Hydrodynamics

Class 4

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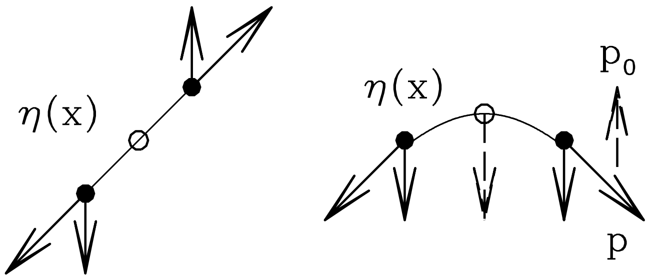
Surface Tension



Tangential force along surface $\Longrightarrow$ normal force if slope varies.
$\eta_{xx} < 0 \Longrightarrow F_z < 0$ to be counterbalanced by $p > p_0$:

$$p_0 - p = \sigma \frac{\partial^2 \eta}{\partial x^2}$$

Equation satisfied at surface:

$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla} \phi|^2 = \frac{p_0 - p}{\rho} - g\eta$$

becomes:

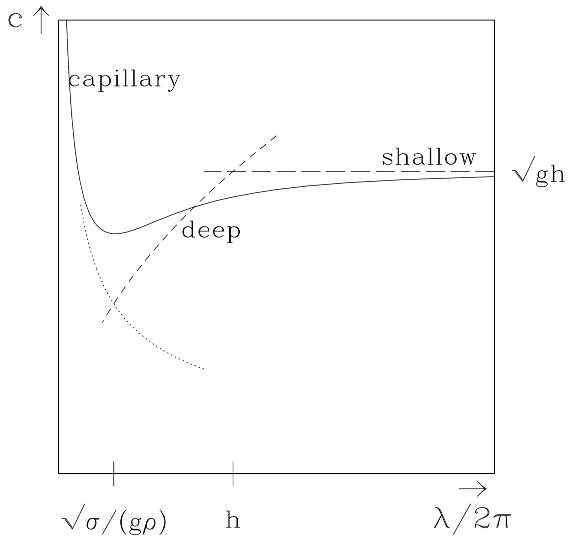
$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla} \phi|^2 = \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} - g\eta$$

$$
\begin{aligned}
\phi_t(z=0) &= \tfrac{\sigma}{\rho}\eta_{xx} - g\eta \\
\frac{-\omega^2\epsilon}{k}\,\frac{\cosh(kh)}{\sinh(kh)}\,\sin(kx-\omega t) &= \epsilon\left(-\tfrac{\sigma}{\rho}k^2 - g\right)\sin(kx-\omega t) \\
\omega^2 &= \left(gk + \tfrac{\sigma}{\rho}k^3\right)\tanh(kh)
\end{aligned}
$$

Velocities for gravity-capillary waves

$$
\omega^2 = \left(gk + \tfrac{\sigma}{\rho}k^3\right)\tanh(kh)
$$

	capillary	deep water	shallow water
wavelength	$\lambda = \frac{2\pi}{k} \ll \sqrt{\frac{\sigma}{\rho g}}$	$\lambda = \frac{2\pi}{k} \ll h$	$\lambda = \frac{2\pi}{k} \gg h$
phase velocity $c = \dfrac{\omega}{k}$	$\sqrt{\dfrac{\sigma}{\rho}k}$	$\sqrt{\dfrac{g}{k}}$	$\sqrt{gh}$
group velocity $c_g = \dfrac{d\omega}{dk}$	$\dfrac{3}{2}\sqrt{\dfrac{\sigma}{\rho}k}$	$\dfrac{1}{2}\sqrt{\dfrac{g}{k}}$	$\sqrt{gh}$



• Capillary waves have small wavelengths and travel quickly;
these are the leading waves when a stone is dropped in a pond.

• For air-water interface at 20°, crossover between capillary and deep-water gravity waves is at $\lambda = \sqrt{\sigma/(g\rho)} = 1.73$ cm, with phase speed $c =$23.2 cm/sec.

• In deep water, longer wavelengths travel faster and pulse spreads out.

• In shallow water, waves with different wavelengths have the same speed $\Longrightarrow$ wave pulses keep their size and shape $\Longrightarrow$ possibility of tsunami.

