Hydrodynamics

Class 5

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Steady Irrotational Flow in Two Dimensions

incompressible flow $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \Longrightarrow \boldsymbol{u} = \boldsymbol{\nabla}\times\psi\hat{\boldsymbol{e}}_{\boldsymbol{z}} = \hat{\boldsymbol{e}}_{\boldsymbol{x}}\psi_y - \hat{\boldsymbol{e}}_{\boldsymbol{y}}\psi_x$

irrotational flow $\boldsymbol{\nabla}\times\boldsymbol{u} = 0 \Longrightarrow \boldsymbol{u} = \boldsymbol{\nabla}\phi = \hat{\boldsymbol{e}}_{\boldsymbol{x}}\phi_x + \hat{\boldsymbol{e}}_{\boldsymbol{y}}\phi_y$

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi = \phi_x\psi_x + \phi_y\psi_y = u(-v) + v(u) = 0$$

Curves of constant ϕ are perpendicular to curves of constant ψ

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

Cauchy-Riemann equations. Can define an analytic complex function:

$$w(z) = \phi(x,y) + i\psi(x,y)$$

Can define a derivative dw/dz which is independent of direction:

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv\\ \frac{\partial w}{\partial(iy)} &= \frac{\partial\phi}{\partial(iy)} + i\frac{\partial\psi}{\partial(iy)} = \frac{1}{i}v + i\frac{1}{i}u = u - iv\end{aligned}$$

Some properties

$$|\boldsymbol{u}|^2 = u^2 + v^2 = (u - iv)(u + iv) = \left|\frac{dw}{dz}\right| \left|\frac{d\bar{w}}{dz}\right|$$

$$\begin{aligned} u + iv &= (u_r + iu_\theta)(\cos\theta + i\sin\theta) \\ &= u_r\cos\theta - u_\theta\sin\theta + i(u_r\sin\theta + u_\theta\cos\theta) \end{aligned}$$

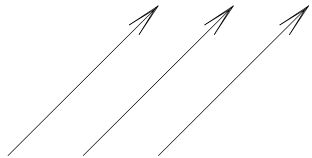
$$\begin{aligned} u - iv &= u_r\cos\theta - u_\theta\sin\theta - i(u_r\sin\theta + u_\theta\cos\theta) \\ &= u_r(\cos\theta - i\sin\theta) + u_\theta(-\sin\theta - i\cos\theta) \\ &= u_r(\cos\theta - i\sin\theta) - iu_\theta(\cos\theta - i\sin\theta) \\ &= (u_r - iu_\theta)e^{-i\theta} \end{aligned}$$

Basic flows:

Uniform flows

recall $\dfrac{dw}{dz} = u - iv$

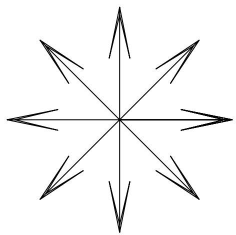
$w(z)$	$w'(z)$	u	v
Uz	U	U	0
iUz	iU	0	$-U$
$Uze^{-i\alpha}$	$Ue^{-i\alpha}$	$U\cos\alpha$	$U\sin\alpha$

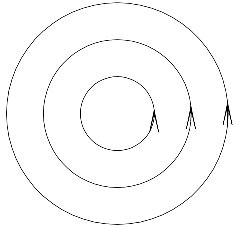


Sources, sinks, and vortices

recall $\dfrac{dw}{dz} = (u_r - iu_\theta)e^{-i\theta}$

$w(z)$	$w'(z)$	u_r	u_θ	
$c\log z$	$\dfrac{c}{z} = \dfrac{c}{re^{i\theta}}$	$\dfrac{c}{r}$	0	source or sink
$ic\log z$	$\dfrac{ic}{z} = \dfrac{ic}{re^{i\theta}}$	0	$-\dfrac{c}{r}$	point vortex





Recall $\nabla \times (c/r)\hat{\boldsymbol{e}}_\theta = 0$ for $r > 0$

Flow in a sector

$$
\begin{aligned}
w(z) &= Uz^n = Ur^n e^{in\theta} \\
&= Ur^n \cos n\theta + iUr^n \sin n\theta \\
&= \phi + i\psi
\end{aligned}
$$

