligned}$$

$$= -J[\psi, \Delta \psi]$$

$$\partial_t \Delta \psi + J[\psi, \Delta \psi] = Pr[\partial_x T + \Delta^2 \psi]$$

Linear stability analysis

Linearized equations:

$$\partial_t \Delta\psi = Pr[\partial_x T + \Delta^2 \psi]$$
$$\partial_t T = Ra\, \partial_x \psi + \Delta T$$

Solutions:

$$\psi(x,z,t) = \hat{\psi} \sin qx \ \sin k\pi z\, e^{\lambda t} \qquad q \in \mathcal{R},\ k \in \mathcal{Z}^+,\ \lambda \in \mathcal{C}$$
$$T(x,z,t) = \hat{T} \cos qx \ \sin k\pi z\, e^{\lambda t}$$

$\uparrow$ functions $\qquad$ $\uparrow$ scalars

$$\gamma^2 \equiv q^2 + (k\pi)^2$$

$$-\lambda\gamma^2\hat{\psi} = Pr[-q\hat{T} + \gamma^4\hat{\psi}]$$
$$\lambda\hat{T} = Ra\; q\; \hat{\psi} - \gamma^2\hat{T}$$

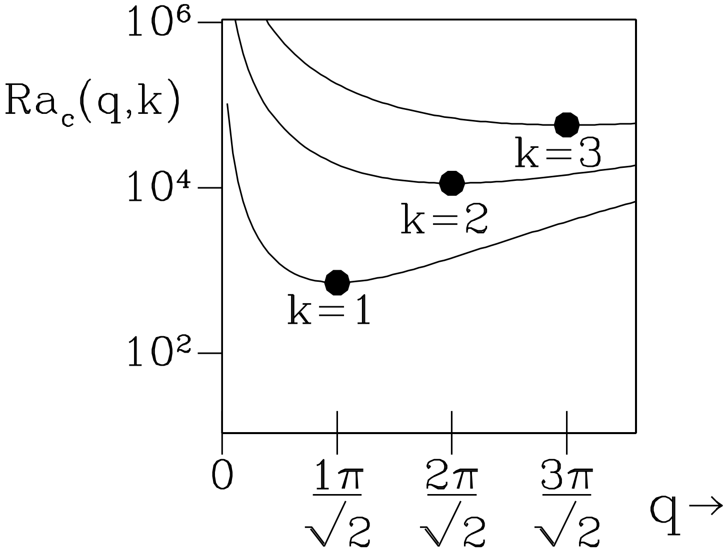
$$\lambda \begin{bmatrix} \hat{\psi} \\ \hat{T} \end{bmatrix} = \begin{bmatrix} -Pr\,\gamma^2 & Pr\,q/\gamma^2 \\ Ra\,q & -\gamma^2 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{T} \end{bmatrix}$$

Seek: $\lambda = 0$

$$Pr\,\gamma^4 - Pr\,Ra\,\frac{q^2}{\gamma^2} = 0$$

$$Ra = \frac{\gamma^6}{q^2} = \frac{(q^2 + (k\pi)^2)^3}{q^2} \equiv Ra_c(q, k)$$

Convection Threshold



Conductive state unstable at (q, k) for $Ra > Ra_c(q, k)$

Conductive state stable if

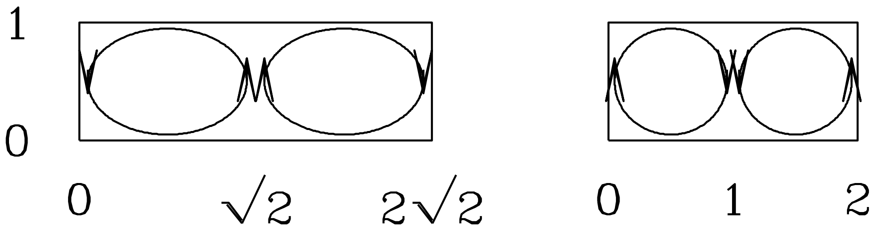
$$Ra < \min_{\substack{q \in \mathcal{R} \\ k \in \mathcal{Z}^+}} Ra_c(q,k)$$

$$\begin{aligned}
0 &= \frac{\partial Ra_c(q,k)}{\partial q} = \frac{q^2 3(q^2+(k\pi)^2)^2 2q - 2q(q^2+(k\pi)^2)^3}{q^4} \\
&= \frac{2(q^2+(k\pi)^2)^2}{q^3}(3q^2 - (q^2+(k\pi)^2)) \\
\Longrightarrow\ & q^2 = \frac{(k\pi)^2}{2}
\end{aligned}$$

$$Ra_c\left(q = \frac{k\pi}{\sqrt{2}}, k\right) = \frac{(k\pi)^2/2 + (k\pi)^2)^3}{(k\pi)^2/2} = \frac{27}{4}(k\pi)^4$$

$$Ra_c \equiv Ra_c\left(q = \frac{\pi}{\sqrt{2}}, k=1\right) = \frac{27}{4}(\pi)^4 = 657.5$$

