Hydrodynamics

Class 3

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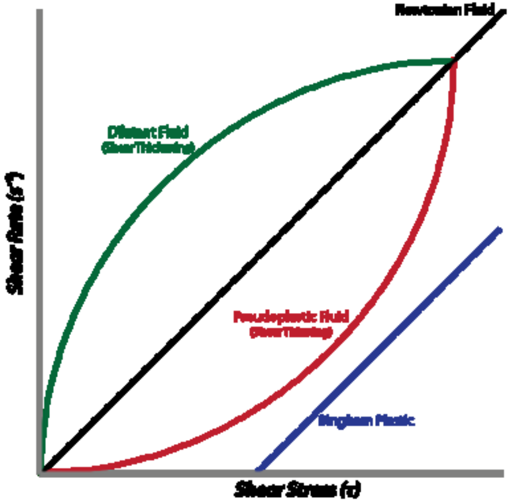
Viscosity, continued: non-Newtonian Fluids

$$\boldsymbol{\nabla}\cdot\tau = -\boldsymbol{\nabla} p + \boldsymbol{\nabla}\cdot\sigma$$

Newtonian fluids: shear rate tensor $\boldsymbol{\sigma}$ is proportional to shear stress tensor $\boldsymbol{e}$.
Showed that $\sigma_{ij} = \mu(e_{ij} + e_{ji}) + \lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}$.
This is not the only possibility $\Longrightarrow$ non-Newtonian fluids



from Wikipedia http://en.wikipedia.org/wiki/Non-Newtonian_fluid

Viscosity, continued: Reynolds number

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}\frac{p}{\rho} + \nu \boldsymbol{\nabla}^2 \boldsymbol{u}$$

Scale/non-dimensionalize:

$$\boldsymbol{u} = U\tilde{\boldsymbol{u}} \qquad \boldsymbol{x} = L\tilde{\boldsymbol{x}} \qquad t = \tilde{t}L/U \qquad \boldsymbol{\nabla} = \tilde{\boldsymbol{\nabla}}/L \qquad \frac{p}{\rho} = \frac{U^2\tilde{p}}{\tilde{\rho}}$$

$$\begin{aligned}
\frac{\partial\, U\tilde{\boldsymbol{u}}}{\partial\, \tilde{t}(L/U)} + \left(U\tilde{\boldsymbol{u}} \cdot \frac{\tilde{\boldsymbol{\nabla}}}{L}\right) U\tilde{\boldsymbol{u}} \;&=\; -\frac{\tilde{\boldsymbol{\nabla}}}{L}\frac{U^2\tilde{p}}{\tilde{\rho}} + \nu \frac{\tilde{\boldsymbol{\nabla}}^2}{L^2} U\tilde{\boldsymbol{u}} \\
\frac{U^2}{L}\left[\frac{\partial \boldsymbol{u}}{\partial \tilde{t}} + (\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}})\tilde{\boldsymbol{u}}\right] \;&=\; -\frac{U^2}{L}\tilde{\boldsymbol{\nabla}}\frac{\tilde{p}}{\tilde{\rho}} + \frac{\nu U}{L^2}\tilde{\boldsymbol{\nabla}}^2\tilde{\boldsymbol{u}} \\
\frac{\partial \tilde{\boldsymbol{u}}}{\partial \tilde{t}} + (\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{\nabla}})\tilde{\boldsymbol{u}} \;&=\; -\tilde{\boldsymbol{\nabla}}\frac{\tilde{p}}{\tilde{\rho}} + \underbrace{\frac{\nu}{UL}}_{\frac{1}{Re}}\tilde{\boldsymbol{\nabla}}^2\tilde{\boldsymbol{u}}
\end{aligned}$$

Reynolds number Re measures importance of viscous effects

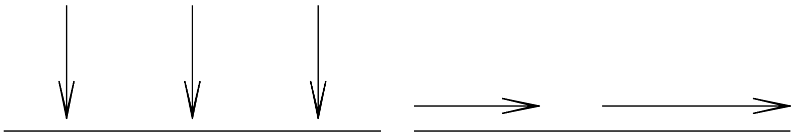
Viscosity, continued: boundary conditions

