Hydrodynamics

Class 8

Laurette TUCKERMAN

laurette@pmmh.espci.fr

Another ordinary differential equation with boundary layer

$$\epsilon y'' + a(x)y' + b(x)y = 0, \qquad y(0) = A,\ y(1) = B$$
$$a(x) > 0 \text{ over } [0,1], \qquad \epsilon \to 0$$

Assume boundary layer near $x = 0$. Outer equation, using B.C. $y(x = 1) = B$:

$$\begin{aligned}
a(x)y'_{\text{out}} + b(x)y_{\text{out}} &= 0 \\
\frac{dy_{\text{out}}}{dx} &= -\frac{b(x)}{a(x)}\,y_{\text{out}} \\
\int_x^1 \frac{dy_{\text{out}}}{y_{\text{out}}} &= -\int_x^1 dt\,\frac{b(t)}{a(t)} \\
\ln y_{\text{out}}(1) - \ln y_{\text{out}}(x) &= -\int_x^1 dt\,\frac{b(t)}{a(t)} \\
\ln y_{\text{out}}(x) &= \ln B + \int_x^1 dt\,\frac{b(t)}{a(t)} \\
y_{\text{out}}(x) &= B\exp\left[\int_x^1 dt\,\frac{b(t)}{a(t)}\right]
\end{aligned}$$

Inside boundary layer near $x = 0$, solution varies rapidly.

Set $X = x/\delta$, with $\delta \ll 1$.

$$\frac{\epsilon}{\delta^2}\frac{d^2 y_{\text{in}}}{dX^2} + a\frac{1}{\delta}\frac{dy_{\text{in}}}{dX} + by_{\text{in}} = 0$$

Terms are of order ϵ/δ^2, $1/\delta$, 1.
For non-trivial equation, want two terms to dominate (= "dominant balance")

$$\epsilon/\delta^2 \sim 1/\delta \quad \Longrightarrow \delta \sim \epsilon$$

$$\frac{1}{\delta}\frac{d^2 y_{\text{in}}}{dX^2} + a\frac{1}{\delta}\frac{dy_{\text{in}}}{dX} + by_{\text{in}} = 0$$

$$\frac{d^2 y_{\text{in}}}{dX^2} + a\frac{dy_{\text{in}}}{dX} = 0$$

Inside boundary layer, set $a(X\delta) = a(0) = \alpha$ and $y_{\text{in}}(0) = A$

$$y_{\text{in}} = Ce^{-\alpha X} + D \Longrightarrow A = C + D \Longrightarrow D = A - C \Longrightarrow y_{\text{in}} = A + C[e^{-\alpha X} - 1]$$

Matching in *overlap region* needed to determine C. Consider $x = O(\epsilon^{1/2})$.

$$\text{Then } X = x/\delta \sim \epsilon^{1/2}/\epsilon = \epsilon^{-1/2} \to \infty \text{ as } \epsilon \to 0$$

$$\text{so } y_{\text{in}} = A + C[e^{-\alpha X} - 1] \sim \boxed{A - C}$$

$$\text{and } x \sim \epsilon^{1/2} \to 0 \text{ as } \epsilon \to 0$$

$$\text{so } y_{\text{out}} = B \exp\left[\int_x^1 dt\, \frac{b(t)}{a(t)}\right] \sim B \exp\left[\int_0^1 dt\, \frac{b(t)}{a(t)}\right] \equiv \boxed{BK}$$

$$\text{Therefore } A - C = BK$$

$$\begin{aligned} \text{so } y_{\text{in}}(X) &= A + (A - BK)[e^{-\alpha X} - 1] \\ &= A + Ae^{-\alpha X} - A - BK[e^{-\alpha X} - 1] \\ &= Ae^{-\alpha X} + BK[1 - e^{-\alpha X}] \end{aligned}$$

Any $x = O(\epsilon^{\gamma})$ for $0 < \gamma < 1$ also works in this case.

$$\begin{aligned} y_{\text{unif}}(x) &= \overbrace{Ae^{-\alpha x/\epsilon} + BK[1 - e^{-\alpha x/\epsilon}]}^{y_{\text{in}}} + \overbrace{B \exp\left[\int_x^1 dt\, \frac{b(t)}{a(t)} dt\right]}^{y_{\text{out}}} - \overbrace{BK}^{y_{\text{overlap}}} \\ &= (A - BK)e^{-a(0)x/\epsilon} + B \exp\left[\int_x^1 dt\, \frac{b(t)}{a(t)} dt\right] \end{aligned}$$

What if we assume instead that the boundary layer is at $x = 1$?