Internal Gravity Waves = Inertial Waves

Stratification: $\rho_0(z)$ with $\rho'(z) < 0$ (salinity increases with depth)
Base state is in equilibrium:

$$0 = -\frac{dp_0}{dz} - \rho_0 g \Longrightarrow p_0(z) = -g \int^z dz' \rho(z')$$

System governed by

$$\begin{aligned} \rho \left[\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right] \boldsymbol{u} &= -\boldsymbol{\nabla} p - \rho g \hat{\boldsymbol{e}}_{\boldsymbol{z}} \\ \left[\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right] \rho &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \end{aligned}$$

Set

$$\begin{aligned} p(x,z,t) &= p_0(z) + p_1(x,z,t) \\ \rho(x,z,t) &= \rho_0(z) + \rho_1(x,z,t) \\ \boldsymbol{u}(x,z,t) &= u_1(x,z,t)\hat{\boldsymbol{e}}_{\boldsymbol{x}} + w_1(x,z,t)\hat{\boldsymbol{e}}_{\boldsymbol{z}} \end{aligned}$$

Zero-order terms already in equilibrium. Retain first order:

$$\begin{aligned}
\rho_0 \partial_t \boldsymbol{u}_1 + \cancel{\rho_1 \partial_t \boldsymbol{u}_0} + \cancel{(\boldsymbol{u}_0 \cdot \boldsymbol{\nabla})\boldsymbol{u}_1} + \cancel{(\boldsymbol{u}_1 \cdot \boldsymbol{\nabla})\boldsymbol{u}_0} &= -\boldsymbol{\nabla} p_1 - \rho_1 g \hat{\boldsymbol{e}}_{\boldsymbol{z}} \\
\partial_t \rho_1 + \cancel{\boldsymbol{u}_0 \cdot \boldsymbol{\nabla} \rho_1} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} \rho_0 &= 0 \\
\boldsymbol{\nabla} \cdot \boldsymbol{u}_1 &= 0
\end{aligned}$$

$$\begin{aligned}
\rho_0 \partial_t u_1 &= -\partial_x p_1 \\
\rho_0 \partial_t w_1 &= -\partial_z p_1 - \rho_1 g \\
\partial_t \rho_1 + \cancel{u_1 \partial_x \rho_0} + w_1 \partial_z \rho_0 &= 0 \\
\partial_x u_1 + \partial_z w_1 &= 0
\end{aligned}$$

Eliminate u_1, ρ_1, p_1 in favor of w_1:

$$\begin{aligned}
\partial_x u_1 + \partial_z w_1 = 0 &\implies \partial_x u_1 = -\partial_z w_1 \\
\rho_0 \partial_t u_1 = -\partial_x p_1 &\implies \partial_{xx} p_1 = -\rho_0 \partial_{tx} u_1 = \rho_0 \partial_{tz} w_1 \\
\partial_t \rho_1 + w_1 \rho_0' = 0 &\implies \partial_t \rho_1 = -\rho_0' w_1 \\
\rho_0 \partial_t w_1 = -\partial_z p_1 - \rho_1 g &\implies \rho_0 \partial_{ttxx} w_1 = -\partial_{ztxx} p_1 - g \partial_{txx} \rho_1
\end{aligned}$$

leading to:

$$\begin{aligned}
\rho_0 \partial_{ttxx} w_1 &= -\partial_{zt}(\rho_0 \partial_{tz} w_1) - g \partial_{xx}(-\rho_0' w_1) \\
&= -\rho_0' \partial_{ttz} w_1 - \rho_0 \partial_{ttzz} w_1 + g \rho_0' \partial_{xx} w_1 \\
\rho_0 \partial_{tt} \nabla^2 w_1 &= \rho_0' (g \partial_{xx} w_1 - \partial_{ttz} w_1)
\end{aligned}$$

$$\partial_{tt}\nabla^2 w_1 = \frac{\rho_0'}{\rho_0}(g\partial_{xx} - \partial_{ttz})w_1$$

Define *Brunt-Vaäisälä frequency*:

$$N(z) = \sqrt{-g\frac{\rho_0'(z)}{\rho_0(z)}} \quad \text{(recall } \rho_0' < 0\text{)} \qquad \Longrightarrow \qquad \frac{\rho_0' g}{\rho_0} = -N^2$$

