Hydrodynamics

Class 6

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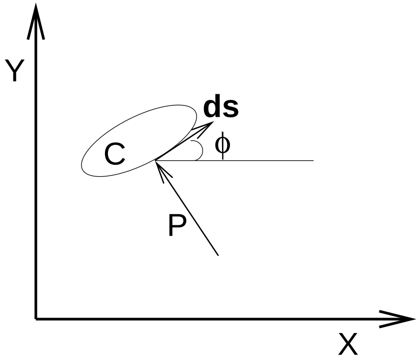
Blasius Theorem

$$\text{Drag} - i\,\text{Lift} = F_x - iF_y = \frac{i\rho}{2} \oint \left(\frac{dw}{dz}\right)^2 dz$$

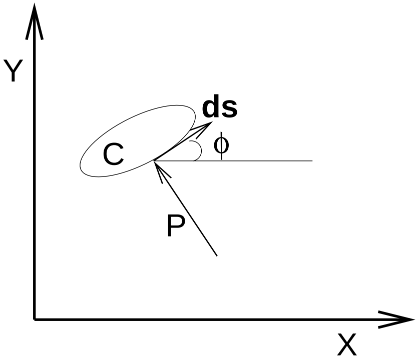
Complex integral useful because:
- integral determined by residue theorem!
- can move contour to more convenient one (e.g., circle)!



2D irrotational incompressible flow around body with boundary C

On small boundary segment ds,

$$\begin{aligned}
dz = dx + idy &= (\cos\phi + i\sin\phi)\,ds = e^{i\phi}\,ds \\
\frac{dw}{dz} = u - iv &= |\boldsymbol{u}|\,(\cos\phi - i\sin\phi) = |\boldsymbol{u}|\,e^{-i\phi} \\
\left(\frac{dw}{dz}\right)^2 &= |\boldsymbol{u}|^2\,e^{-2i\phi} \\
\left(\frac{dw}{dz}\right)^2 dz &= |\boldsymbol{u}|^2\,e^{-2i\phi}e^{i\phi}\,ds = |\boldsymbol{u}|^2\,e^{-i\phi}ds
\end{aligned}$$



2D irrotational incompressible flow around body with boundary C

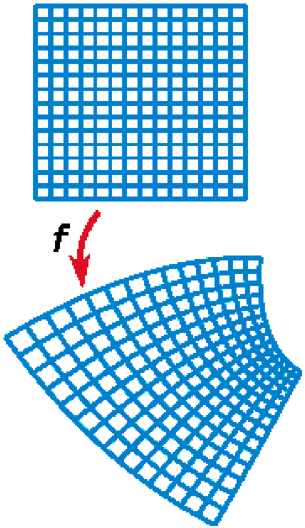
$$
\begin{aligned}
dF_x - idF_y &= p[-\cos(\pi/2-\phi) - i\sin(\pi/2-\phi)]\,ds \\
&= p\,[-\cancel{\cos(\pi/2)}\cos\phi - \sin(\pi/2)\sin\phi \\
&\quad -i\,(\sin(\pi/2)\cos\phi - \cancel{\cos(\pi/2)}\sin\phi)]\;ds \\
&= p(-\sin\phi - i\cos\phi)\,ds \\
&= -ip(-i\sin\phi + \cos\phi)\,ds \\
&= -ipe^{-i\phi}\,ds \\
&= -i(p_\infty + \frac{\rho}{2}(U_\infty^2 - |\boldsymbol{u}|^2))e^{-i\phi}\,ds \\
&= Ke^{-i\phi}\,ds + \frac{i\rho}{2}|\boldsymbol{u}|^2 e^{-i\phi}\,ds \\
&= Ke^{-i\phi}\,ds + \frac{i\rho}{2}\left(\frac{dw}{dz}\right)^2 dz \\
F_x - iF_y &= \cancel{K\oint e^{-i\phi}\,ds} + \frac{i\rho}{2}\oint\left(\frac{dw}{dz}\right)^2 dz
\end{aligned}
$$