Outer equation, using BC $y(x=0) = A$:

$$\begin{aligned}
a(x)y'_{\text{out}} + b(x)y_{\text{out}} &= 0 \\
\frac{dy_{\text{out}}}{dx} &= -\frac{b(x)}{a(x)}\,y_{\text{out}} \\
\int_0^x \frac{dy_{\text{out}}}{y_{\text{out}}} &= -\int_0^x dt\,\frac{b(t)}{a(t)} \\
\ln y_{\text{out}}(x) - \ln y_{\text{out}}(0) &= -\int_0^x dt\,\frac{b(t)}{a(t)} \\
\ln y_{\text{out}}(x) &= \ln A - \int_0^x dt\,\frac{b(t)}{a(t)} \\
y_{\text{out}}(x) &= A\exp\left[-\int_0^x dt\,\frac{b(t)}{a(t)}\right]
\end{aligned}$$

Inside boundary layer near $x = 1$, solution varies rapidly.

Set $X = (1 - x)/\delta$, with $\delta \ll 1$. $d/dx \longrightarrow d/dX$ changes sign.

$$\frac{\epsilon}{\delta^2}\frac{d^2 y_{\text{in}}}{dX^2} - a\frac{1}{\delta}\frac{dy_{\text{in}}}{dX} + by_{\text{in}} = 0$$

Terms are of order ϵ/δ^2, $1/\delta$, 1.
For non-trivial equation, want two terms to dominate (= "dominant balance")

$$\epsilon/\delta^2 \sim 1/\delta \quad \Longrightarrow \delta \sim \epsilon$$

$$\frac{1}{\delta}\frac{d^2 y_{\text{in}}}{dX^2} - a\frac{1}{\delta}\frac{dy_{\text{in}}}{dX} + by_{\text{in}} = 0$$

$$\frac{d^2 y_{\text{in}}}{dX^2} - a\frac{dy_{\text{in}}}{dX} = 0$$

Inside boundary layer, set $a = a(1) = \alpha$ and $y_{\text{in}}(1) = B$

$$y_{\text{in}} = Ce^{\alpha X} + D \Longrightarrow B = C + D \Longrightarrow D = B - C \Longrightarrow y_{\text{in}} = B + C[e^{\alpha X} - 1]$$

Matching in overlap region needed to determine C. Consider $1 - x = O(\epsilon^{1/2})$.

Then $X = (1-x)/\delta \sim \epsilon^{1/2}/\epsilon = \epsilon^{-1/2} \to \infty$ as $\epsilon \to 0$

so $y_{\text{in}} = B + C[e^{\alpha X} - 1] \to \infty$

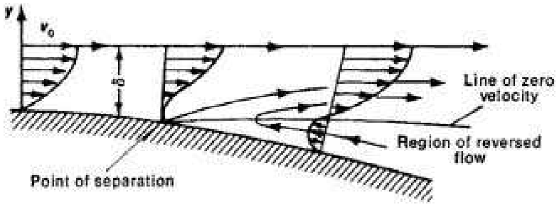
and $x \sim 1 - \epsilon^{1/2} \to 1$ as $\epsilon \to 0$

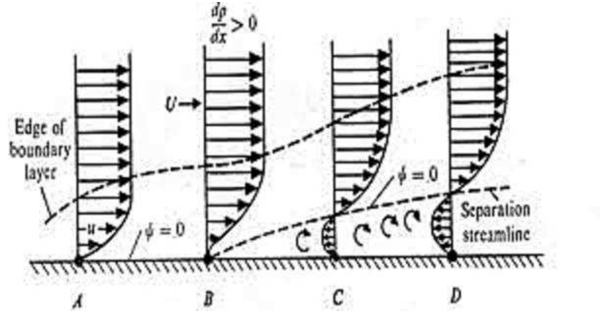
so $y_{\text{out}} = A \exp\left[-\int_0^1 dt\, \frac{b(t)}{a(t)}\right] = AK$

$$\boxed{\text{No match possible}}$$

Boundary layer is at $x = 1$ if $a(x) < 0$.