Incompressible and Newtonian fluids: $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - g\hat{\boldsymbol{e}}_{\boldsymbol{z}} + \nu \boldsymbol{\nabla}^2 \boldsymbol{u}$$

Second order in space: specify BCs at all boundaries on u, v, w

Specify normal and tangential velocity at solid boundaries



normal velocity tangential velocity

Euler equations: $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - g\hat{\boldsymbol{e}}_{\boldsymbol{z}}$$

Without viscosity, fluid layers can slide over one another $\Longrightarrow$

Specify only normal velocity at solid boundaries

Waves

Assume *potential flow* $\Longrightarrow \boldsymbol{u} = \boldsymbol{\nabla}\phi$
Assume incompressible $\Longrightarrow \nabla^2\phi = 0$
Implies irrotational ($\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = 0$)
Assume no viscosity (bulk viscous term $\mu\boldsymbol{\nabla}^2\boldsymbol{u}$ does not generate vorticity, i.e. its curl is zero, but vorticity can be generated by viscous stresses at boundaries).

$$\frac{\partial \boldsymbol{u}}{dt} + \tfrac{1}{2}\boldsymbol{\nabla}|u|^2 = -\frac{1}{\rho}\boldsymbol{\nabla}p - g\hat{\boldsymbol{e}}_{\boldsymbol{z}}$$

$$\frac{\partial \boldsymbol{\nabla}\phi}{dt} + \boldsymbol{\nabla}\left(\tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2\right) = -\boldsymbol{\nabla}\left(\frac{p}{\rho}\right) - \boldsymbol{\nabla}(gz)$$

$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 = -\frac{p}{\rho} - gz + F(t)$$

Define surface height $\eta(x, y, t)$. At $z = \eta$, $p = p_0 = 0$. Absorb $F(t)$ into ϕ_t:

$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 = -\frac{p}{\rho} - g\eta + F(t)$$

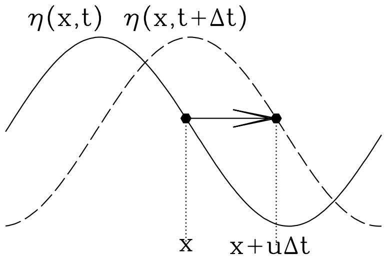
$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 = -g\eta$$

Effect of horizontal motion

$$\eta(x+u\Delta t, t+\Delta t) = \eta(x,t)$$

$$\eta(x,t) + \frac{\partial \eta}{\partial t}\Delta t + \frac{\partial \eta}{\partial x} u \Delta t = \eta(x,t)$$

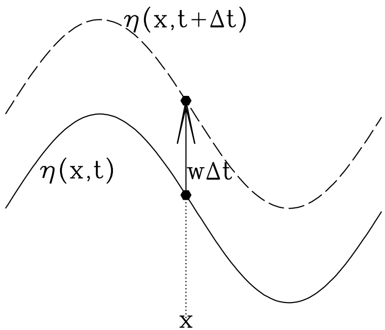
$$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} u$$



Effect of vertical motion

$$\eta(x, t+\Delta t) = \eta(x,t) + w\Delta t$$

$$\frac{\partial \eta}{\partial t} = w$$



Combined effect

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z}$$

Velocity field

$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$	$\nabla^2\phi = 0$	throughout	incompressibility
$w = 0$	$\phi_z = 0$	at bottom $z = -h$	no penetration

$\nabla^2\phi = 0$ in 2D with periodic BCs in x ($\lambda = 2\pi/k$) and $\phi_z = 0$ at $z = -h \Longrightarrow$ trigonometric in x and exponential in z.

$$\begin{aligned}
\phi(x,z,t) &= a(x,t)\sinh(kz) + b(x,t)\cosh(kz) \\
0 = w(z=-h) &= ka(x,t)\cosh(kh) - kb(x,t)\sinh(kh) \Longrightarrow \\
& \quad b(x,t) = a(x,t)/\tanh(kh) \\
\phi(x,z,t) &= a(x,t)[\sinh(kz) + \cosh(kz)/\tanh(kh)] \\
&= a(x,t)\frac{\cosh(k(z+h))}{\sinh(kh)} \\
w = \phi_z(x,z,t) &= ka(x,t)\frac{\sinh(k(z+h))}{\sinh(kh)}
\end{aligned}$$