Body around $z = 0$. Contour very far away. No singularities outside body.
Flow around body is superposition of uniform flow, sources and sinks, vortex, doublet, $\ldots$, approximately centered at 0.
Sources and sinks cancel since contour of body is closed streamline.

$$
\begin{aligned}
w(z) &= Uz + \cancel{\frac{m}{2\pi}\log z} + \frac{i\Gamma}{2\pi}\log z + \frac{\mu}{z} + \ldots \\
\frac{dw}{dz} &= U + \frac{i\Gamma}{2\pi z} - \frac{\mu}{z^2} + \ldots \\
\left(\frac{dw}{dz}\right)^2 &= U^2 + \frac{2Ui\Gamma}{2\pi z} - \left(\frac{\Gamma}{2\pi z}\right)^2 - \frac{2U\mu}{z^2} + \ldots \\
F_x - iF_y &= \frac{i\rho}{2}\oint\left(\frac{dw}{dz}\right)^2 dz \\
&= \frac{i\rho}{2}\oint\left[U^2 + \frac{2Ui\Gamma}{2\pi z} - \left(\frac{\Gamma}{2\pi z}\right)^2 - \frac{2U\mu}{z^2} + \ldots\right] dz \\
&= \frac{i\rho}{2}\,2\pi i\,\frac{2Ui\Gamma}{2\pi} \\
&= -i\rho U\Gamma \\
\text{Lift} = F_y &= \rho U\Gamma
\end{aligned}
$$

Kutta-Joukowski Lift Theorem

Conformal Mapping



Transform $\zeta = (\xi, \eta) \to z = (x, y)$
(simple geometry) $\to$ (actual geometry)

Define $\tilde{\phi}(\xi, \eta) = \phi(x(\xi, \eta), y(\xi, \eta))$

$$(\partial_x^2 + \partial_y^2)\phi(x, y) = 0 \Longrightarrow (\partial_\xi^2 + \partial_\eta^2)\tilde{\phi}(\xi, \eta) = 0$$

$$(\partial_x^2 + \partial_y^2)\psi(x, y) = 0 \Longrightarrow (\partial_\xi^2 + \partial_\eta^2)\tilde{\psi}(\xi, \eta) = 0$$

$w(z) = \phi + i\psi$ complex potential in z-plane $\Longrightarrow$
$\tilde{w}(\zeta) = \tilde{\phi} + i\tilde{\psi}$ complex potential in ζ-plane (drop tildes)

Relation between velocity in z-plane and velocity in ζ-plane:

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz}$$

Strength of sources and vortices is conserved

$$m = \oint \boldsymbol{u} \cdot \boldsymbol{n} dl = \oint (u\,dy - v\,dx)$$

$$\Gamma = \oint \boldsymbol{u} \cdot d\boldsymbol{\ell} = \oint (u\,dx + v\,dy)$$

$$\begin{aligned}
\Gamma_z + i m_z &= \oint (u\,dx + v\,dy) + i \oint (u\,dy - v\,dx) \\
&= \oint (u - iv)(dx + i\,dy) \\
&= \oint \frac{dw}{dz}\,dz = \oint \frac{dw}{d\zeta}\frac{d\zeta}{dz}\,dz = \oint \frac{dw}{d\zeta}\,d\zeta = \Gamma_\zeta + i m_\zeta
\end{aligned}$$

Joukowski transformation

$$z = \zeta + \frac{R^2}{\zeta}$$

$$\frac{dz}{d\zeta} = 1 - \frac{R^2}{\zeta^2} = \begin{cases} 0 & \text{if} \quad \zeta = \pm c \\ \infty & \text{if} \quad \zeta = 0 \end{cases}$$

Both $|\zeta| < R$ and $|\zeta| > R$ are mapped into entire plane.