Boundary Layer Separation





Boundary layer equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{with} \quad -\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx} = \frac{1}{2}\frac{d(U^2)}{dx}$$

At wall, $u = v = 0$, so

$$\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{\text{wall}} = \frac{1}{\rho}\frac{dp}{dx} = -\frac{1}{2}\frac{d(U^2)}{dx} \quad \text{so } \textit{deceleration} \implies \frac{d(U^2)}{dx} < 0 \implies \left.\frac{\partial^2 u}{\partial y^2}\right|_{\text{wall}} > 0$$

At top edge of boundary layer, usually $u \to U$ and $\dfrac{\partial u}{\partial y} \to 0^+$ and $\dfrac{\partial^2 u}{\partial y^2} < 0$

$$\left\{ \begin{array}{l} \textit{favorable} \\ \textit{adverse} \end{array} \right\} \text{pressure gradient: } \frac{dp}{dx} \left\{ \begin{array}{c} < \\ > \end{array} \right\} 0 \Longrightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_{\text{wall}} \left\{ \begin{array}{c} < \\ > \end{array} \right\} 0$$

•In *favorable* case, can have $\dfrac{\partial^2 u}{\partial y^2} < 0$ over entire boundary layer.

$\dfrac{\partial u}{\partial y}$ decreases monotonically from positive value at wall to 0^+ at bdy layer edge.

•In *adverse* case, $\dfrac{\partial^2 u}{\partial y^2} > 0$ at wall and $\dfrac{\partial^2 u}{\partial y^2} < 0$ at bdy layer edge, so

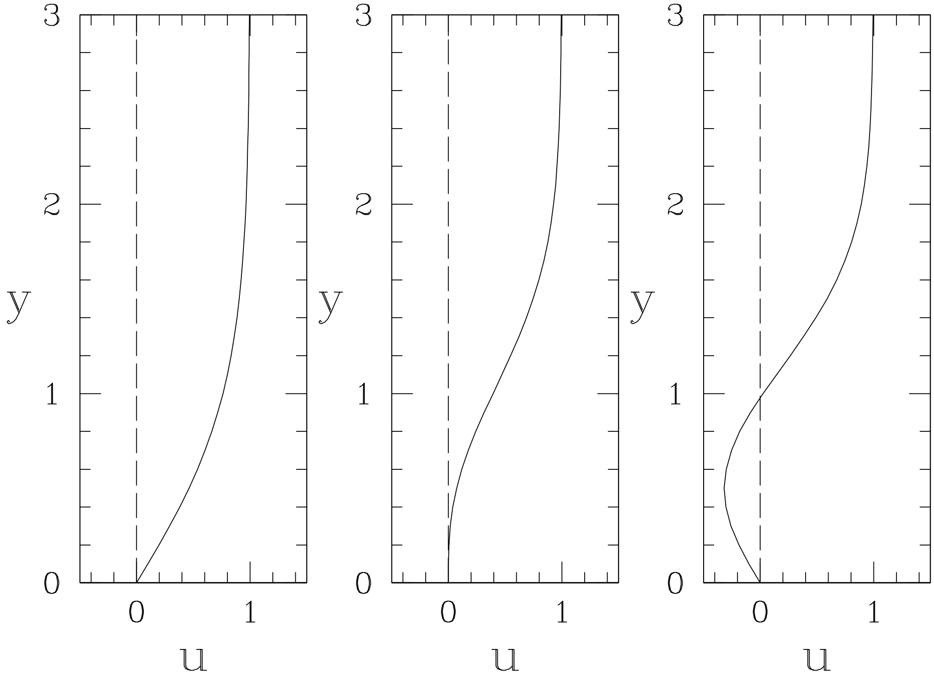
$\dfrac{\partial^2 u}{\partial y^2}$ changes sign in boundary layer at an inflection point.

If $\dfrac{\partial^2 u}{\partial y^2}$ is *sufficiently positive* at wall, then $\dfrac{\partial u}{\partial y}\Big|_{\text{wall}} < 0$ so $u|_{\text{wall}} < 0 \Longrightarrow \boxed{\text{Separation}}$

Boundary layer equations not valid beyond separation point.