Rigid Boundaries



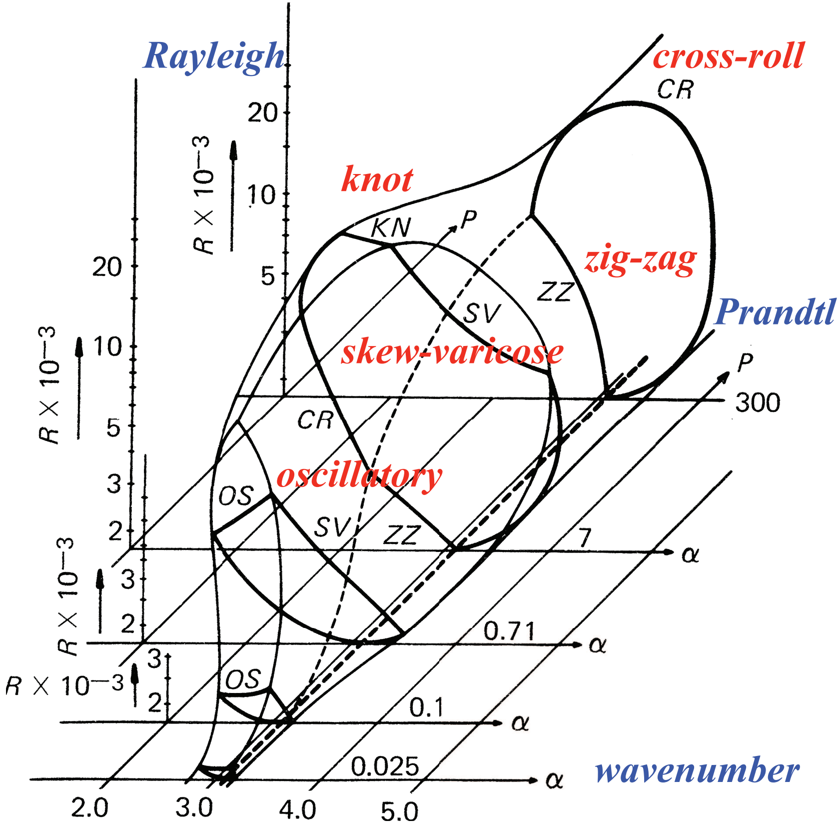
Calculation follows the same principle, but more complicated.

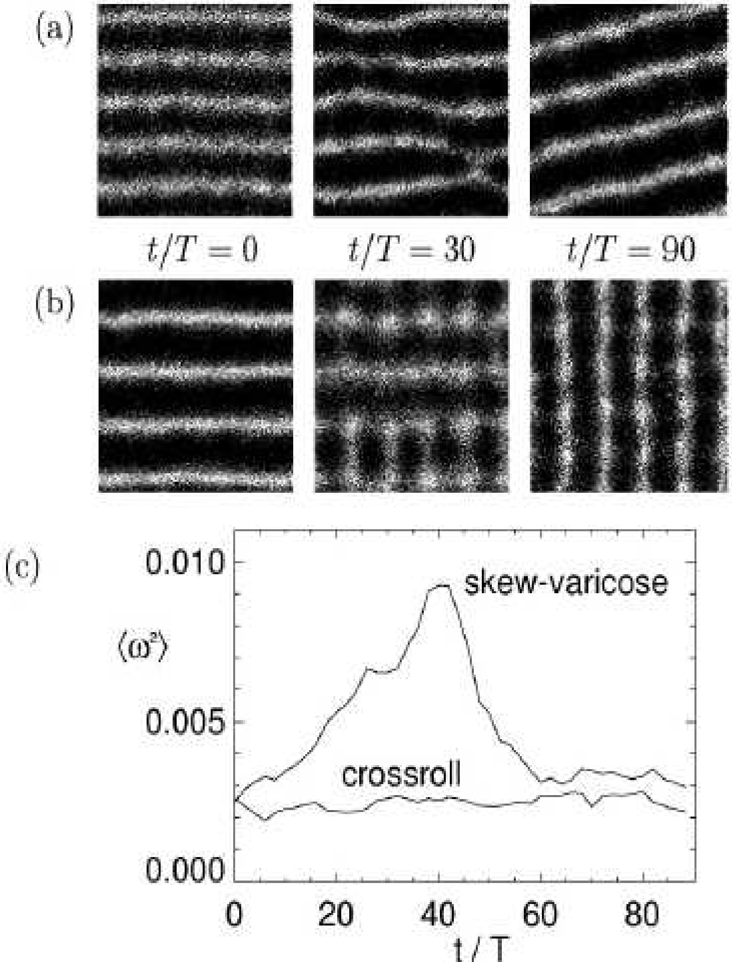
Boundaries damp perturbations $\Longrightarrow$ higher threshold

