Hydrodynamics

Class 7

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Boundary Layers

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

Hypotheses for boundary layer equations:

(1) Variation in y is faster than variation in $x \Longrightarrow \partial_y \gg \partial_x$

Incompressibility then implies $u \gg v$ and in fact:

$$\frac{O(u)}{O(v)} = \frac{O(\partial_y)}{O(\partial_x)} \equiv \frac{1}{\delta}$$

Define: $U_\infty \equiv O(u) \Longrightarrow O(v) = \delta U_\infty$ $\qquad \frac{1}{L} \equiv O(\partial_x) \Longrightarrow O(\partial_y) = \frac{1}{L\delta}$

Incompressibility also implies that the two inertial terms are of the same order:

$$u\partial_x \sim v\partial_y \sim \frac{U_\infty}{L}$$

Non-dimensionalize:

$$
\begin{aligned}
\frac{U_\infty^2}{L}\left(\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right) &= -\frac{1}{L}\,\frac{P_0}{\rho}\,\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu U_\infty}{L^2}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{\delta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right) \\
\frac{\delta U_\infty^2}{L}\left(\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}\right) &= -\frac{1}{\delta L}\,\frac{P_0}{\rho}\,\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\nu \delta U_\infty}{L^2}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{\delta^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right) \\
\frac{U_\infty}{L}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} &= -\frac{1}{U_\infty^2}\,\frac{P_0}{\rho}\,\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu}{L U_\infty}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{\delta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right) \\
\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} &= -\frac{1}{\delta^2 U_\infty^2}\,\frac{P_0}{\rho}\,\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\nu}{L U_\infty}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{\delta^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right) \\
\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} &= 0
\end{aligned}
$$

(2) Largest viscous term should be comparable with inertial terms:

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{1}{U_\infty^2}\frac{P_0}{\rho}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu}{LU_\infty}\left(\cancel{\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}} + \frac{1}{\delta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

Set $\quad \frac{\nu}{LU_\infty}\frac{1}{\delta^2} = 1 \Longrightarrow \boxed{\delta = \sqrt{\frac{\nu}{LU_\infty}} = \sqrt{\frac{1}{Re}}}$

Set $\quad \frac{1}{U_\infty^2}\frac{P_0}{\rho} = 1 \Longrightarrow P_0 = \rho U_\infty^2$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{1}{\delta^2}\underbrace{\frac{1}{U_\infty^2}\frac{P_0}{\rho}}_{1}\frac{\partial \tilde{p}}{\partial \tilde{y}} + \underbrace{\frac{\nu}{LU_\infty}}_{\delta^2}\left(\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{\delta^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right)$$

$$\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = \boxed{-\frac{1}{\delta^2}\frac{\partial \tilde{p}}{\partial \tilde{y}}} + \delta^2\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}$$

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}} \qquad \Longrightarrow \tilde{p} = \tilde{p}(x)$$

Boundary layer equations:

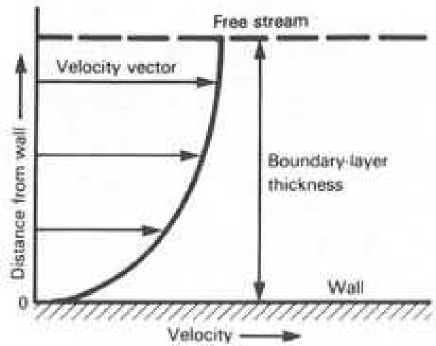
$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{d\tilde{p}}{d\tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$
$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

In dimensional terms:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary conditions are:

$$u(x, y=0) = 0$$
$$v(x, y=0) = 0$$
$$u(x, y=\infty) = U_\infty(x)$$
$$u(x=0, y) = U_e(y)$$



Implications:

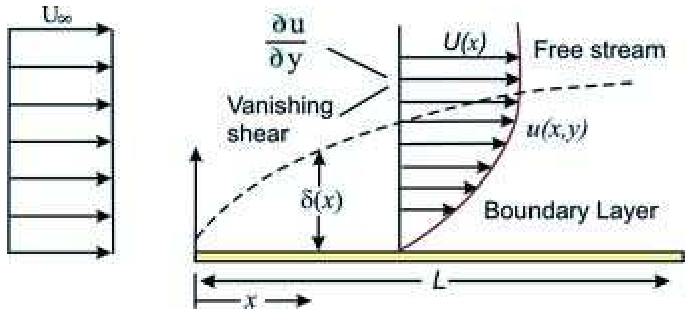
Equations are *parabolic* (∂_y^2 only), not *elliptic* ($\partial_x^2 + \partial_y^2$)

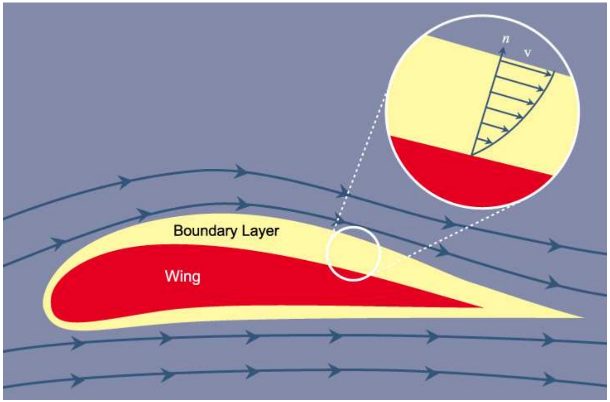
$\Longrightarrow$ BCs at two y-values but only at entry $x = 0$