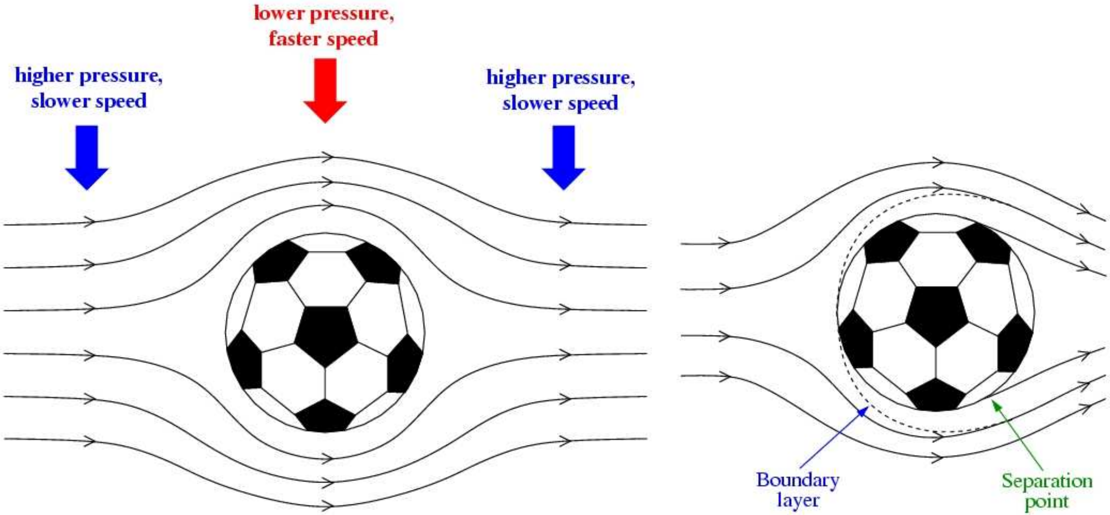
Behind blunt body, steep adverse pressure gradient $\Longrightarrow$ immediate separation.

Behind streamlined body, mild pressure gradient can be overcome.

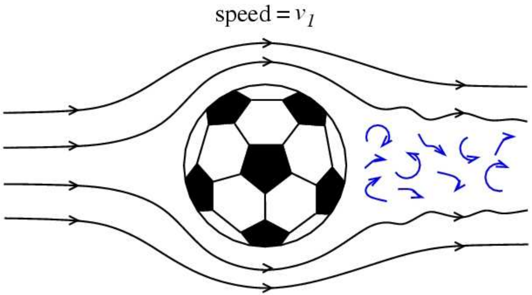


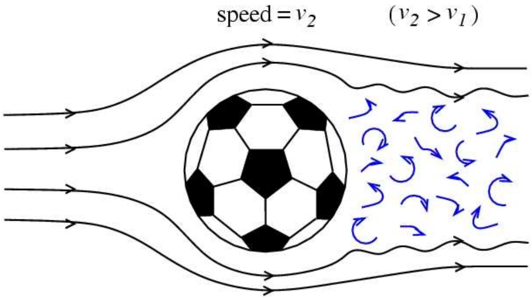
3
2
y
1
0
0
1
u

Soccer Ball

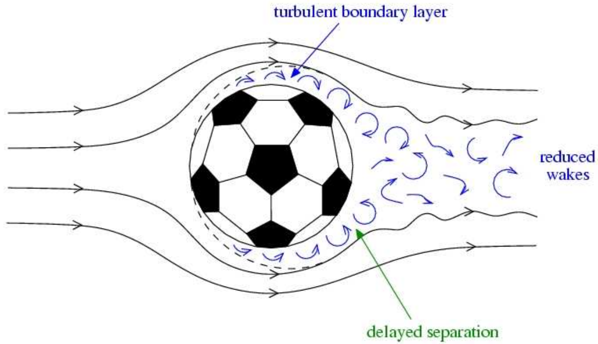


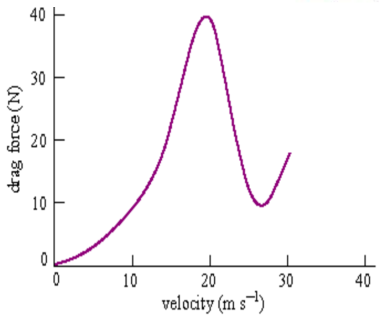
From M.K. Yip, Hong Kong University,
http://www.physics.hku.hk/~phys0607