$q_c \downarrow \Longrightarrow \ell_c = \pi/q_c \uparrow \Longrightarrow$ rolls $\approx$ circular

	Ra_c	q_c	ℓ_c
stress-free boundaries	$\frac{27}{4}\pi^4 = 657.5$	$\frac{\pi}{\sqrt{2}}$	1.4
rigid boundaries	≈ 1700	$\approx \pi$	≈ 1

Instabilities of straight rolls: “Busse balloon”





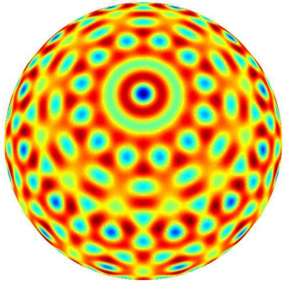
skew-varicose instability

cross-roll instability

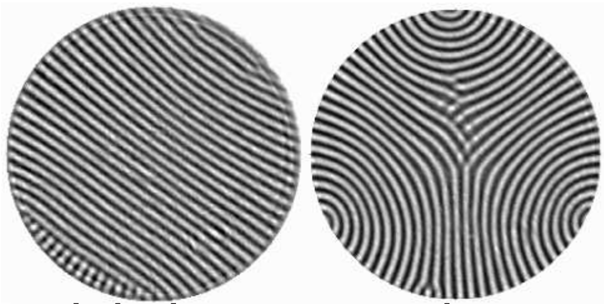
Continuum-type stability balloon in oscillated granulated layers, J. de Bruyn, C. Bizon, M.D. Shattuck, D. Goldman, J.B. Swift & H.L. Swinney, Phys. Rev. Lett. 1998.

Complex spatial patterns in convection

Experimental spiral defect chaos
Egolf, Melnikov, Pesche, Ecke
Nature **404** (2000)

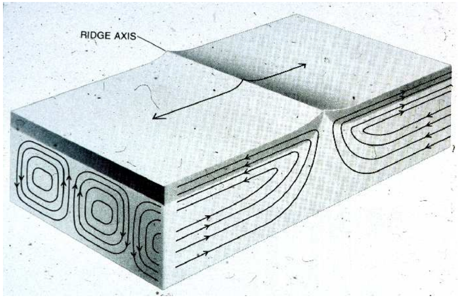


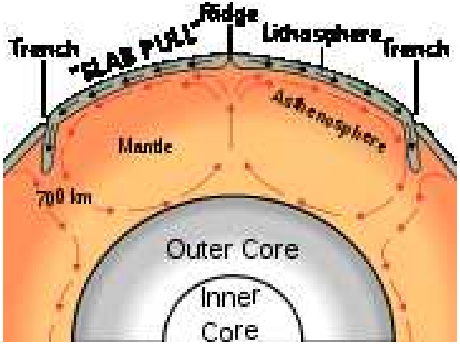
Spherical harmonic $\ell = 28$
P. Matthews
Phys. Rev. E. **67** (2003)



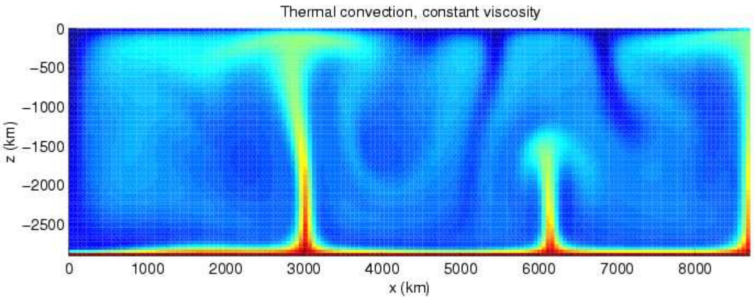
Convection in cylindrical geometry. Bajaj et al. J. Stat. Mech. (2006)