We are interested in $|\zeta| \geq R$ (outside cylinder)

For large $|\zeta|$, we have $z \rightarrow \zeta$ (mapping approaches identity)

$$\zeta = \frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2}$$

(Choose $+\sqrt{\ }$ so that $z \rightarrow \zeta$ as $|z| \rightarrow \infty$)

Circle $\zeta = ae^{i\phi}$, $a > R$:

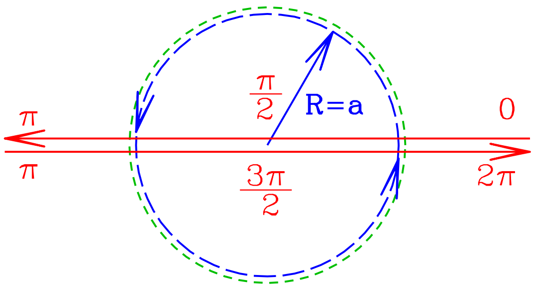
$$z = \zeta + \frac{R^2}{\zeta} = ae^{i\phi} + \frac{R^2}{a}e^{-i\phi} = \left(a + \frac{R^2}{a}\right)\cos\phi + i\left(a - \frac{R^2}{a}\right)\sin\phi = x + i\,y$$

Circle of radius a in ζ-plane $\Longrightarrow$ ellipse with major and minor semi-axes $a \pm R^2/a$ in z-plane:

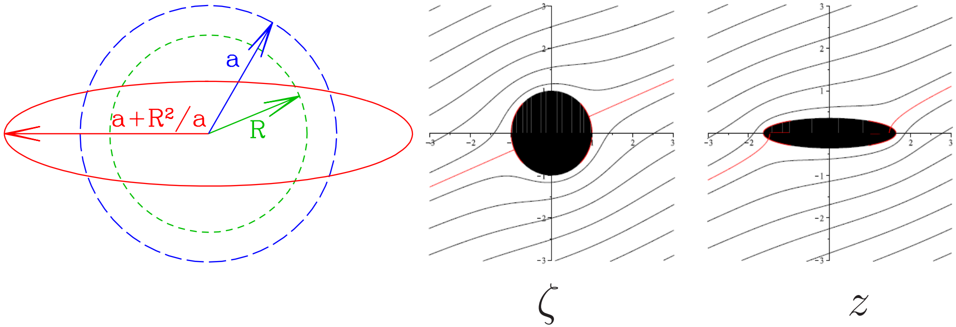
$$\frac{x^2}{(a + R^2/a)^2} + \frac{y^2}{(a - R^2/a)^2} = 1$$

Circle $\zeta = Re^{i\phi}$:

$$z = \zeta + \frac{R^2}{\zeta} = Re^{i\phi} + \frac{R^2}{R}e^{-i\phi} = R(e^{i\phi} + e^{-i\phi}) = 2R\cos\phi$$



Circle of radius R in ζ-plane $\Longrightarrow$ line segment $[-2R, 2R]$ in z-plane



Flow around circular cylinder of radius $a \Longrightarrow$
flow around an elliptical cylinder with semi=axes $a \pm R^2/a$

Uniform flow at angle α : $f(\zeta) = U\zeta e^{-i\alpha}$

Using $\quad w(\zeta) = f(\zeta) + \overline{f(a^2/\bar{\zeta})} \quad$ leads to $\quad w(\zeta) = U\left(\zeta e^{-i\alpha} + \dfrac{a^2}{\zeta e^{-i\alpha}}\right)$