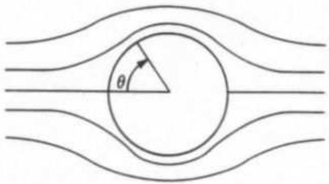


Speed increases $\Longrightarrow$ turbulent wake of laminar boundary layer widens $\Longrightarrow$ more drag

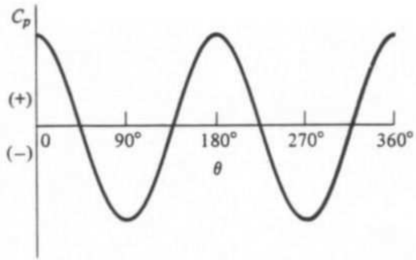




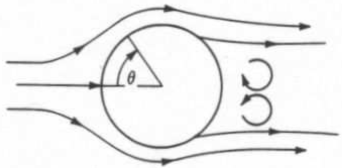
Speed increases past critical value $\Longrightarrow$ boundary layer becomes turbulent $\Longrightarrow$ separates further $\Longrightarrow$ smaller wake $\Longrightarrow$ less drag



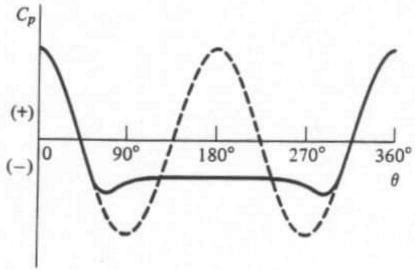
(a)



(b)



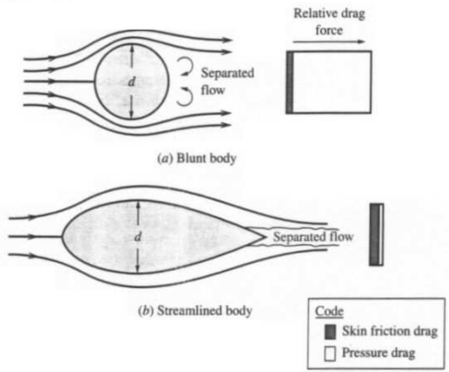
(a)



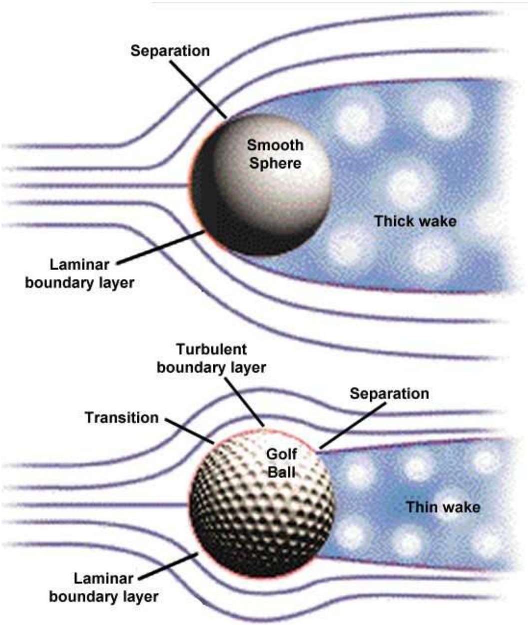
(b)

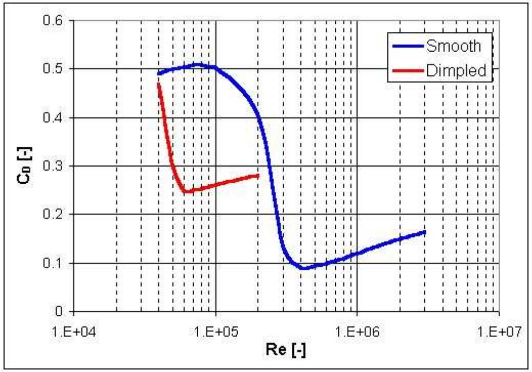
Airflow around a ball for different Reynolds number