$p(x)$, $U_\infty(x)$ are inherited from outer irrotational problem

$$\Longrightarrow -\frac{1}{\rho}\frac{dp}{dx} = U_\infty \frac{dU_\infty}{dx}$$

Spatially developing boundary layer has $\delta(x)$, $Re = Re(x)$



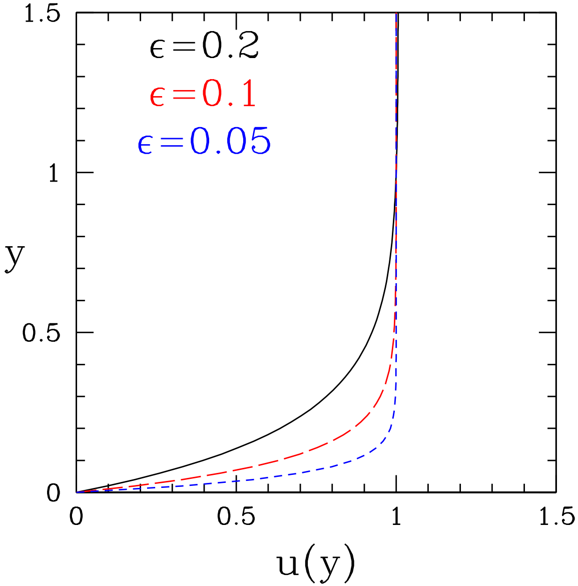


Ordinary Differential Equation with boundary layer

$$\epsilon u'' + u' = 0 \qquad u(0) = 0 \qquad u(1) = 1$$

$$u(y) = e^{\lambda y} \text{ where } \epsilon\lambda^2 + \lambda = 0 \Longrightarrow \lambda = 0 \text{ or } \lambda = -1/\epsilon$$

$$\begin{aligned} u(y) &= a + be^{-y/\epsilon} \\ 0 = u(0) &= a + b \\ 1 = u(1) &= a(1 - e^{-1/\epsilon}) \\ u(y) &= \frac{1 - e^{-y/\epsilon}}{1 - e^{-1/\epsilon}} \end{aligned}$$



$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} = 0 \qquad u(0) = 0 \qquad u(1) = 1$$

Outer solution:

Consider problem with $\epsilon = 0 \Longrightarrow u'_{\text{out}} = 0 \Longrightarrow u =$ constant

Guess that boundary layer is at $y = 0$, so set $u_{\text{out}} = u(1) = 1$

Inner solution: Define stretched variable $Y \equiv y/\epsilon$

$$\epsilon \frac{1}{\epsilon^2} \frac{d^2u_{\text{in}}}{dY^2} + \frac{1}{\epsilon} \frac{du_{\text{in}}}{dY} = 0$$

$$\frac{d^2u_{\text{in}}}{dY^2} + \frac{du_{\text{in}}}{dY} = 0 \implies u_{\text{in}}(Y) = a + be^{-Y}$$

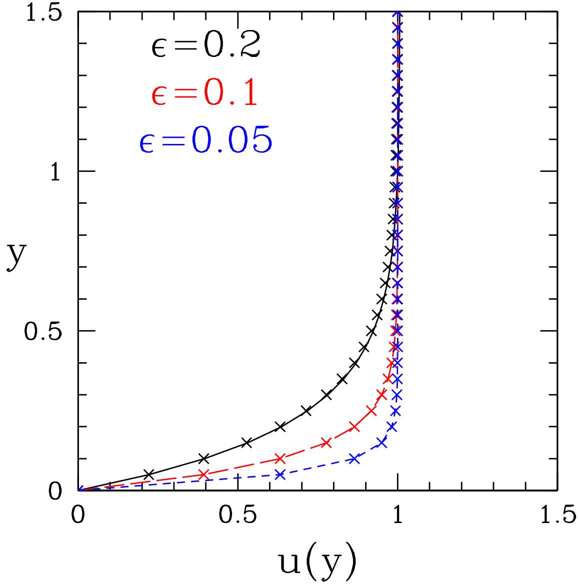
$$u_{\text{in}}(Y = 0) = 0 = a + b \implies u_{\text{in}}(Y) = a(1 - e^{-Y})$$

Matching:

$$\lim_{Y \to \infty} u_{\text{in}}(Y) = \lim_{y \to 0} u_{\text{out}}(y) = u_{\text{overlap}}$$

$$a = 1$$

$$\begin{aligned} u_{\text{approx}}(y) &= u_{\text{in}} + u_{\text{out}} - u_{\text{overlap}} \\ &= (1 - e^{-y/\epsilon}) + 1 - 1 = 1 - e^{-y/\epsilon} \end{aligned}$$



Curves : $u_{\text{exact}}(y) = \dfrac{1 - e^{-y/\epsilon}}{1 - e^{-1/\epsilon}}$

Crosses : $u_{\text{approx}}(y) = 1 - e^{-y/\epsilon}$

What happens if we try to put the boundary layer at $y = 1$?