Invert using $\zeta = \frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2}$:

$$w(z) = U\left[ze^{-i\alpha} + \left(\frac{a^2}{R^2}e^{i\alpha} - e^{-i\alpha}\right)\left(\frac{z}{2} - \sqrt{\frac{z^2}{4} - R^2}\right)\right]$$

$$
\begin{aligned}
\frac{w}{U} &= \zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}} \qquad \text{using} \qquad \zeta = \frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2} \\
&= \left[\frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2}\right] e^{-i\alpha} + \frac{a^2}{\left[\frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2}\right] e^{-i\alpha}} \\
&= \left[\frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2}\right] e^{-i\alpha} + \frac{a^2 \left[\frac{z}{2} - \sqrt{\frac{z^2}{4} - R^2}\right]}{\left[\frac{z^2}{4} - \left(\frac{z^2}{4} - R^2\right)\right] e^{-i\alpha}} \\
&= z e^{-i\alpha} + \left[-\frac{z}{2} + \sqrt{\frac{z^2}{4} - R^2}\right] e^{-i\alpha} + \frac{a^2}{R^2} \left[\frac{z}{2} - \sqrt{\frac{z^2}{4} - R^2}\right] e^{i\alpha} \\
&= z e^{-i\alpha} + \left(\frac{a^2}{R^2} e^{i\alpha} - e^{-i\alpha}\right) \left[\frac{z}{2} - \sqrt{\frac{z^2}{4} - R^2}\right]
\end{aligned}
$$

Circulation and stagnation points

Add circulation: $w(\zeta) = U\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta e^{-i\alpha}}\right) + \frac{i\Gamma}{2\pi}\log\zeta$

($\log(\zeta e^{-i\alpha}) = \log\zeta - \cancel{i\alpha}$, since constant is unimportant)

$$\begin{aligned}
\frac{dw}{d\zeta} &= U\left(e^{-i\alpha} - \frac{a^2}{\zeta^2 e^{-i\alpha}}\right) + \frac{i\Gamma}{2\pi\zeta} \\
&= U\left(e^{-i\alpha} - \frac{a^2}{r^2 e^{2i\theta} e^{-i\alpha}}\right) + \frac{i\Gamma}{2\pi r e^{i\theta}} \\
(u_r - iu_\theta)e^{-i\theta} &= \left(U\left(e^{i(\theta-\alpha)} - \frac{a^2}{r^2}e^{-i(\theta-\alpha)}\right) + \frac{i\Gamma}{2\pi r}\right)e^{-i\theta}
\end{aligned}$$

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos(\theta - \alpha) \qquad -u_\theta = U\left(1 + \frac{a^2}{r^2}\right)\sin(\theta - \alpha) + \frac{\Gamma}{2\pi r}$$

At $r = a$, $\quad u_r = 0 \qquad u_\theta = -2U\sin(\theta - \alpha) - \frac{\Gamma}{2\pi a}$

$$\Longrightarrow \sin(\theta_{\text{stag}} - \alpha) = -\frac{\Gamma}{4\pi U a}$$

Kutta condition

Recall

$$\begin{aligned} \frac{dw}{dz} &= \frac{dw}{d\zeta}\frac{d\zeta}{dz} \\ \frac{dz}{d\zeta} &= 1 - \frac{R^2}{\zeta^2} = 0 \Longrightarrow \frac{d\zeta}{dz} = \infty \text{ at } \zeta = \pm R \end{aligned}$$

To prevent $\dfrac{dw}{dz} = \infty$ at $z = \pm 2R$, must have $\dfrac{dw}{d\zeta} = 0$ at $\zeta = \pm R$

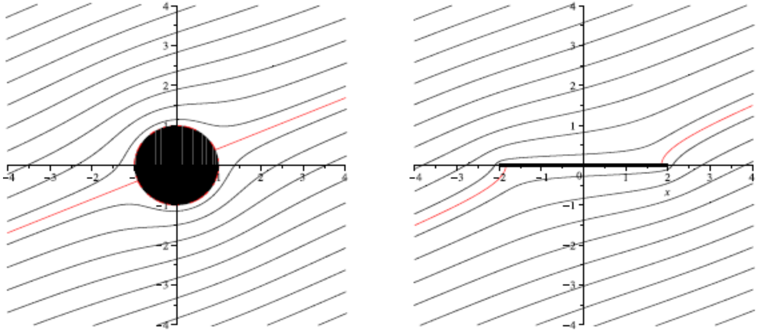
If ζ domain is $|\zeta| > a > R$, then singular point is not in domain.