Reynolds number	Boundary layer	Type of wake	Main drag force
<2000	Laminar	Small laminar	Viscous
2000 - 100000	Laminar	Large turbulent	Pressure
>100000	Turbulent	Small turbulent	Pressure



Golf Ball

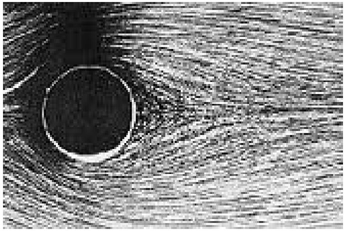




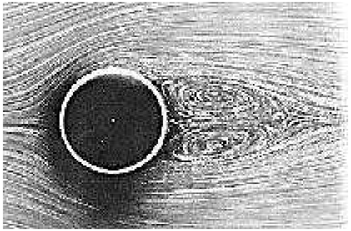
Dimples $\Longrightarrow$ turbulent bdy layer $\Longrightarrow$ delays separation $\Longrightarrow$ reduces drag

From
http://www.aerospaceweb.org/question/aerodynamics/q0215.shtml

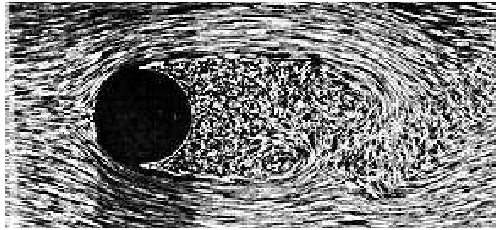
Wake of circular cylinder: visualisations by Taneda