Flow is along curves on which ψ is constant

$$\psi_0 = Ur^n \sin n\theta$$

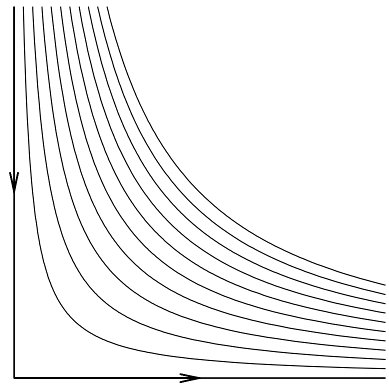
$$\text{e.g.} \quad \psi = 0 \quad \text{on} \quad \theta = \pi/n$$

$n = 1 \Longrightarrow$ uniform flow

$n = 2 \Longrightarrow$ flow in a 90° corner

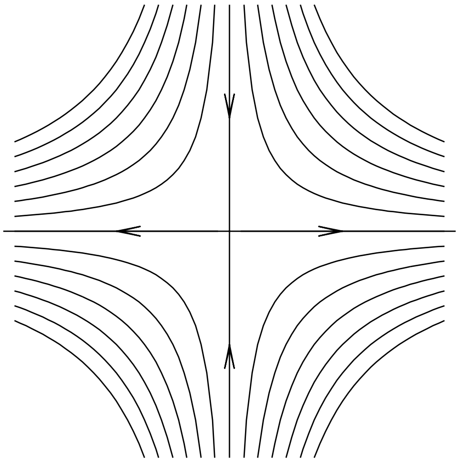
recall $\quad \dfrac{dw}{dz} = (u_r - iu_\theta)e^{-i\theta}$

$$
\begin{aligned}
\frac{dw}{dz} &= nz^{n-1} = nUr^{n-1}e^{in\theta}e^{-i\theta} \\
&= (nUr^{n-1}\cos n\theta + nUr^{n-1}\sin n\theta)e^{-i\theta} \\
&= (u_r - iu_\theta)e^{-i\theta}
\end{aligned}
$$



Stagnation Point

$$
\begin{aligned}
xy &= \psi \\
z^2 &= x^2 - y^2 + 2ixy = \phi + i\psi = w(z) \\
\frac{dw}{dz} &= u - iv = 2z = 2x + 2iy \\
& u = 2x \quad \text{and} \quad v = -2y
\end{aligned}
$$



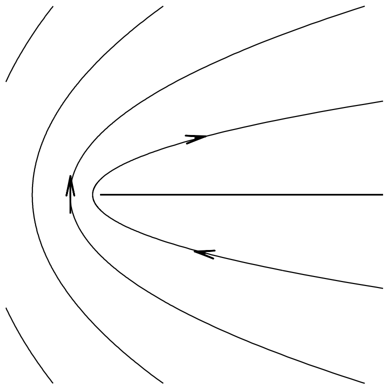
Flow around an edge

$$
\begin{aligned}
w(z) &= U z^{1/2} = U r^{1/2} e^{i\theta/2} \\
&= U r^{1/2} \cos\theta/2 + iU r^{1/2} \sin\theta/2 \\
&= \phi + i\psi
\end{aligned}
$$

Flow is along curves on which ψ is constant

$$\psi_0 = U r^{1/2} \sin\theta/2$$

$$\text{e.g.} \quad \psi = 0 \quad \text{on} \quad \theta = 0, 2\pi$$



Doublet

Coalescence of a source and a sink:

$$
\begin{aligned}
w(z) &= c\log(z+\epsilon) - c\log(z-\epsilon) \\
&= c\log\left(\frac{z+\epsilon}{z-\epsilon}\right) \\
&= c\log\left(\frac{1+\epsilon/z}{1-\epsilon/z}\right) \\
&= c\log\left(\left(1+\frac{\epsilon}{z}\right)\left(1+\frac{\epsilon}{z}+\left(\frac{\epsilon}{z}\right)^2+\dots\right)\right) \\
&= c\log\left(1+\frac{2\epsilon}{z}+2\left(\frac{\epsilon}{z}\right)^2+\dots\right) \\
&= c\frac{2\epsilon}{z}+\dots
\end{aligned}
$$

Take $c \to \infty$ and $\epsilon \to 0$ such that $2c\epsilon \to \mu$ $\implies$ $\boxed{w(z) = \frac{\mu}{z}}$

$$\frac{1}{1-\epsilon} = 1+\epsilon+\epsilon^2+\dots \qquad \log(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \dots$$

Doublet (continued)

$$w(z) = \frac{\mu}{z} = \frac{\mu \bar{z}}{z \bar{z}}$$

$$\phi + i\psi = \frac{\mu(x - iy)}{x^2 + y^2}$$

$$\phi = \frac{\mu x}{x^2 + y^2} \qquad \psi = \frac{-\mu y}{x^2 + y^2}$$

Streamlines satisfy:

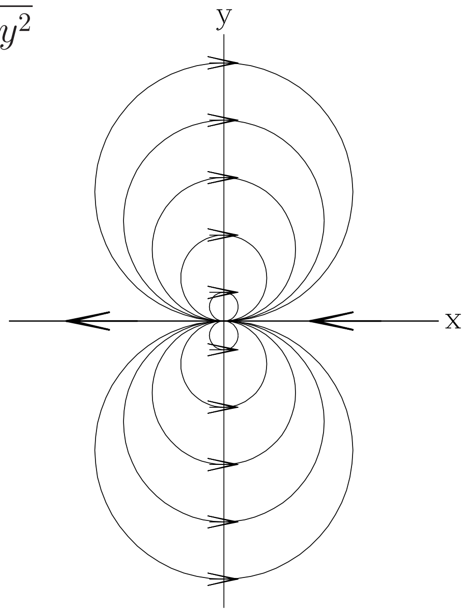
$$\psi = \frac{-\mu y}{x^2 + y^2}$$

$$x^2 + y^2 = \frac{-\mu}{\psi} y$$

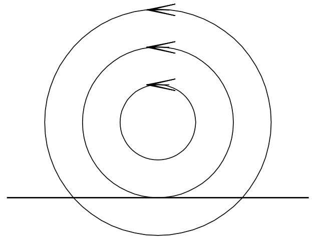
$$x^2 + y^2 + \frac{\mu}{\psi} y = 0$$

$$x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$

Circle centered at $(x, y) = \left(0, \frac{-\mu}{2\psi}\right)$ of radius $\left|\frac{\mu}{2\psi}\right|$



Vortex near a wall: method of images



Vortex:

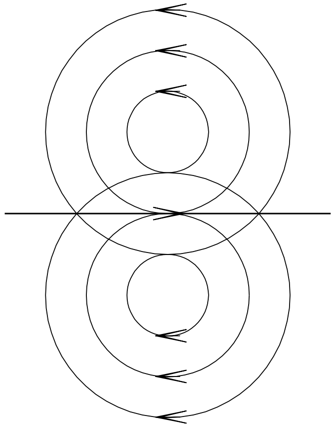
$$w(z) = ic\log(z - id)$$

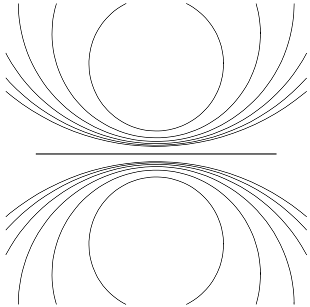
Streamlines penetrate wall at $y = 0$

Add equal and opposite "image vortex" at $z = -id$:

$$\begin{aligned} w(z) &= ic\log(z - id) - ic\log(z + id) = ic\log\left(\frac{z - id}{z + id}\right) \\ &= ic\left(\log\left|\frac{z - id}{z + id}\right| + i\arg\left(\frac{z - id}{z + id}\right)\right) \\ &= \phi + i\psi \end{aligned}$$

Streamlines satisfy $\left|\frac{z - id}{z + id}\right| = \exp\left(\frac{\psi}{c}\right) = \text{constant}$





More generally, choose $w(z) = f(z) + \overline{f(\overline{z})}$

On x-axis, $z = \overline{z}$, so $w(z) = f(z) + \overline{f(z)} = \phi + i\psi$

$\Longrightarrow w(z)$ is real and ψ is zero $\Longrightarrow x$-axis is a streamline.

$$\begin{aligned} w(z) &= ic\log(z - id) + \overline{ic\log(\overline{z} - id)} \\ &= ic\log(z - id) - ic\,\overline{\log(\overline{z} + id)} \\ &= ic\log(z - id) - ic\log(z - id) \end{aligned}$$

Note
$$\begin{aligned} \log z &= \log(re^{i\theta}) = \log r + i\theta \\ \overline{\log \overline{z}} &= \overline{\log(re^{-i\theta})} = \overline{\log r - i\theta} = \log r + i\theta = \log z \end{aligned}$$