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} = 0 \qquad u(0) = 0 \qquad u(1) = 1$$

Outer solution:

Consider problem with $\epsilon = 0 \Longrightarrow u'_{\text{out}} = 0 \Longrightarrow u =$ constant

Guess that boundary layer is at $y = 1$, so set $u_{\text{out}} = u(0) = 0$

Inner solution: Define stretched variable $Y \equiv (1-y)/\epsilon$

$$\epsilon \frac{1}{\epsilon^2} \frac{d^2u_{\text{in}}}{dY^2} - \frac{1}{\epsilon} \frac{du_{\text{in}}}{dY} = 0$$

$$\frac{d^2u_{\text{in}}}{dY^2} - \frac{du_{\text{in}}}{dY} = 0 \Longrightarrow u_{\text{in}}(Y) = a + be^Y$$

$$u_{\text{in}}(Y = 0) = 1 = a + b \Longrightarrow u_{\text{in}}(Y) = a + (1-a)e^Y$$

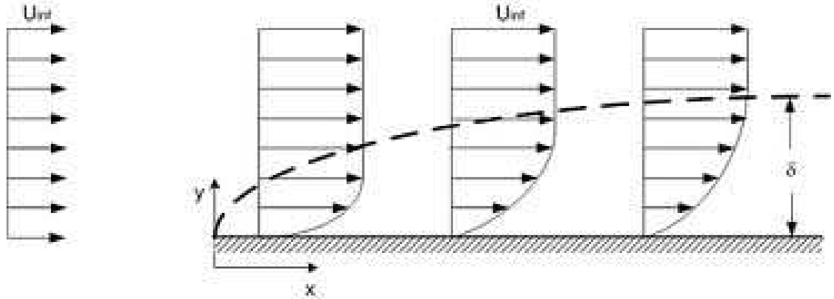
Matching:

$$\lim_{Y \to \infty} u_{\text{in}}(Y) = \lim_{y \to 1} u_{\text{out}}(y)$$

$$\left\{ \begin{array}{ccc} \infty & \text{if} & a \neq 1 \\ 1 & \text{if} & a = 1 \end{array} \right\} = 0$$

Not possible

Blasius boundary layer



For $x < 0$
$u = U_\infty$, $v = 0$

Surface at $y = 0$, $x \geq 0 \Longrightarrow u = v = 0$
In free stream, $U(x) = U_\infty \Longrightarrow dp/dx = 0$

Previously defined:

$$\delta = \sqrt{\frac{\nu}{LU_\infty}} = \sqrt{\frac{1}{Re}} \qquad \tilde{y} = \frac{y}{L}\frac{1}{\delta} = \frac{y}{L}\sqrt{\frac{LU_\infty}{\nu}} = y\sqrt{\frac{U_\infty}{\nu L}}$$

Spatially developing boundary layer has $L \to x$ so $\delta \to \delta(x)$, $Re \to Re(x)$

Now define:

$$\eta = \frac{y}{\delta(x)} \quad \to \quad y\sqrt{\frac{U_\infty}{\nu x}}$$

$$\eta = \frac{y}{\delta(x)}$$

$$\eta_x = -y\,\frac{\delta'(x)}{\delta(x)^2} = -\eta\,\frac{\delta'(x)}{\delta(x)} \qquad \text{and} \qquad \eta_y = \frac{1}{\delta(x)}$$

Seek *similarity solution* of the form

$$\begin{aligned}
u(x,y) &= UF(\eta(x,y)) = Uf'(\eta) \\
u_x &= Uf''(\eta)\,\eta_x = -Uf''(\eta)\,\eta\,\frac{\delta'(x)}{\delta(x)} = -v_y \\
v &= \int dy\; v_y(\eta) = \int dy\, Uf''(\eta)\,\eta\,\frac{\delta'(x)}{\delta(x)} \\
&= U\delta'(x)\int \frac{dy}{\delta(x)} f''(\eta)\,\eta = U\delta'(x)\int d\eta f''(\eta)\,\eta \\
&= U\delta'(x)\left[f'(\eta)\,\eta - \int d\eta f'(\eta)\right] = U\delta'(\eta f' - f) \\
u_y &= Uf''(\eta)\,\eta_y = Uf''(\eta)/\delta(x) \\
u_{yy} &= Uf'''(\eta)\,\eta_y/\delta(x) = Uf'''(\eta)/\delta(x)^2
\end{aligned}$$

$$uu_x + vu_y = \nu u_{yy}$$

$$\begin{aligned}
uu_x &= \cancel{(Uf')\left(-Uf''\eta\,\delta'/\delta\right)} \\
vu_y &= \left(U\delta'\left(\cancel{\eta f'} - f\right)\right)\ (Uf''/\delta) \\
\nu u_{yy} &= \nu U f'''/\delta^2
\end{aligned}$$

$$uu_x + vu_y = \nu u_{yy}$$

$$\begin{aligned}
-U^2 f f'' \delta'/\delta &= \nu U f'''/\delta^2 \\
-\boxed{\frac{U\delta'\delta}{\nu}} f f'' &= f'''
\end{aligned}$$

In order for a similarity solution to exist, we require

$$\frac{U\delta'(x)\delta(x)}{\nu} = g(\eta) = C$$

(Since $U\delta'\delta$ is independent of y, $g(\eta)$ must be the constant function.)