Re=9.6

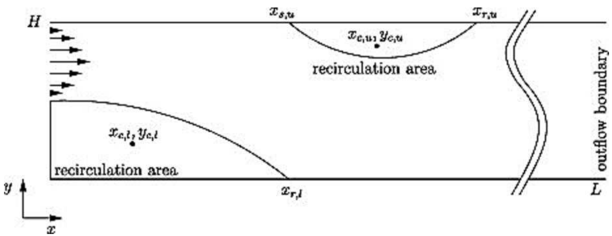


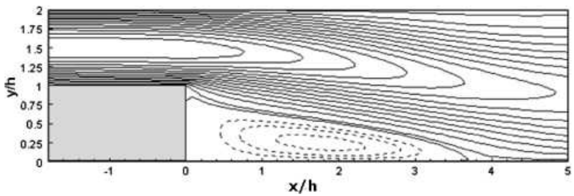
Re=26

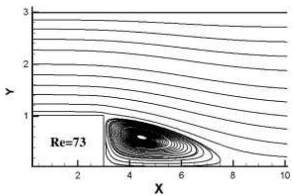


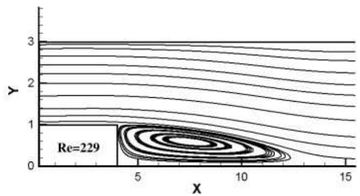
Re=2000

Backward Facing Step





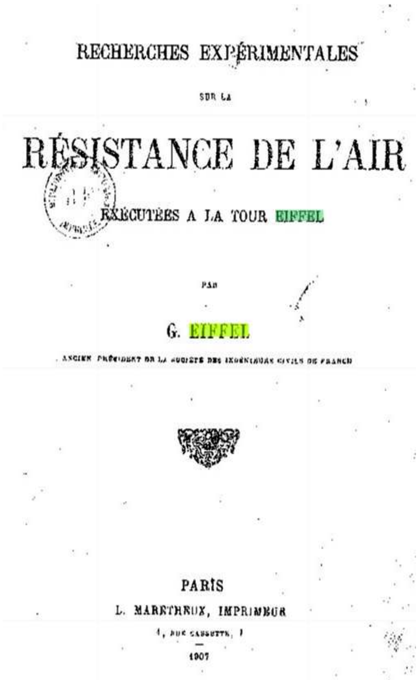




Re=73

Re = 229

M. Eiffel discovers the drag crisis by dropping balls from his tower:



RECHERCHES EXPÉRIMENTALES

SUR LA

RÉSISTANCE DE L'AIR

EXÉCUTÉES A LA TOUR EIFFEL

PAR

G. EIFFEL

ANCIEN PRÉSIDENT DE LA SOCIÉTÉ DES INGÉNIEURS CIVILS DE FRANCE

PARIS

L. MARETHEUX, IMPRIMEUR

1, RUE CASSETTE, 1

1907

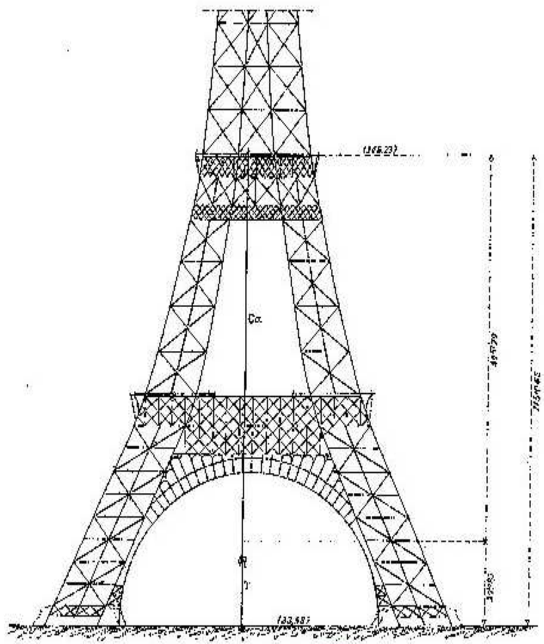


Fig. 1. — Installation à la Tour Eiffel.

Thin Film Theory

Viscous fluid between rigid boundaries at $z = 0$ and $z = h(x, y)$

Characteristic horizontal scales: velocity U, length $L \gg h$

Incompressibility:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad \Longrightarrow \quad \frac{w}{h} \sim \frac{U}{L} \quad \Longrightarrow \quad w \sim \frac{h}{L} U \ll U$$

No-slip boundary conditions:

$$\left\{ \begin{array}{ll} u, v = 0 & \text{at } \ z = 0, h \\ u, v \sim U & \text{near middle} \end{array} \right\} \Longrightarrow \frac{\partial(u, v)}{\partial z} \sim \frac{U}{h} \gg \frac{U}{L} \sim \frac{\partial(u, v)}{\partial(x, y)}$$

Can neglect inertial terms if:

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla}) \sim \frac{U}{L} \ll \frac{\nu}{h^2} \sim \nu \frac{\partial^2}{\partial z^2} \quad \Longleftrightarrow \quad Re = \frac{UL}{\nu} \ll \left(\frac{L}{h} \right)^2$$

Pressure mainly depends on horizontal coordinates:

$$\frac{\partial p}{\partial z} = \mu \frac{\partial^2 w}{\partial z^2} \sim \mu \frac{hU/L}{h^2} \ll \mu \frac{U}{h^2} \sim \mu \frac{\partial^2 (u, v)}{\partial z^2} = \frac{\partial p}{\partial(x, y)}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}(x,y) \Longrightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(x,y)z^2 + A(x,y)z + B(x,y)$$

$$\frac{\partial^2 v}{\partial z^2}(x,y) = \frac{1}{\mu}\frac{\partial p}{\partial y} \Longrightarrow v = \frac{1}{2\mu}\frac{\partial p}{\partial y}(x,y)z^2 + C(x,y)z + D(x,y)$$

Hele-Shaw cell

$$h \text{ constant} \Longrightarrow \left\{ \begin{array}{l} u = \dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}z(z-h) \\ v = \dfrac{1}{2\mu}\dfrac{\partial p}{\partial y}z(z-h) \end{array} \right\} \Longrightarrow \boldsymbol{u}_{\text{hor}} = \boldsymbol{\nabla}_{\text{hor}}\phi \;\text{ with }\; \phi = \frac{1}{2\mu}z(z-h)p$$

$\Longrightarrow$ Potential flow – despite low Reynolds number

Circulation:

$$\Gamma = \oint \boldsymbol{u}\cdot \boldsymbol{ds} = \frac{1}{2\mu}z(z-h)\oint \boldsymbol{\nabla}P\cdot\boldsymbol{ds} = 0$$

Thin-film flow down an inclined plane

2D with tilted coordinates $\Longrightarrow -\boldsymbol{g} = -g(-\sin\alpha\hat{\boldsymbol{e}}_{\boldsymbol{x}} + \cos\alpha\hat{\boldsymbol{e}}_{\boldsymbol{z}})$

Solid surface at $z = 0$ with no-slip boundary conditions.