Source near wall

Source at $z = id$ plus image source at $z = -id$

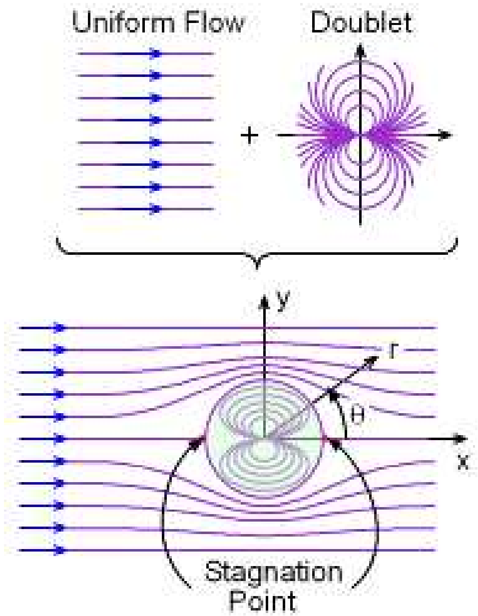
$$
\begin{aligned}
w(z) &= c\log(z - id) + \overline{c\log(\bar{z} - id)} \\
&= c\log(z - id) + c\,\overline{\log(\bar{z} - id)} \\
&= c\log(z - id) + c\;\log(z + id) \\
&= c\log((z - id)(z + id)) \\
&= c\log((z^2 - (id)^2) \\
&= c\log(x^2 - y^2 + d^2 + 2ixy) \\
&= \phi + i\psi
\end{aligned}
$$

Flow around a circular cylinder

$$w(z) = f(z) + \overline{f(a^2/\bar{z})}$$

On cylinder of radius a

$$z\bar{z} = a^2 \Longrightarrow \frac{a^2}{\bar{z}} = \frac{a^2}{a^2/z} = z \Longrightarrow f(z) + f(a^2/\bar{z}) = f(z) + \overline{f(z)}$$



Uniform flow past a circular cylinder:

$$\begin{aligned} w(z) &= Uz + \overline{U\frac{a^2}{\bar{z}}} \\ &= Uz + U\frac{a^2}{z} \end{aligned}$$

Flow around a circular cylinder (continued)

$$
\begin{aligned}
w(z) &= Uz + U\frac{a^2}{z} \\
w'(z) &= U - U\frac{a^2}{z^2} = U - \frac{Ua^2}{r^2}e^{-2i\theta} \\
(u_r - iu_\theta)e^{-i\theta} &= U\left(e^{i\theta} - \frac{a^2}{r^2}e^{-i\theta}\right)e^{-i\theta} \\
&= U\left(\left(1 - \frac{a^2}{r^2}\right)\cos\theta + i\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)e^{-i\theta} \\
&= U\left(\left(1 - \frac{a^2}{r^2}\right)\cos\theta + i\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)e^{-i\theta}
\end{aligned}
$$

On cylinder $a^2/r^2 = 1$ so $u_r = 0$ and $u_\theta = -2U\sin\theta$

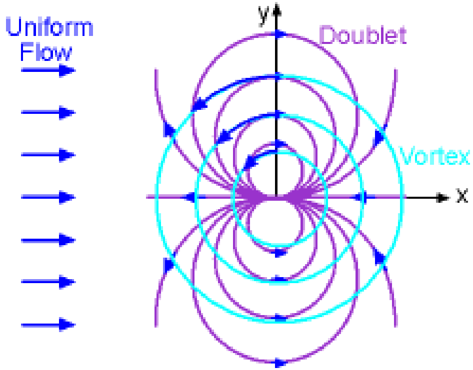
$$
u_\theta \begin{cases} < 0 & \text{for} \quad 0 \leq \theta \leq \pi \\ = 0 & \text{at} \quad \theta = 0, \pi \\ > 0 & \text{for} \quad \pi \leq \theta \leq 2\pi \end{cases}
$$

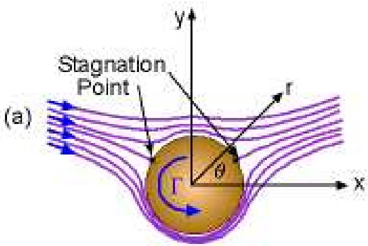
Flow around a circular cylinder (continued)

Add circulation (point vortex):