Geophysics



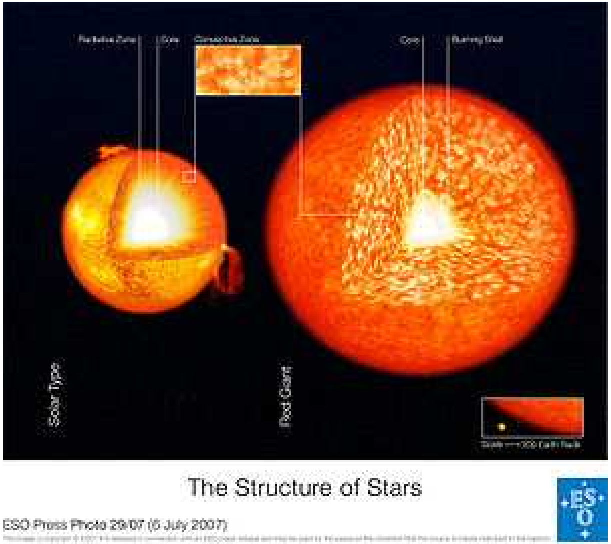


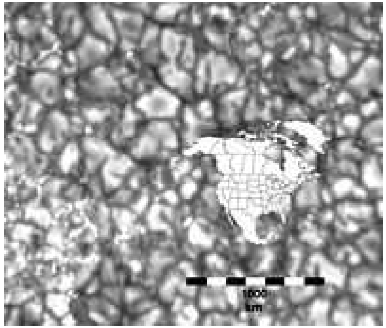
Convection and plate tectonics



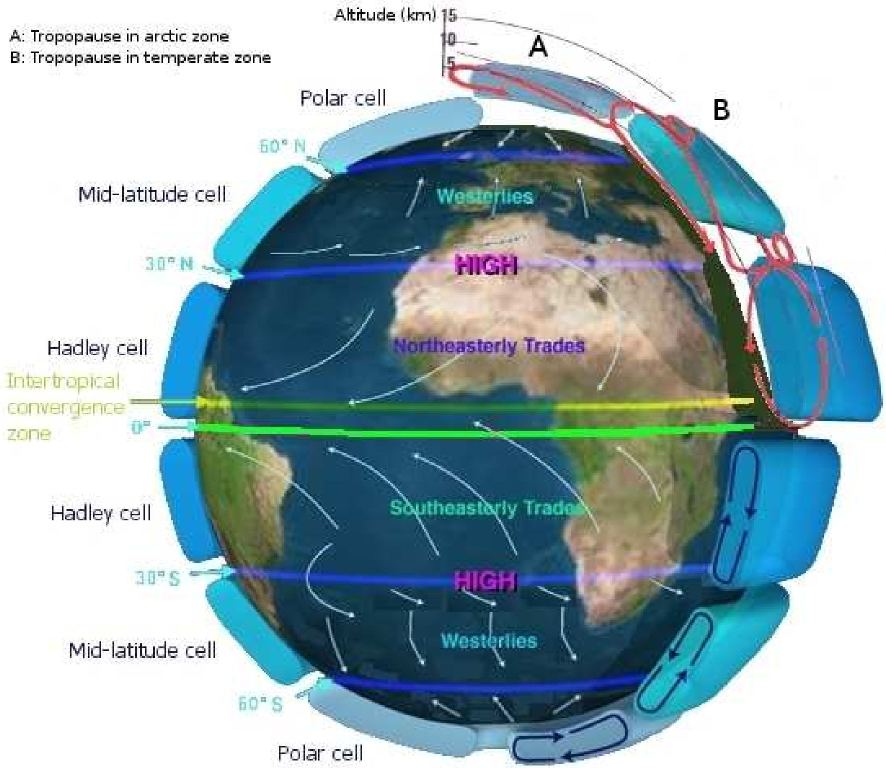
Numerical simulation of convection in earth’s mantle, showing plumes and thin boundary layers. By H. Schmeling, Wikimedia Commons.