At surface $z = \eta(x,y,t)$:

$\eta_t + u\eta_x + v\eta_y = w$	$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z$	kinematics
$\phi_t + \frac{1}{2}\vert\boldsymbol{u}\vert^2 = -g\eta$	$\phi_t + \frac{1}{2}\vert\boldsymbol{\nabla}\phi\vert^2 = -g\eta$	dynamics

$$\begin{aligned} \phi(x,z,t) &= a(x,t)\frac{\cosh(k(z+h))}{\sinh(kh)} \\ w = \phi_z(x,z,t) &= ka(x,t)\frac{\sinh(k(z+h))}{\sinh(kh)} \end{aligned}$$

Small-amplitude waves: neglect quadratic terms and evaluate at $z = 0$:

$\eta_t = w$	$\eta_t = \phi_z$	kinematics
$\phi_t = -g\eta$	$\phi_t = -g\eta$	dynamics

Seek solution of form $\eta = \epsilon \sin(kx - \omega t) \Longrightarrow$

$$\begin{aligned} w = \phi_z(z=0) &= \eta_t \\ ka(x,t) &= -\omega\epsilon\cos(kx-\omega t) \\ \phi(x,z,t) = -\frac{\omega\epsilon}{k} & \quad \cos(kx-\omega t)\frac{\cosh(k(z+h))}{\sinh(kh)} \end{aligned}$$

$$\begin{aligned} \phi_t(z=0) &= -g\eta \\ \frac{-\omega^2\epsilon}{k\sinh(kh)}\sin(kx-\omega t) &= -g\epsilon\sin(kx-\omega t) \end{aligned}$$

Dispersion relation: $\omega^2 = gk\tanh(kh)$ Phase velocity: $c = \omega/k$

Pressure

$$\phi(x,z,t) = -\frac{\omega\epsilon}{k}\frac{\cosh(k(z+h))}{\sinh(kh)}\cos(kx-\omega t)$$

$$\begin{aligned}
p + \rho g z &= -\rho\frac{\partial\phi}{dt} + \frac{\rho}{2}|\boldsymbol{\nabla}\phi|^2 \approx -\rho\frac{\partial\phi}{dt} \\
&= -\frac{\omega^2\rho\epsilon}{k}\quad\frac{\cosh(k(z+h))}{\sinh(kh)}\sin(kx-\omega t) \\
&= -\frac{gk\tanh(kh)\rho\epsilon}{k}\quad\frac{\cosh(k(z+h))}{\sinh(kh)}\sin(kx-\omega t) \\
&= -g\rho\epsilon\frac{\cosh(k(z+h))}{\cosh(kh)}\sin(kx-\omega t)
\end{aligned}$$

Penetration into depth: at $z = -h$,

$$p + \rho g z = -g\rho\epsilon\, e^{-kh}\ \sin(kx-\omega t)$$

Particle paths for traveling waves

$$\begin{aligned}
\phi(x,z,t) &= -\frac{\omega\epsilon}{k}\cos(kx-\omega t)\,\frac{\cosh((k+h)z)}{\sinh(kh)} \\
\frac{dx}{dt} = u = \phi_x &= \omega\epsilon\sin(kx-\omega t)\,\frac{\cosh((k+h)z)}{\sinh(kh)} \\
\frac{dz}{dt} = w = \phi_z &= -\omega\epsilon\cos(kx-\omega t)\,\frac{\sinh((k+h)z)}{\sinh(kh)}
\end{aligned}$$