Try $\quad \delta = ax^q \quad \Longrightarrow \quad \dfrac{Uaqx^{q-1}ax^q}{\nu} = C \quad \Longrightarrow \quad 2q-1 = 0 \quad \Longrightarrow \quad q = \dfrac{1}{2}$

$$\frac{U\frac{1}{2}a^2}{\nu} = C = \frac{1}{2} \quad \Longrightarrow \quad a^2 = \frac{\nu}{U} \quad \Longrightarrow \quad \delta(x) = \sqrt{a^2 x} = \sqrt{\frac{\nu x}{U}}$$

(Can take $C = \frac{1}{2}$ without loss of generality.)

Blasius equation:

$$f''' + \frac{1}{2} f f'' = 0$$

Boundary conditions:

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U}}} \qquad \text{so} \qquad \begin{cases} y = 0 & \leftrightarrow \quad \eta = 0 \\ y = \infty & \leftrightarrow \quad \eta = \infty \\ x = 0 & \leftrightarrow \quad \eta = \infty \end{cases}$$

$$\begin{aligned} 0 = u(x, y = 0) = U f'(0) \quad &\Longrightarrow \quad f'(0) = 0 \\ 0 = v(x, y = 0) = U\delta'(\eta f' - f) \quad &\Longrightarrow \quad f(0) = 0 \\ U = u(x, y = \infty) = U f'(\infty) \quad &\Longrightarrow \quad f'(\infty) = 1 \\ U = u(x = 0, y) = U f'(\infty) \quad &\Longrightarrow \quad f'(\infty) = 1 \end{aligned}$$

$$f''' = -\tfrac{1}{2} f f'' \qquad f(0) = 0 \qquad f'(0) = 0 \qquad f'(\infty) = 1$$

$$\begin{aligned}
f(0) &= 0 \\
f'(0) &= 0 \\
f''(0) &= A \qquad (A = 0.332 \text{ determined by } f'(\infty) = 1) \\
f'''(0) &= -\tfrac{1}{2} f(0) f''(0) = 0 \\
f^{(iv)}(0) &= -\tfrac{1}{2}(f'(0) f''(0) + f'(0) f'(0)) = 0 \\
f^{(v)}(0) &= -\tfrac{1}{2}(f(0) f^{(iv)}(0) + 2 f'(0) f'''(0) + f''(0) f''(0)) \\
&= -\tfrac{1}{2}(0 + 0 + A^2) = -\tfrac{1}{2} A^2 \\
f(\eta) &\simeq \tfrac{1}{2} A \eta^2 - \tfrac{1}{120}\tfrac{1}{2} A^2 \eta^5 + \ldots \\
f'(\eta) &\simeq A\eta - \tfrac{1}{48} A^2 \eta^4 + \ldots \\
u &= U f' \simeq U A \eta = U A y \sqrt{\tfrac{U}{\nu x}} \\
v &= \sqrt{\tfrac{U\nu}{x}} (\eta f' - f) \simeq \sqrt{\tfrac{U\nu}{x}}\, A \eta^2 = \sqrt{\tfrac{U\nu}{x}} A y^2 \tfrac{U}{\nu x} = \sqrt{\tfrac{U^3 \nu}{x^3}}\, A y^2
\end{aligned}$$

Boundary layer thickness:

$$u(\eta_{99}) = 0.99\,U \Longrightarrow \eta_{99} = 4.9 \Longrightarrow y_{99} = 4.9\sqrt{\frac{\nu x}{U}}$$

Wall shear stress:

x component of force per unit area imparted by flow to plate with normal y:

$$\tau = \rho\nu\frac{\partial u}{\partial y} = \rho\nu\, U A \left(\frac{U}{\nu x}\right)^{1/2} = \rho U^2 A \left(\frac{\nu}{U x}\right)^{1/2} = \frac{\rho U^2 A}{\sqrt{Re_x}}$$

Skin friction coefficient: $$C_f = \frac{\tau}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}$$

Drag force per unit length (in z) on one side of plate:

$$\begin{aligned} D &= \int_0^L \tau(x)\,dx = \int_0^L \rho\nu U A \left(\frac{U}{\nu x}\right)^{1/2} dx \\ &= 2\rho\nu U A \left(\frac{UL}{\nu}\right)^{1/2} = 2\rho U^2 L A \left(\frac{\nu}{LU}\right)^{1/2} = \frac{2A}{Re_L}\,\rho U^2 L = \frac{0.664}{Re_L}\,\rho U^2 L \end{aligned}$$

Drag coefficient: $$C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \frac{1.33}{\sqrt{Re_L}}$$