Suppose:

$$\rho_0(z) \sim e^{-\alpha z} \qquad \text{so that } N(z) \text{ is constant and} \quad \frac{\rho_0'}{\rho_0} = -\alpha$$

Then:

$$\partial_{tt}\nabla^2 w_1 = -N^2\partial_{xx}w_1 + \alpha\partial_{ttz}w_1$$

Substitute $e^{i(kx+\ell z-\omega t)}e^{\gamma z}W_1$

$$\begin{aligned}
-\omega^2(-k^2 + (i\ell+\gamma)^2)W_1 &= -N^2(-k^2)W_1 + \alpha(-\omega^2)(i\ell+\gamma)W_1 \\
-\omega^2(-k^2 - \ell^2 + \gamma^2 + 2i\ell\gamma) &= N^2k^2 - \alpha\omega^2(i\ell+\gamma)
\end{aligned}$$

$$-\omega^2(2i\ell\gamma) = -\alpha\omega^2 i\ell \qquad\qquad -\omega^2(-k^2 - \ell^2 + \gamma^2) = N^2k^2 - \alpha\omega^2\gamma$$

$$\omega^2(2i\ell\gamma) = \alpha\omega^2 i\ell \qquad \omega^2(k^2+\ell^2-\gamma^2) = N^2k^2 - \alpha\omega^2\gamma$$

$$\gamma = \frac{\alpha}{2} \qquad \omega^2\left(k^2+\ell^2+\gamma(\alpha-\gamma)\right) = N^2k^2$$

$$\omega^2\left(k^2+\ell^2+\frac{\alpha}{2}\frac{\alpha}{2}\right) = N^2k^2$$

$$\omega^2 = \frac{N^2k^2}{k^2+\ell^2+\left(\frac{\alpha}{2}\right)^2}$$

If $\alpha^2 \ll k^2+\ell^2$, i.e. if scale height $1/\alpha$ is much larger than wavelength

$$\lambda = \frac{2\pi}{\sqrt{k^2+\ell^2}} \ll \frac{1}{\alpha} \Longrightarrow \omega^2 = \frac{N^2k^2}{k^2+\ell^2}$$

$$\omega = \frac{Nk}{(k^2+\ell^2)^{1/2}} \equiv N\cos\theta$$

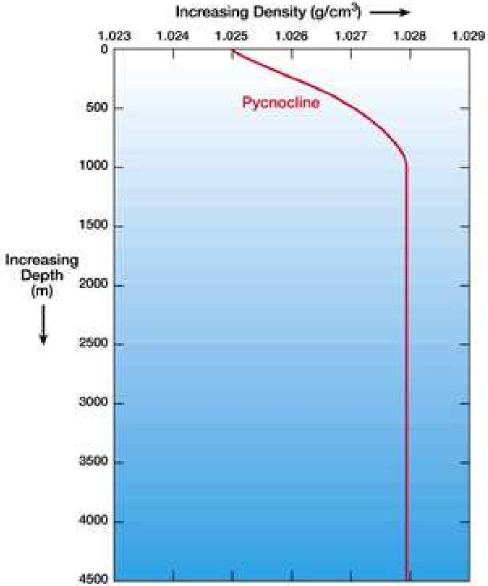
Phase velocity (2D) $\quad \boldsymbol{c} \equiv \dfrac{\omega}{k^2+\ell^2}(k\hat{\boldsymbol{e}}_{\boldsymbol{x}}+\ell\hat{\boldsymbol{e}}_{\boldsymbol{z}}) = \dfrac{Nk}{(k^2+\ell^2)^{3/2}}(k\hat{\boldsymbol{e}}_{\boldsymbol{x}}+\ell\hat{\boldsymbol{e}}_{\boldsymbol{z}})$