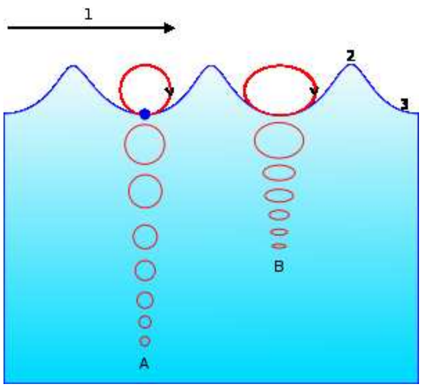
For *small excursion* from (x_0, z_0):

$$\begin{aligned}
\frac{dx}{dt} &= \omega\epsilon\sin(kx_0-\omega t)\frac{\cosh(k(z_0+h))}{\sinh(kh)} \equiv \omega\sin(kx_0-\omega t)A \\
x(t) &= x_0 + \cos(kx_0-\omega t)A \\
\frac{dz}{dt} &= -\omega\epsilon\cos(kx_0-\omega t)\frac{\cosh(k(z_0+h))}{\sinh(kh)} \equiv -\omega\cos(kx_0-\omega t)B \\
z(t) &= z_0 + \sin(kx_0-\omega t)B \\
1 &= \left(\frac{x(t)-x_0}{A}\right)^2 + \left(\frac{z(t)-z_0}{B}\right)^2 \qquad \text{ellipses}
\end{aligned}$$

$$1 = \left(\frac{x(t) - x_0}{A}\right)^2 + \left(\frac{z(t) - z_0}{B}\right)^2 \qquad \text{ellipses}$$

$$A = \epsilon \frac{\cosh(k(z_0 + h))}{\sinh(kh)} \qquad B = \epsilon \frac{\sinh(k(z_0 + h))}{\sinh(kh)}$$

deep $kh \gg 1 \implies$ circles $\cosh(k(z_0 + h)) \approx \sinh(k(z_0 + h))$ so $A \approx B$	shallow $kh \ll 1 \implies$ flat ellipses $\cosh(k(z_0 + h)) \gg \sinh(k(z_0 + h))$ so $A \gg B$



From Wikipedia http://en.wikipedia.org/wiki/Water_waves

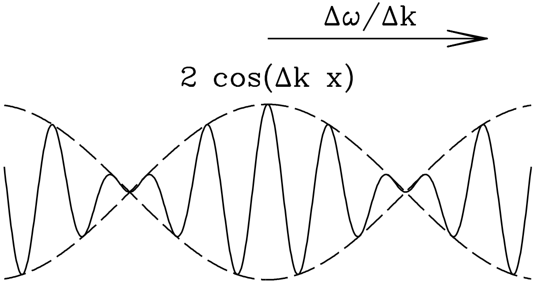
Justified since $A, B \sim \epsilon$ so $(x(t) - x_0), (z(t) - z_0)$ are small.

Group velocity

$$\cos(A) + \cos(B) = 2\cos((A+B)/2)\cos((A-B)/2)$$

$$\begin{aligned}
\eta(x,t) &= \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) \\
&= 2\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right) \\
&= 2\cos(kx-\omega t)\cos\left(\Delta k\ x - \Delta\omega t\right) \\
&= 2\cos\left(k\left(x - \frac{\omega}{k}t\right)\right)\cos\left(\Delta k\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right)
\end{aligned}$$

nodes move with speed ω/k envelope moves with speed $\Delta\omega/\Delta k$



Group velocity and wavepackets

$$\eta(x,0) = \int dk\ \hat{\eta}_k\ \exp(ikx) \qquad \text{centered around } k_0 \text{ with } \omega(k_0) \equiv \omega_0$$

$$\begin{aligned}
\eta(x,t) &= \int dk\ \hat{\eta}_k\ \exp(i(kx - \omega(k)t)) \\
&= \int dk\ \hat{\eta}_k\ \exp[i(k_0 x - \omega(k_0)t)] \exp[i((k-k_0)x - (\omega(k) - \omega(k_0))t)] \\
&= \exp[i(k_0 x - \omega(k_0)t)] \int dk\ \hat{\eta}_k\ \exp[i((k-k_0)x - \omega'(k_0)(k-k_0)t)] \\
&= \exp[i(k_0 x - \omega(k_0)t)] \int dk\ \hat{\eta}_k\ \exp[i(k-k_0)(x - \omega'(k_0)t)]
\end{aligned}$$

nodes move with speed $\omega(k_0)/k_0$ envelope moves with speed $\omega'(k_0)$

Velocities for gravity waves

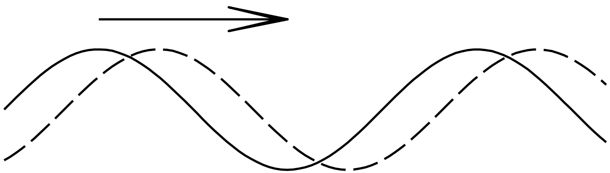
$$\omega = \sqrt{gk \tanh(kh)}$$

	shallow water $kh \ll 1$	deep water $kh \gg 1$
phase velocity $\frac{\omega}{k}$	$\sqrt{gh}$	$\sqrt{\frac{g}{k}}$
group velocity $\frac{d\omega}{dk}$	$\sqrt{gh}$	$\frac{1}{2}\sqrt{\frac{g}{k}}$