If $a = R$ (flat plate in z domain), must set circulation Γ such that stagnation point θ_{stag} is at sharp trailing edge $\theta = 0$

$$\frac{\Gamma}{4\pi U a} = \sin(\alpha - \theta_{\mathrm{stag}}) = \sin\alpha$$

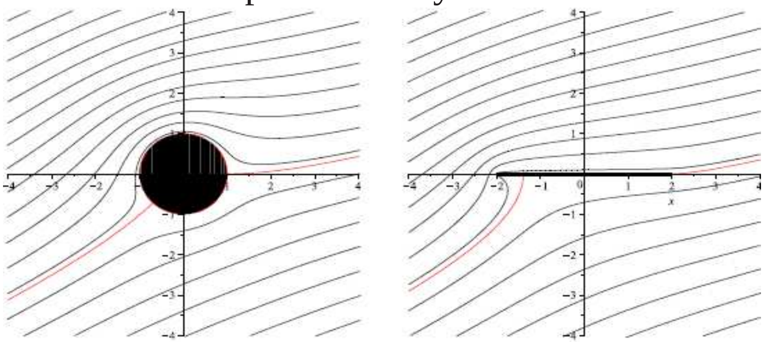
What about singularity at sharp leading edge $\theta = \pi$?

No circulation



ζ z

Circulation prescribed by Kutta condition



ζ z

Lift Coefficient

Dimensionless lift coefficient is:

$$C_L = \frac{\text{Lift}}{\frac{1}{2}\rho U^2 \ell}$$

where ℓ is the length or chord of the airfoil

Blasius Theorem: lift $= \rho U \Gamma$, leading to

$$C_L \equiv \frac{\text{Lift}}{\frac{1}{2}\rho U^2 \ell} = \frac{\rho U \Gamma}{\frac{1}{2}\rho U^2 \ell} = \frac{2\Gamma}{U\ell}$$

Flat plate: flow with circulation taken to satisfy Kutta condition:

$$\Gamma = 4\pi U a \sin\alpha$$

Flat plate is line segment $[-2a, 2a]$ with length $\ell = 4a$

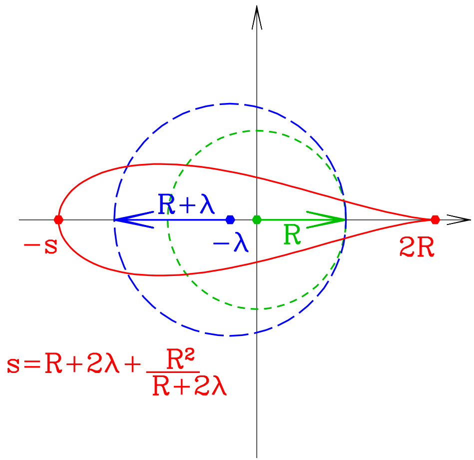
$$C_L = \frac{2\Gamma}{U\ell} = \frac{2(4\pi U a \sin\alpha)}{U(4a)} = 2\pi \sin\alpha$$

Symmetric Joukowski airfoil

Use Joukowski transformation

$$z = \zeta + \frac{R^2}{\zeta}$$

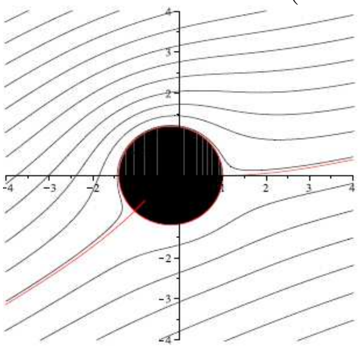
on a *shifted circle*

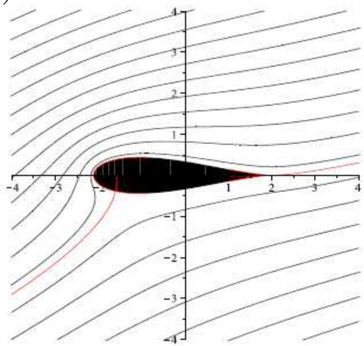


$$\zeta = -\lambda + (R+\lambda)e^{i\phi}$$

Here is the airfoil shape:

$$z = -\lambda + (R+\lambda)e^{i\phi} + \frac{R^2}{-\lambda + (R+\lambda)e^{i\phi}}$$





Flow around symmetric Joukowski airfoil

$$w(\zeta) = U\left((\zeta+\lambda)e^{-i\alpha} + \frac{(R+\lambda)^2}{\zeta+\lambda}e^{i\alpha}\right) + \frac{i\Gamma}{2\pi}\log(\zeta+\lambda)$$

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz} = \left(U\left(e^{-i\alpha} - \left(\frac{R+\lambda}{\zeta+\lambda}\right)^2 e^{i\alpha}\right) + \frac{i\Gamma}{2\pi(\zeta+\lambda)}\right)\frac{1}{1-R^2/\zeta^2}$$

Singularities at $\zeta = \pm R$. But $\zeta = -R$ is inside wing, not flow region, so OK!

Choose Γ to place stagnation point at $\zeta = R$, eliminating remaining singularity

At $\zeta = +R$,

$$\frac{dw}{d\zeta} = -2Ui\sin\alpha + \frac{i\Gamma}{2\pi(R+\lambda)}$$

$$\frac{\Gamma}{4\pi U(R+\lambda)} = \sin\alpha$$

According to Blasius Theorem, lift is then

$$\rho U\Gamma = 4\pi\rho U^2(R+\lambda)\sin\alpha$$

No lift for $\alpha = 0$! Small lift for α small!

Cambered Joukowski Airfoil

Use Joukowski transformation

$$z = \zeta + \frac{R^2}{\zeta}$$

on a *circle shifted left and up*

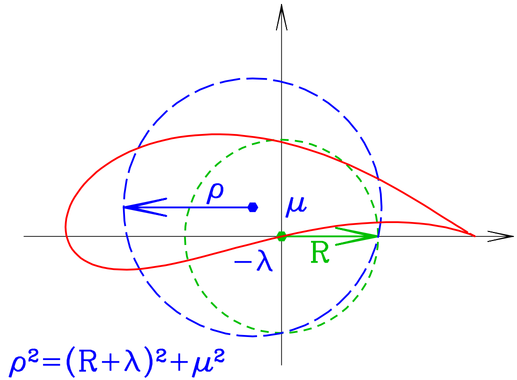
$$\zeta = -\lambda + i\,\mu + \sqrt{(R+\lambda)^2 + \mu^2}\; e^{i\phi}$$