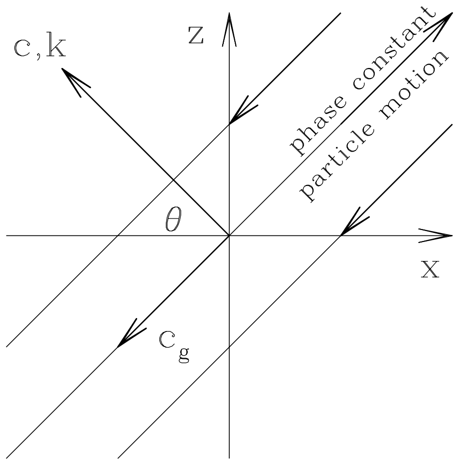
Group velocity (2D) $\quad \boldsymbol{c}_g \equiv \dfrac{\partial\omega}{\partial k}\hat{\boldsymbol{e}}_{\boldsymbol{x}} + \dfrac{\partial\omega}{\partial\ell}\hat{\boldsymbol{e}}_{\boldsymbol{z}} = \dfrac{N\ell}{(k^2+\ell^2)^{3/2}}(\ell\hat{\boldsymbol{e}}_{\boldsymbol{x}}-k\hat{\boldsymbol{e}}_{\boldsymbol{z}})$

Phase and group velocity are perpendicular: $\boldsymbol{c} \perp \boldsymbol{c}_g$

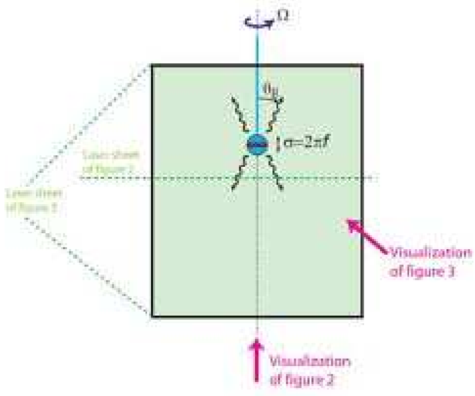
Fluid velocity $\quad \boldsymbol{k} \cdot \boldsymbol{u} = 0 \Longrightarrow \boldsymbol{u} \perp \boldsymbol{c} \quad , \quad \boldsymbol{u} \parallel \boldsymbol{c}_g$

$\rho_0(z)$

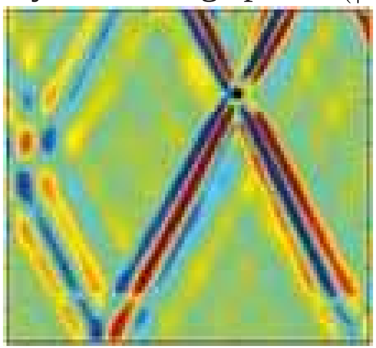


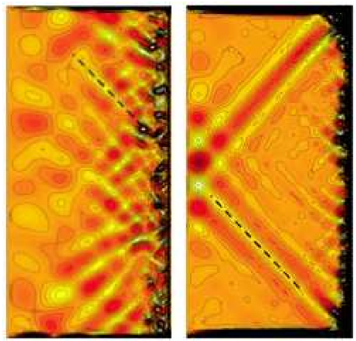


Inertial waves produced in laboratory experiments



vertically oscillating sphere ($|\omega| \leq N$) generates inertial waves





visualization of waves produced at FAST, Orsay and IRPHE, Marseille

Waves in rotating fluids

Uniformly rotating velocity of small amplitude:

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

where $p = \text{pressure} - \rho(\Omega^2 r^2/2 - gz)$

$$\frac{\partial^2}{\partial t^2}\nabla^2 p + (2\Omega)^2 \frac{\partial^2}{\partial z^2} p = 0$$

Setting

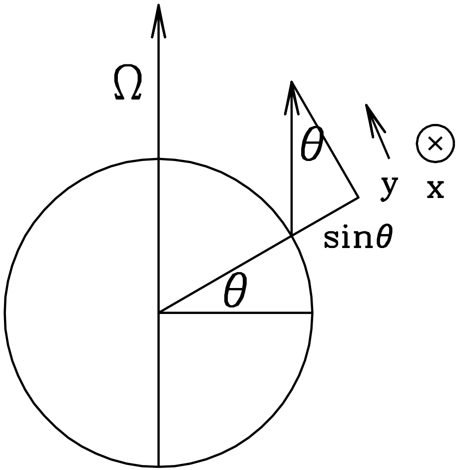
$$p = e^{i(kx+\ell y+mz-\omega t)}$$

$$\omega^2(k^2+\ell^2+m^2)p - (2\Omega)^2 m^2 p = 0$$

$$\omega^2 = (2\Omega)^2 \frac{m^2}{k^2+\ell^2+m^2}$$

Beta plane

Project $\boldsymbol{\Omega}$ from earth's rotation to component perpendicular to earth's surface:



$$
\begin{aligned}
f &= 2\Omega\sin\theta = 2\Omega\sin(\theta_0+\Delta\theta) \\
&= 2\Omega(\sin\theta_0\cos\Delta\theta + \sin\Delta\theta\cos\theta_0) \\
&\approx 2\Omega(\sin\theta_0 + \Delta\theta\cos\theta_0) \\
&= 2\Omega\sin\theta_0 + 2\Omega\cos\theta_0\Delta\theta \\
&= 2\Omega\sin\theta_0 + 2\Omega\cos\theta_0\frac{y}{R} \\
&= 2\Omega\sin\theta_0 + \frac{2\Omega\cos\theta_0}{R}y \\
&= 2\Omega\sin\theta_0 + \beta y \\
&= f_0 + \beta y
\end{aligned}
$$

Approximate earth's surface with Cartesian coordinates (x, y) and (u, v)

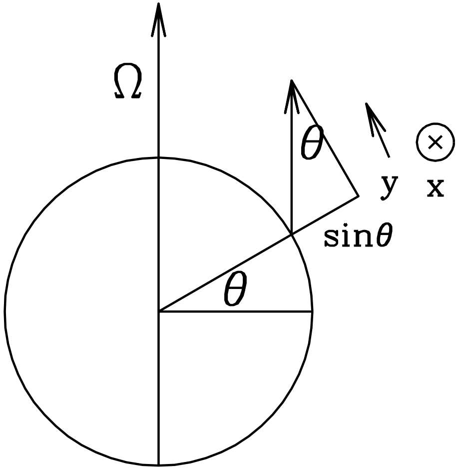
$$
\begin{aligned}
\partial_t u + \partial_x p/\rho &= 2\Omega v = f v = (f_0+\beta y)v \\
\partial_t v + \partial_y p/\rho &= -2\Omega u = -f u = -(f_0+\beta y)u
\end{aligned}
$$

Dependence $f(y)$ leads to *Rossby waves* – important in atmostphere and ocean

Rossby waves

$$
\begin{aligned}
\partial_t \boldsymbol{u} &= -2\Omega(y)\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \boldsymbol{u} - \frac{1}{\rho}\boldsymbol{\nabla} p \\
\hat{\boldsymbol{e}}_{\boldsymbol{z}} \cdot \boldsymbol{\nabla}\times\partial_t \boldsymbol{u} &= -\hat{\boldsymbol{e}}_{\boldsymbol{z}} \cdot \boldsymbol{\nabla}\times(2\Omega(y)\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \boldsymbol{u}) \\
&= -\partial_x(2\Omega(y)\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \boldsymbol{u})_y + \partial_y(2\Omega(y)\hat{\boldsymbol{e}}_{\boldsymbol{z}} \times \boldsymbol{u})_x \\
&= -\partial_x(2\Omega(y)u) + \partial_y(-2\Omega(y)v) \\
&= -2\Omega(y)(\partial_x u + \partial_y v) - \beta v \\
\partial_t(\partial_x v - \partial_y u) &= -\beta v \\
\partial_t(\partial_{xx} v - \partial_{yx} u) &= -\beta \partial_x v \\
\partial_t(\partial_{xx} v + \partial_{yy} v) &= -\beta \partial_x v \\
\partial_t \nabla^2 v &= -\beta \partial_x v \\
v \sim e^{i(kx+\ell y-\omega t)} \implies -i\omega(-k^2 - \ell^2) &= -i\beta k \\
\omega &= \frac{-\beta k}{k^2 + \ell^2}
\end{aligned}
$$

Rossby waves



$$\omega = \frac{-\beta k}{k^2 + \ell^2}$$

$\beta = 2\Omega \cos\theta_0 / R > 0$

$\Longrightarrow \omega$ and k are of opposite signs.

$v \sim e^{i(kx + \ell y - \omega t)}$

Rossby wave crests travel in direction of decreasing x, i.e. westwards

wave speed is inversely proportional to $|k|$