Falkner-Skan solutions

$$uu_x + vu_y = UU_x + \nu u_{yy} \qquad\qquad u_x + v_y = 0$$

Generalize to $U(x)$ non-constant

$$\begin{aligned}
u(x,y) &= U(x)F(\eta(x,y)) = U(x)f'(\eta) \\
u_x &= U(x)f''(\eta)\,\eta_x + U'(x)f'(\eta) = -U(x)f''(\eta)\,\eta\,\frac{\delta'(x)}{\delta(x)} + U'(x)f'(\eta) = -v_y \\
v &= \int dy\; v_y(\eta) = \int dy\;\left[U(x)f''(\eta)\,\eta\,\frac{\delta'(x)}{\delta(x)} - U'(x)f'(\eta)\right] \\
&= \int \frac{dy}{\delta(x)}\;[U(x)f''(\eta)\,\eta\,\delta'(x) - U'(x)\delta(x)f'(\eta)] \\
&= \int d\eta\;[U(x)f''(\eta)\,\eta\,\delta'(x) - U'(x)\delta(x)f'(\eta)] \\
&= U(x)\delta'(x)\left[f'(\eta)\,\eta - \int d\eta\; f'(\eta)\right] - U'\delta(x)\int d\eta\; f'(\eta) \\
&= U\delta'(\eta f' - f) - U'f = U\delta'\eta f' - (U\delta)'f \\
u_y &= U(x)f''(\eta)\,\eta_y = U(x)f''(\eta)/\delta(x) \\
u_{yy} &= U(x)f'''(\eta)\,\eta_y/\delta(x) = U(x)f'''(\eta)/\delta(x)^2
\end{aligned}$$

$$
\begin{aligned}
uu_x + vu_y &= UU_x + \nu u_{yy} \\
uu_x &= (Uf')\left(\cancel{-Uf''\eta\delta'/\delta} + U'f'\right) \\
vu_y &= (\cancel{U\delta'\eta f'} - (U\delta)'f)\ (Uf''/\delta) \\
\nu u_{yy} &= \nu U f'''/\delta^2
\end{aligned}
$$

$$
\begin{aligned}
(Uf')\ (U'f') - ((U\delta)'f)\ (Uf''/\delta) &= UU' + \nu U f'''/\delta^2 \\
U'\delta^2(f')^2 - (U\delta)'\delta f f'' &= U'\delta^2 + \nu f'''
\end{aligned}
$$

In order for a similarity solution to exist, we require:

$$
\frac{(U\delta)'\delta}{\nu} = \alpha \qquad \text{and} \qquad \frac{U'\delta^2}{\nu} = \beta
$$

Solutions exist for many forms of $U(x)$, satisfying:

$$
f''' + \beta(1 - (f')^2) + \alpha f f'' = 0
$$

$$\frac{(U\delta)'\delta}{\nu} = \alpha \qquad \text{and} \qquad \frac{U'\delta^2}{\nu} = \beta$$

$$U(x) = ax^n \implies U' = anx^{n-1} \implies \delta^2 = \frac{\nu\beta}{anx^{n-1}} \implies \delta = \left(\frac{\nu\beta}{an}\right)^{\frac{1}{2}} x^{\frac{1-n}{2}}$$

$$\implies U\delta = ax^n \left(\frac{\nu\beta}{an}\right)^{\frac{1}{2}} x^{\frac{1-n}{2}} = a\left(\frac{\nu\beta}{an}\right)^{\frac{1}{2}} x^{\frac{n+1}{2}}$$

$$\implies (U\delta)' = \frac{n+1}{2}a\left(\frac{\nu\beta}{an}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}}$$

$$\implies (U\delta)'\delta = \frac{n+1}{2}a\left(\frac{\nu\beta}{an}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}} \left(\frac{\nu\beta}{an}\right)^{\frac{1}{2}} x^{\frac{-(n-1)}{2}}$$

$$= \frac{n+1}{2}\,\frac{\nu\beta}{n} = \alpha\,\nu$$

$$f''' + n(1-(f')^2) + \frac{n+1}{2}ff'' = 0$$

$$f''' + \beta(1 - (f')^2) + \alpha f f'' = 0$$

$$\alpha = 0, \quad \beta = 1 \quad \Longrightarrow \quad \begin{cases} \delta(x)\, U(x) = C_1 \\ U'(x) = C_2 U^2(x) \end{cases} \Longrightarrow U(x) \sim \tfrac{1}{x}$$

$$f''' + 1 - (f')^2 = 0$$

Closed form solution! $\quad f'(\eta) = 3\left[\dfrac{1 - ce^{-\sqrt{2}\eta}}{1 + ce^{-\sqrt{2}\eta}}\right]^2 \qquad$ with $c = \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}} = 0.101\ldots$

Corresponds to sink at leading edge of plate $\Longrightarrow$

$$U(x) = \frac{m}{x} \qquad U\frac{dU}{dx} = \frac{m}{x}\left(\frac{-m}{x^2}\right) \qquad \delta = x\sqrt{\frac{\nu}{|m|}} \qquad \eta = \frac{y}{x}\sqrt{\frac{|m|}{\nu}}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{m^2}{x^3} + \nu\frac{\partial^2 u}{\partial y^2}$$

Ekman Layers

Disk at $\left\{ \begin{array}{c} z = -L \\ z = +L \end{array} \right\}$ rotates at angular velocity $\left\{ \begin{array}{c} \Omega^{\rm lower} \\ \Omega^{\rm upper} \end{array} \right\}$

Rapid rotation and inviscid flow $\Longrightarrow$ Taylor-Proudman $\Longrightarrow \partial_z \boldsymbol{u} = 0$

Incompatible with $\Omega^{\rm lower} \neq \Omega^{\rm upper} \Longrightarrow$ boundary layers

Assume steady flow and rapid rotation, with $\left| \dfrac{\Omega^{\rm lower} - \Omega^{\rm upper}}{\Omega^{\rm lower} + \Omega^{\rm upper}} \right| \ll 1$