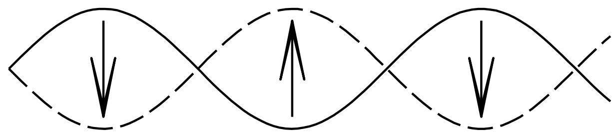
Traveling and Standing Waves

Linear problem so can add solutions

$$\begin{aligned}\eta &= \epsilon(\sin(kx-\omega t)+\sin(kx+\omega t))/2\\ &= \epsilon\left[\sin(kx)\cos(\omega t)-\cos(kx)\sin(\omega t)+\sin(kx)\cos(\omega t)+\cos(kx)\sin(\omega t)\right]/2\\ &= \epsilon\sin(kx)\cos(\omega t)\end{aligned}$$



traveling wave
$\sin(kx-\omega t)$



standing wave
$\sin(kx)\cos(\omega t)$

$$\begin{aligned}\phi(x,z,t) &= -\frac{\omega\epsilon}{k}\,\frac{\cosh((k+h)z)}{\sinh(kh)}\,\frac{1}{2}(\cos(kx-\omega t)+\cos(kx+\omega t))\\ &= -\frac{\omega\epsilon}{k}\,\frac{\cosh((k+h)z)}{\sinh(kh)}\cos(kx)\cos(\omega t)\end{aligned}$$

Particle paths for standing waves

$$
\begin{aligned}
\phi(x,z,t) &= -\frac{\omega\epsilon}{k}\,\frac{\cosh((k+h)z)}{\sinh(kh)}\cos(kx)\cos(\omega t) \\
\frac{dx}{dt} = u = \phi_x &= \omega\epsilon\,\frac{\cosh((k+h)z)}{\sinh(kh)}\sin(kx)\cos(\omega t) \\
\frac{dz}{dt} = w = \phi_z &= -\omega\epsilon\,\frac{\sinh((k+h)z)}{\sinh(kh)}\cos(kx)\cos(\omega t)
\end{aligned}
$$

For *small excursion* from (x_0, z_0):

$$
\begin{aligned}
\frac{dx}{dt} = &= \omega\epsilon\,\frac{\cosh((k+h)z_0)}{\sinh(kh)}\sin(kx_0)\sin(\omega t) \\
x(t) &= x_0 - \epsilon\,\frac{\cosh((k+h)z_0)}{\sinh(kh)}\sin(kx_0)\cos(\omega t) \\
\frac{dz}{dt} = &= -\omega\epsilon\,\frac{\sinh((k+h)z_0)}{\sinh(kh)}\cos(kx_0)\cos(\omega t) \\
z(t) &= z_0 - \epsilon\,\frac{\sinh((k+h)z_0)}{\sinh(kh)}\cos(kx_0)\sin(\omega t)
\end{aligned}
$$

For $kx_0 = 0, \ldots n\pi, \ldots \Longrightarrow \sin(kx_0) = 0$:

$$\begin{aligned} x(t) &= x_0 \\ z(t) &= z_0 - (-1)^n \omega\epsilon \, \frac{\sinh((k+h)z_0)}{\sinh(kh)} \sin(\omega t) \end{aligned}$$

i.e., vertical oscillations.

For $kx_0 = (2n+1)\pi/2 \Longrightarrow \cos(kx_0) = 0$:

$$\begin{aligned} x(t) &= x_0 + (-1)^n \omega\epsilon \, \frac{\cosh((k+h)z_0)}{\sinh(kh)} \sin(kx_0) \cos(\omega t) \\ z(t) &= z_0 \end{aligned}$$

i.e. horizontal oscillations.