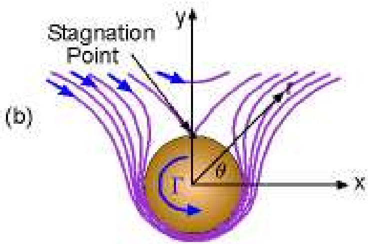
$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z$$

$$w'(z) = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}$$

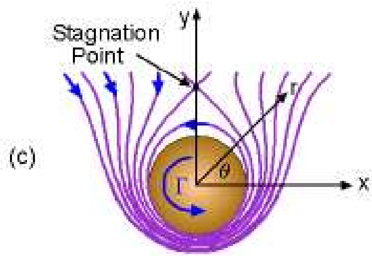




$$\frac{\Gamma}{4\pi a U} < 1$$



$$\frac{\Gamma}{4\pi a U} = 1$$



$$\frac{\Gamma}{4\pi a U} > 1$$

Calculate force on cylinder

Bernouilli's theorem states that if flow is uniform somewhere (e.g. far away),

$$p + \frac{\rho}{2}|\boldsymbol{u}|^2 = \text{constant} = p_\infty + \frac{\rho}{2}|\boldsymbol{u}_\infty|^2 = p_\infty + \frac{\rho}{2}U^2$$

On surface of cylinder $z = ae^{i\theta}$,

$$\begin{aligned}
w'(z) &= U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z} = U\left(1 - e^{-2i\theta}\right) - \frac{i\Gamma}{2\pi a}e^{i\theta} \\
&= \left(U\left(e^{i\theta} - e^{-i\theta}\right) - \frac{i\Gamma}{2\pi a}\right)e^{-i\theta} \\
u_\theta &= -2U\sin\theta + \frac{\Gamma}{2\pi a} \\
U^2 - |\boldsymbol{u}|^2 &= U^2 - u_\theta^2 = U^2 - \left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2 \\
&= U^2 - \left(4U^2\sin^2\theta + \left(\frac{\Gamma}{2\pi a}\right)^2 - 4U\sin\theta\frac{\Gamma}{2\pi a}\right)
\end{aligned}$$

Lift: Force in y direction, integrated around cylinder

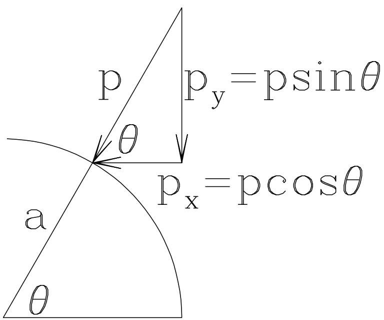
$$L = -\int_0^{2\pi} a\,d\theta\; p\sin\theta = -\int_0^{2\pi} a\,d\theta \sin\theta \left[p_\infty + \frac{\rho}{2}\left(U^2 - |\boldsymbol{u}|^2\right)\right]$$

$$= -\int_0^{2\pi} a\,d\theta \sin\theta \left[p_\infty + \frac{\rho}{2}\left(U^2 - \left(\frac{\Gamma}{2\pi a}\right)^2 - 4U^2\sin^2\theta + 4U\sin\theta\frac{\Gamma}{2\pi a}\right)\right]$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta) \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$\sin n\theta$ and $\cos n\theta$ integrate to zero, so

$$L = -\int_0^{2\pi} a\,d\theta\; \sin\theta\,\frac{\rho}{2}\,4U\sin\theta\frac{\Gamma}{2\pi a} = -\int_0^{2\pi} a\,d\theta\;\frac{\rho}{2}\,\frac{1}{2}\,4U\frac{\Gamma}{2\pi} = -\rho U\Gamma$$



If $\Gamma > 0$, circulatory flow opposes uniform flow $U\hat{\boldsymbol{e}}_x$ above cylinder, and adds to it below cylinder $\Longrightarrow$ velocity $\downarrow$ above and $\uparrow$ below $\Longrightarrow$ pressure $\uparrow$ above and $\downarrow$ below $\Longrightarrow$ force downwards

Drag: Force in x direction:

$$\begin{aligned} D &= \int_0^{2\pi} a\,d\theta\ p\cos\theta = \int_0^{2\pi} a\,d\theta\cos\theta\left[p_\infty + \frac{\rho}{2}\left(U^2 - |\boldsymbol{u}|^2\right)\right] \\ &= -\int_0^{2\pi} a\,d\theta\cos\theta\left[p_\infty + \frac{\rho}{2}\left(U^2 - \left(\frac{\Gamma}{2\pi a}\right)^2 - 4U^2\sin^2\theta + 4U\sin\theta\frac{\Gamma}{2\pi a}\right)\right] \end{aligned}$$

$$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta \qquad \cos\theta\sin^2\theta = \frac{1}{4}(\cos\theta - \cos 3\theta) \qquad \Longrightarrow D = 0$$

Blasius Theorem

$$F_x - iF_y = \frac{i\rho}{2}\oint\left(\frac{dw}{dz}\right)^2 dz$$