Write equations in frame rotating at average $\Omega \equiv \dfrac{\Omega^{\rm lower} + \Omega^{\rm upper}}{2}$

$$\underbrace{\cancel{\partial_t \boldsymbol{u}}}_{\rm steady} + \underbrace{\cancel{(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}}}_{\boldsymbol{u} \ll R\Omega} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla} H + \nu \nabla^2 \boldsymbol{u}$$

$$H \equiv P/\rho + \Phi - R\Omega^2/2$$

In rotating frame, disks rotate at $\pm \hat{\boldsymbol{e}}_\theta \, r \, \Omega \, \epsilon \equiv \pm \hat{\boldsymbol{e}}_\theta \, r \, \dfrac{\Omega^{\rm upper} - \Omega^{\rm lower}}{2}$

Interior equations for (U_r, U_θ, U_z) (inviscid, axisymmetric):

$$
\begin{aligned}
-2\Omega U_\theta &= -\frac{\partial H}{\partial r} \\
2\Omega U_r &= -\frac{1}{r}\frac{\partial H}{\partial \theta} = 0
\end{aligned}
$$

(to use later)

$$
\begin{aligned}
0 &= -\frac{\partial H}{\partial z} \\
0 &= \frac{\partial U_z}{\partial z}
\end{aligned}
$$

Boundary layer equations for (u_r, u_θ, u_z):

$$
\begin{aligned}
-2\Omega u_\theta &= -\frac{\partial H}{\partial r} + \nu \frac{\partial^2 u_r}{\partial z^2} \\
2\Omega u_r &= -\frac{1}{r}\frac{\partial H}{\partial \theta} + \nu \frac{\partial^2 u_\theta}{\partial z^2}
\end{aligned}
$$

(to use later)

$$
\begin{aligned}
0 &= -\frac{\partial H}{\partial z} + \nu \frac{\partial^2 u_z}{\partial z^2} \\
0 &= \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}
\end{aligned}
$$

Combine:

$$
\begin{aligned}
-2\Omega u_\theta &= -2\Omega U_\theta + \nu \partial_z^2 u_r \\
2\Omega u_r &= 2\Omega U_r + \nu \partial_z^2 u_\theta
\end{aligned}
$$

Rewrite as:

$$
\begin{aligned}
-2\Omega(u_\theta - U_\theta) &= \nu \partial_z^2 (u_r - U_r) \\
2\Omega(u_r - U_r) &= \nu \partial_z^2 (u_\theta - U_\theta)
\end{aligned}
$$

$$
\begin{aligned}
-2\Omega(u_\theta - U_\theta) &= \nu\partial_z^2(u_r - U_r) \\
2\Omega(u_r - U_r) &= \nu\partial_z^2(u_\theta - U_\theta) \\
2\,i\,\Omega[(u_r - U_r) + i(u_\theta - U_\theta)] &= \nu\partial_z^2[(u_r - U_r) + i(u_\theta - U_\theta)] \\
2i\alpha^{-2} f &= \partial_z^2 f
\end{aligned}
$$

with $f \equiv (u_r - U_r) + i(u_\theta - U_\theta)$, $\alpha^{-2} \equiv \Omega/\nu$.

Use $(1+i)^2 = 1 + i^2 + 2i = 2i$.

$$
\begin{aligned}
f &= A(r)e^{(1+i)z/\alpha} + B(r)e^{-(1+i)z/\alpha} \\
&= A(r)e^{z/\alpha}(\cos(z/\alpha) + i\sin(z/\alpha)) + B(r)e^{-z\alpha}(\cos(z/\alpha) - i\sin(z/\alpha))
\end{aligned}
$$

Boundary layer of thickness $\alpha = \sqrt{\dfrac{\nu}{\Omega}}$.

Boundary conditions (in frame rotating with angular velocity Ω)

$$\boldsymbol{u}(r, z = \pm L) = \pm \hat{\boldsymbol{e}}_\theta \, r \, \Omega \, \epsilon$$

For boundary layer at bottom, use $Z \equiv (z + L)/\alpha$.

$$\begin{aligned}
(u_r - U_r)(r, Z) &= e^Z (A^R(r) \cos Z - A^I(r) \sin Z) + e^{-Z} (B^R(r) \cos Z + B^I(r) \sin Z) \\
(u_\theta - U_\Theta)(r, Z) &= e^Z (A^I(r) \cos Z + A^R(r) \sin Z) + e^{-Z} (B^I(r) \cos Z - B^R(r) \sin Z)
\end{aligned}$$

Apply matching and boundary conditions (recall $U_r = 0$)

$$\left\{ \begin{array}{c} (u_r - U_r)(r, \infty) = 0 \\ (u_\theta - U_\theta)(r, \infty) = 0 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} A^R(r) = 0 \\ A^I(r) = 0 \end{array} \right\}$$

$$\begin{aligned}
B^R(r) &= (u_r - U_r)(r, 0) = 0 \\
B^I(r) &= (u_\theta - U_\theta)(r, 0) = -r \, \Omega \, \epsilon - U_\theta(r)
\end{aligned}$$

$$\begin{aligned}
u_r(r, Z) &= (-r \, \Omega \epsilon - U_\theta(r)) \, e^{-Z} \sin Z \\
u_\theta(r, Z) &= U_\theta(r) + (-r \, \Omega \epsilon - U_\theta(r)) \, e^{-Z} \cos Z
\end{aligned}$$