Free upper surface at $z = h(x, t)$. Boundary conditions are

$$p = p_{\rm atm} \ \text{ and } \ \mu\frac{\partial u}{\partial z} = 0$$

Not steady, but can show $\boldsymbol{u}$ depends on time mainly via $h(x, t)$

Momentum balance in z

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\cos\alpha \Longrightarrow p = -\rho g\cos\alpha z + f(x, t)$$

Apply boundary condition $p = p_{\rm atm}$ at $z = h$:

$$\begin{aligned}
p_{\rm atm} &= -\rho g\cos\alpha h(z, t) + f(x, t) \\
p &= -\rho g\cos\alpha z + \rho g\cos\alpha h(x, t) + p_{\rm atm} \\
&= \rho g\cos\alpha(h(x, t) - z) + p_{\rm atm}
\end{aligned}$$

Momentum balance in x

$$
\begin{aligned}
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + g\sin\alpha + \nu\frac{\partial^2 u}{\partial z^2} \\
&= -g\cos\alpha\frac{\partial h}{\partial x} + g\sin\alpha + \nu\frac{\partial^2 u}{\partial z^2} \\
&\approx g\sin\alpha + \nu\frac{\partial^2 u}{\partial z^2} \quad \text{if} \ \frac{\partial h}{\partial x} \sim \frac{h}{L} \ll \tan\alpha
\end{aligned}
$$

$$
u = -\frac{g\sin\alpha}{2\nu}z^2 + A(x,t)z + B(x,t)
$$

Apply boundary conditions $u = 0$ at $z = 0$ and $\mu\partial u/\partial z = 0$ at $z = h$:

$$
\begin{aligned}
0 &= u(z=0) = B(x,t) \\
\frac{\partial u}{\partial z} &= -\frac{g\sin\alpha}{\nu}\,z + A(x,t) \Longrightarrow 0 = -\frac{g\sin\alpha}{\nu}\,h + A(x,t) \\
u &= -\frac{g\sin\alpha}{2\nu}\,z^2 + \frac{g\sin\alpha}{\nu}\,h\,z = \frac{g\sin\alpha}{\nu}\left(h(x,t)z - \frac{z^2}{2}\right)
\end{aligned}
$$

Incompressibility

$$
w(z) - w(0) = -\int_0^z dz\,\frac{\partial u}{\partial x} = -\int dz\,\frac{g\sin\alpha}{\nu}\,\frac{\partial h}{\partial x}\,z = -\frac{g\sin\alpha}{2\nu}\,\frac{\partial h}{\partial x}\,z^2
$$

Recall kinematic condition at free surface $z = h(x,t)$:

$$
\begin{aligned}
w|_{z=h} &= \left.\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right|_{z=h} \\
\left.\frac{g\sin\alpha}{2\nu}\,\frac{\partial h}{\partial x}\,z^2\right|_{z=h} &= \left.\frac{\partial h}{\partial t} + \frac{g\sin\alpha}{\nu}\left(hz - \frac{z^2}{2}\right)\frac{\partial h}{\partial x}\right|_{z=h} \\
\frac{g\sin\alpha}{2\nu}\,\frac{\partial h}{\partial x}\,h^2 &= \frac{\partial h}{\partial t} + \frac{g\sin\alpha}{\nu}\left(h^2 - \frac{h^2}{2}\right)\frac{\partial h}{\partial x} \\
0 &= \frac{\partial h}{\partial t} + \frac{g\sin\alpha}{\nu}\,h^2\,\frac{\partial h}{\partial x}
\end{aligned}
$$

Can show that main part of solution approaches

$$h(x,t) = \left(\frac{\nu}{g\sin\alpha}\,\frac{x}{t}\right)^{\frac{1}{2}}$$

and that the leading edge of a drop moves with law

$$x_N(t) = \left(\frac{9A^2 g\sin\alpha}{4\nu}\,t\right)^{\frac{1}{3}}$$

where $A = \int_0^{x_N(t)} dx\; h(x,t)$ is area of drop