Convection cells in the sun





Atmospheric circulation



Latitudinal circulation $\Longrightarrow$ Hadley cell, mid-latitude cell, polar cell

Solutal Convection

Mixture of two species whose density depends on the relative concentration:

$$\rho(C) = \rho_0 \left[1 - \beta(C - C_0)\right]$$

Impose concentration C_0 below and C_1 above. Base state has uniform concentration gradient C^*. Define deviation $\hat{C}$ from base state C^*:

$$C^* = C_0 - (C_0 - C_1)\frac{z}{d} \qquad C = C^* + \hat{C}$$

Governing equations

$$\begin{aligned} \rho_0 \left[\partial_t + (U \cdot \boldsymbol{\nabla})\right] U &= -\boldsymbol{\nabla} \hat{P} + g\rho_0\beta\hat{C}\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \mu\Delta U \\ \boldsymbol{\nabla}\cdot U &= 0 \\ \left[\partial_t + (U \cdot \boldsymbol{\nabla})\right] \hat{C} &= \frac{C_0 - C_1}{d} U \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}} + \kappa_C \Delta \hat{C} \end{aligned}$$

lead to solutal Rayleigh number and Schmidt number:

$Ra_C \equiv \dfrac{(C_0 - C_1)d^3 g\beta}{\kappa_C \nu}$ non-dimensional measure of concentration gradient

$Sc \equiv \dfrac{\nu}{\kappa_C}$ momentum diffusivity / mass diffusivity

The convection in the earth's mantle is solutal.

Thermosolutal Convection

$$\rho(T,C) = \rho_0\left[1-\alpha(T-T_0)+\beta(C-C_0)\right]$$

$$\begin{aligned}
\partial_t \Delta\psi + J[\psi,\Delta\psi] &= Pr[\partial_x T + \partial_x C + \Delta^2\psi] \\
\partial_t T + J[\psi,T] &= Ra\,\partial_x\psi + \Delta T \\
\partial_t C + J[\psi,C] &= Ra_C\,\partial_x\psi + \frac{Pr}{Sc}\Delta C
\end{aligned}$$

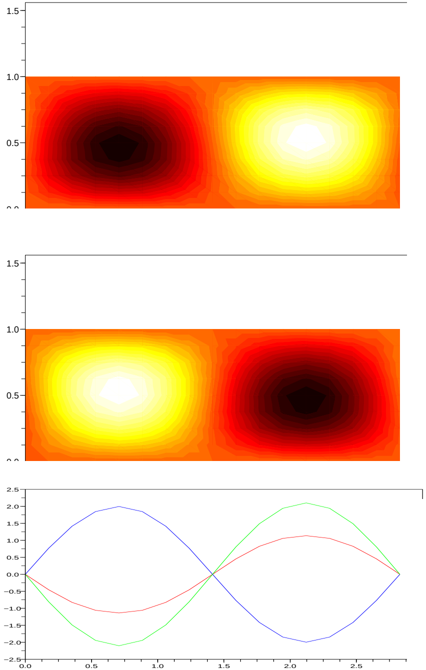
Linear Stability Analysis: set $J[.,.]=0$ and substitute

$$\begin{aligned}
\psi(x,z,t) &= \hat{\psi}\ \sin qx\ \sin k\pi z\ e^{\lambda t} \qquad q\in\mathcal{R},\ k\in\mathcal{Z}^+,\ \lambda\in\mathcal{C} \\
C(x,z,t) &= \hat{C}\ \cos qx\ \sin k\pi z\ e^{\lambda t} \\
T(x,z,t) &= \hat{T}\ \cos qx\ \sin k\pi z\ e^{\lambda t}
\end{aligned}$$

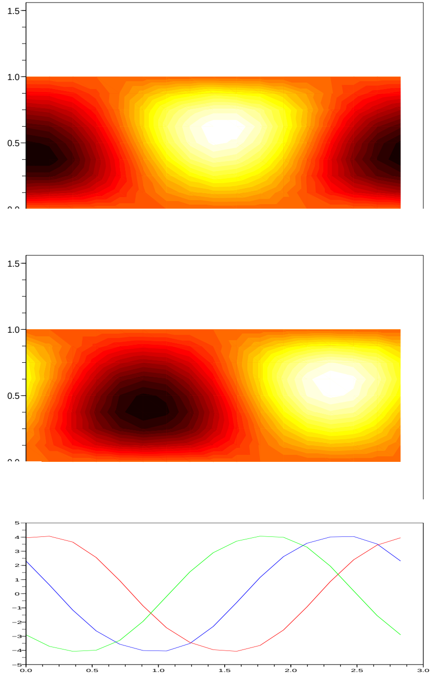
$\uparrow$ functions $\qquad$ $\uparrow$ scalars $\qquad$ $\gamma^2 \equiv q^2 + (k\pi)^2$