Incompressibility in lower boundary layer at $z = -L \Longrightarrow$
(Recall $Z = (z+L)/\alpha$ and define inviscid vorticity $\omega_I \equiv \frac{1}{r}\partial_r(rU_\theta)$)

$$\begin{aligned}
\partial_z u_z &= -\frac{1}{r}\partial_r(ru_r) = -\frac{1}{r}\partial_r\left[r(-r\,\Omega\epsilon - U_\theta(r))\,e^{-Z}\ \sin Z)\right] \\
&= \left(\frac{1}{r}2r\,\Omega\epsilon + \frac{1}{r}\partial_r(rU_\theta)\right)\ e^{-Z}\ \sin Z = (2\Omega\epsilon + \omega_I)\,e^{-Z}\sin Z
\end{aligned}$$

$$\begin{aligned}
u_z(r,Z) - u_z(r,0) &= (2\Omega\epsilon + \omega_I)\int_0^Z dz\, e^{-Z'}\sin Z' \\
&= (2\Omega\epsilon + \omega_I)\int_0^Z \alpha\, dZ'\, e^{-Z'}\sin Z' \\
&= (2\Omega\epsilon + \omega_I)\,\frac{\alpha}{2}\,[-e^{-Z}(\cos Z' + \sin Z')]_0^Z \\
&= (2\Omega\epsilon + \omega_I)\,\frac{\alpha}{2}\,[1 - e^{-Z}(\cos Z + \sin Z)]
\end{aligned}$$

$$\begin{aligned}
[e^{-Z}(c\cos Z + d\sin Z))]' &= e^{-Z}(-c\sin Z + d\cos Z - c\cos Z - d\sin Z) \\
&= e^{-Z}(-(c+d)\sin Z + (d-c)\cos Z) \Longrightarrow c = d = -\tfrac{1}{2}
\end{aligned}$$

$$U_z = u_z(r, Z=\infty) = (2\Omega\epsilon{+}\omega_I)\frac{\alpha}{2}[1{-}e^{-Z}(\cos Z{+}\sin Z)](Z=\infty) = (2\Omega\epsilon{+}\omega_I)\frac{\alpha}{2}$$

For boundary layer at top, use $Z \equiv (z - L)/\alpha$. Relative to frame rotating with average angular velocity Ω, we have

$$\begin{aligned} (u_r - U_r)(r, Z) &= e^Z (A^R(r)\cos Z - A^I(r)\sin Z) + e^{-Z}(B^R(r)\cos Z + B^I(r)\sin Z) \\ (u_\theta - U_\theta)(r, Z) &= e^Z (A^I(r)\cos Z + A^R(r)\sin Z) + e^{-Z}(B^I(r)\cos Z - B^R(r)\sin Z) \end{aligned}$$

Apply matching and boundary conditions (recall $U_r = 0$)

$$\left\{ \begin{array}{c} (u_r - U_r)(r, -\infty) = 0 \\ (u_\theta - U_\theta)(r, -\infty) = 0 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} B^R(r) = 0 \\ B^I(r) = 0 \end{array} \right\}$$

$$\begin{aligned} A^R(r) &= (u_r - U_r)(r, 0) = -U_r(r) = 0 \\ A^I(r) &= (u_\theta - U_\theta)(r, 0) = r\,\Omega\,\epsilon - U_\theta(r) \end{aligned}$$

$$\begin{aligned} u_r(r, Z) &= (r\,\Omega\epsilon - U_\theta(r))\, e^Z \sin Z \\ u_\theta(r, Z) &= U_\theta(r) + (r\,\Omega\epsilon - U_\theta(r))\, e^Z \cos Z \end{aligned}$$

Incompressibility in upper boundary layer at $z = +L \Longrightarrow$

(Recall $Z = (z - L)/\alpha$ and define inviscid vorticity $\omega_I \equiv \frac{1}{r}\partial_r(rU_\theta)$)

$$\begin{aligned} \partial_z u_z &= -\frac{1}{r}\partial_r(ru_r) = -\frac{1}{r}\partial_r\left[r\left(r\,\Omega\epsilon - U_\theta(r)\right)\, e^Z\, \sin Z\right] \\ &= \left(-\frac{1}{r}2r\Omega\epsilon + \frac{1}{r}\partial_r(rU_\theta)\right)\, e^Z\, \sin Z = (-2\Omega\epsilon + \omega_I)\, e^Z\, \sin Z \end{aligned}$$

$$\begin{aligned} u_z(r,Z) - u_z(r,0) &= (-2\Omega\epsilon + \omega_I)\int_0^Z dz\, e^{Z'} \sin Z' \\ &= (-2\Omega\epsilon + \omega_I)\,\alpha\int_0^Z dZ'\, e^{Z'} \sin Z' \\ &= (-2\Omega\epsilon + \omega_I)\,\frac{\alpha}{2}\left[e^Z(\cos Z' - \sin Z')\right]_0^Z \\ &= (2\Omega\epsilon - \omega_I)\,\frac{\alpha}{2}\left[1 - e^Z(\sin Z - \cos Z)\right] \end{aligned}$$

$$\begin{aligned} \left[e^Z(c\cos Z + d\sin Z))\right]' &= e^Z(-c\sin Z + d\cos Z + c\cos Z + d\sin Z) \\ &= e^Z((-c+d)\sin Z + (c+d)\cos Z) \Longrightarrow -c = d = \tfrac{1}{2} \end{aligned}$$

$$U_z = u_z(r, Z = -\infty) = (2\Omega\epsilon - \omega_I)\frac{\alpha}{2}[1 - e^Z(\sin Z - \cos Z)]_{Z=-\infty} = (2\Omega\epsilon - \omega_I)\frac{\alpha}{2}$$