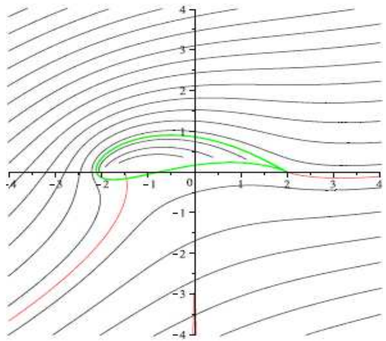
Kutta condition $\Longrightarrow$

$$\Gamma = 4\pi\sqrt{(R+\lambda)^2 + \mu^2}\;\sin(\alpha + \beta)$$

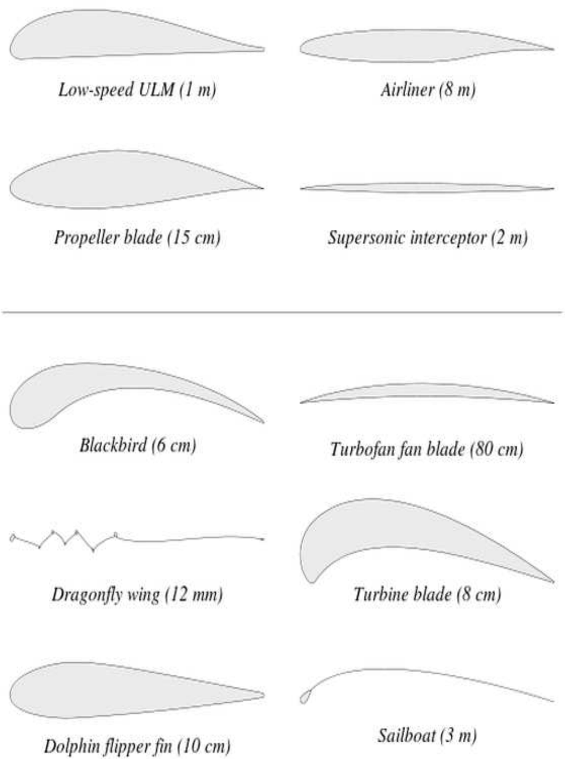
where $\qquad \beta = \arctan\left(\frac{\mu}{R+\lambda}\right)$

Hence there is lift even for α small.

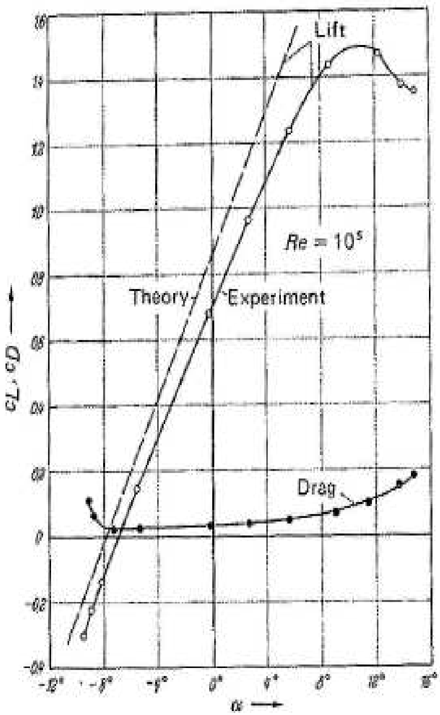


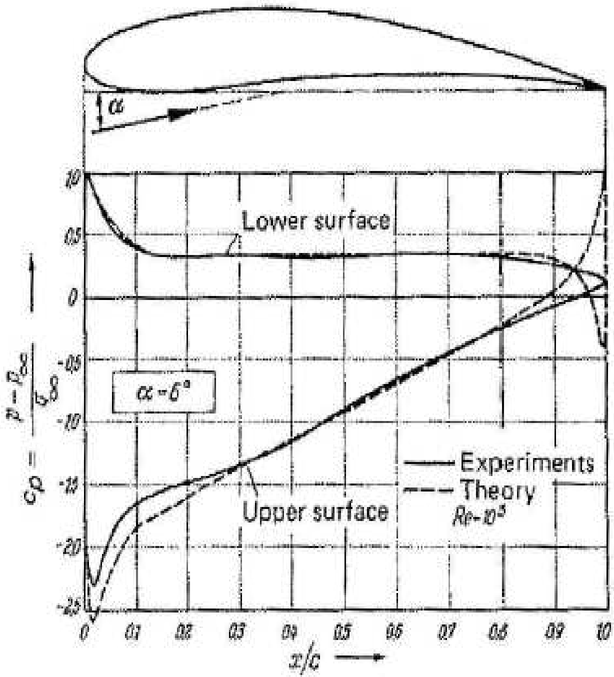


Examples of airfoils



Comparison of lift, drag, and pressure with experiment





Separation