$$\begin{aligned}
-\lambda\gamma^2\hat{\psi} &= -qPr\left[\hat{T}+\hat{C}+\gamma^4\hat{\psi}\right] \\
\lambda\hat{T} &= Ra\,q\,\hat{\psi} - \gamma^2\hat{T} \\
\lambda\hat{C} &= Ra_C\,q\,\hat{\psi} - \frac{Pr}{Sc}\gamma^2\hat{C}
\end{aligned}$$

If thermal and solutal convection oppose each other, λ can be complex
$\Longrightarrow$ time-dependent states, like traveling and standing waves:



Standing waves



Travelling waves

Marangoni Convection: Mechanism

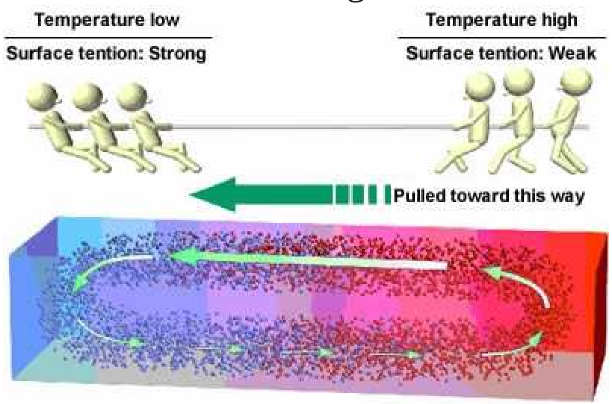
Surface tension varies with temperature. Define Marangoni number.

$$\sigma(T) = \sigma_0[1 + \gamma(T - T_0)] \qquad\qquad Ma \equiv \frac{(T_0 - T_1)d\sigma_0\gamma}{\kappa\mu}$$

Marangoni boundary condition at upper surface:

$$\frac{\partial u}{\partial z} = -\frac{\partial \sigma}{\partial x} = -\gamma\frac{\partial T}{\partial x} \text{ and } w = 0$$

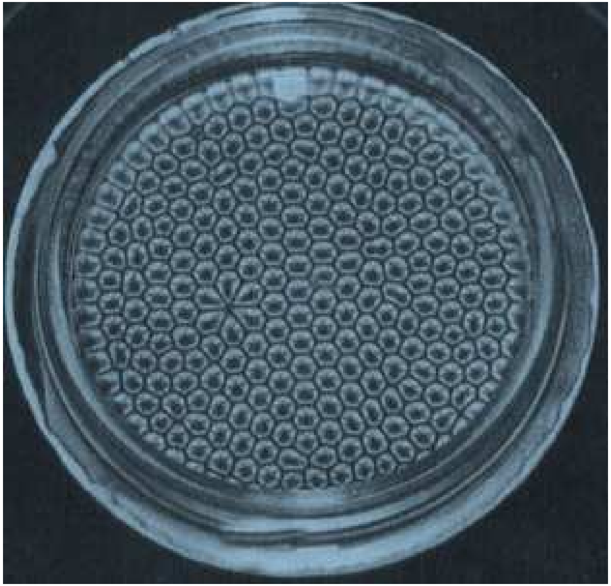
Along free surface, fluid moves towards region with lower surface tension.



Under region with high (low) surface tension, fluid rises (falls).

Marangoni Convection: Pattern

Experimental visualization of Marangoni convection by L. Koschmieder:



Hexagonal cells are formed in Marangoni convection because of asymmetry between top and bottom (theory of dynamical systems and pattern formation).

Marangoni Convection: History

Henri Bénard, “Les tourbillons cellulaires dans une nappe liquide transportant de la chaleur par convection en régime permanent”, Annales de Chimie et de Physique, **23** 62–144 (1901).

Bénard (1901) and Rayleigh (1916) thought that the motion was due to $\rho(T)$. Block (1956) and Pearson (1958) showed that Bénard’s cells were due to $\sigma(T)$. Bénard’s experiment showed hexagonal cells. In fact, buoyancy-driven ($\rho(T)$) convection usually forms rolls, not hexagons. But we still call buoyancy-driven convection “Rayleigh-Bénard convection”, even though Bénard did not see it.