At bottom of inviscid interior,

$$U_z = u_z^{\text{upper}}(r, Z = -\infty) = (2\Omega\epsilon - \omega_I)\frac{\alpha}{2}$$

At top of inviscid interior,

$$U_z = u_z^{\text{lower}}(r, Z = \infty) = (2\Omega\epsilon + \omega_I)\frac{\alpha}{2}$$

Since U_z is independent of z,

$$2\Omega\epsilon - \omega_I = 2\Omega\epsilon + \omega_I \implies \omega_I = 0$$

Measured in non-rotating frame of reference, inviscid vorticity is therefore

$$\frac{1}{r}\partial_r(r(U_\theta + r\Omega)) = \omega_I + 2\Omega = 2\Omega = \Omega^{\text{upper}} + \Omega^{\text{lower}}$$

In the rotating frame of reference,

$$\frac{1}{r}\partial_r(rU_\theta) = 0 \implies \partial_r(rU_\theta) = 0 \implies rU_\theta = c \implies U_\theta = \frac{c}{r} = 0$$

since origin $r = 0$ is in the domain.

Inviscid interior solution:

$$
\begin{aligned}
U_r &= 0 \\
U_\theta &= 0 \\
U_z &= \Omega\epsilon\,\alpha = \frac{\Omega^{\text{upper}} - \Omega^{\text{lower}}}{2}\sqrt{\frac{2\nu}{\Omega^{\text{upper}} + \Omega^{\text{lower}}}}
\end{aligned}
$$

Lower boundary layer solution:

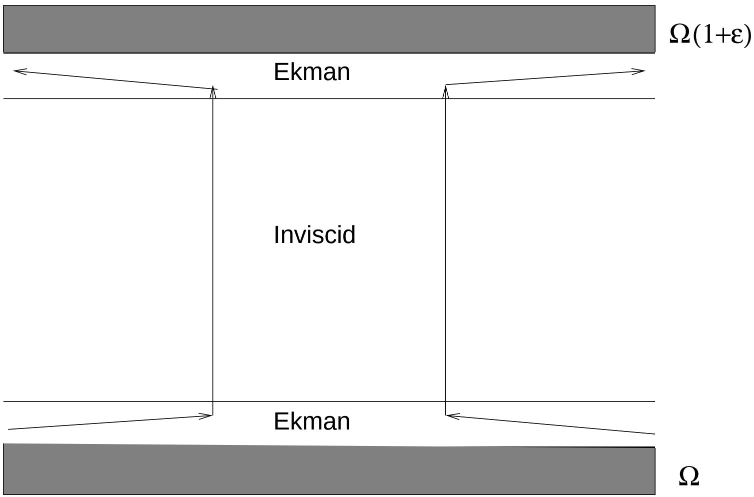
$$
\begin{aligned}
u_r(r,Z) &= (-r\,\Omega\epsilon - U_\theta(r))\,e^{-Z}\sin Z = -r\,\Omega\epsilon\,e^{-Z}\sin Z \\
u_\theta(r,Z) &= U_\theta(r) + (-r\,\Omega\epsilon - U_\theta(r))\,e^{-Z}\cos Z = -r\,\Omega\epsilon\,e^{-Z}\cos Z
\end{aligned}
$$

$$
\begin{aligned}
u_r(r,z) &= -r\,\Omega\epsilon\,e^{-(z+L)/\alpha}\sin((z+L)/\alpha) \\
u_\theta(r,z) &= -r\,\Omega\epsilon\,e^{-(z+L)/\alpha}\cos((z+L)/\alpha)
\end{aligned}
$$

Upper boundary layer solution:

$$
\begin{aligned}
u_r(r,Z) &= (r\,\Omega\epsilon - U_\theta(r))\,e^{Z}\sin Z = r\,\Omega\epsilon\,e^{Z}\sin Z \\
u_\theta(r,Z) &= U_\theta(r) + (r\,\Omega\epsilon - U_\theta(r))\,e^{Z}\cos Z = r\,\Omega\epsilon\,e^{Z}\cos Z
\end{aligned}
$$

$$
\begin{aligned}
u_r(r,z) &= r\,\Omega\epsilon\,e^{(z-L)/\alpha}\sin((z-L)/\alpha) \\
u_\theta(r,z) &= r\,\Omega\epsilon\,e^{(z-L)/\alpha}\cos((z-L)/\alpha)
\end{aligned}
$$



Secondary flow induced by Ekman layers.

Flow is nearly horizontal in boundary layers, nearly vertical in inviscid interior.

Flow moves toward the center at the bottom, where its rotation is slowed,

and toward the exterior at the top, where the rotation is increased.