Hydrodynamics

Class 2

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Streamfunction:

Cartesian coordinates:

In 2D (x, y), if $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, can define ψ such that $\boldsymbol{v} = \boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{e}}_{\boldsymbol{z}}) = \boldsymbol{\nabla}\psi \times \hat{\boldsymbol{e}}_{\boldsymbol{z}}$

$$v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

Satisfies

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \partial_x v_x + \partial_y v_y + \partial_z v_z = \partial_{xy}\psi - \partial_{yx}\psi = 0$$

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\psi = (\boldsymbol{\nabla}\psi \times \hat{\boldsymbol{e}}_{\boldsymbol{z}})\cdot\boldsymbol{\nabla}\psi = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\psi \cdot \hat{\boldsymbol{e}}_{\boldsymbol{z}} = 0$$

$\implies \boldsymbol{v} \perp \boldsymbol{\nabla}\psi \iff \boldsymbol{v}$ is parallel to curves of constant ψ

Can also have $\boldsymbol{v} = \boldsymbol{\nabla}\times(\psi(x, y)\hat{\boldsymbol{e}}_{\boldsymbol{z}}) + v_z(x, y)\hat{\boldsymbol{e}}_{\boldsymbol{z}}$ (sometimes called 2.5D flow)

Cylindrical coordinates:

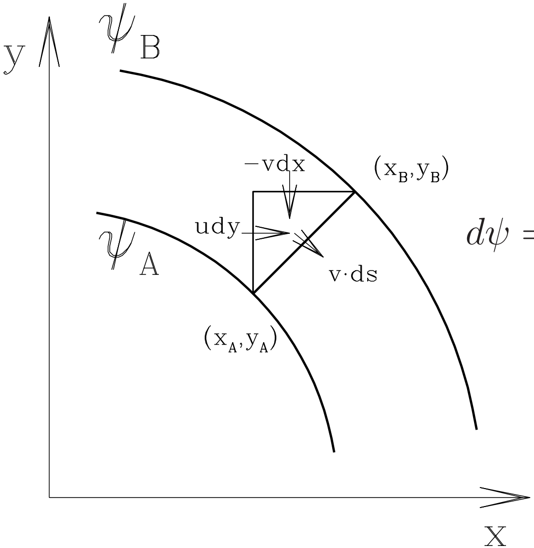
a) $\boldsymbol{v}$ is independent of ϕ

$$\boldsymbol{v} = \hat{\boldsymbol{e}}_\phi \times \frac{\boldsymbol{\nabla}\psi(R, z)}{r} = \boldsymbol{\nabla}\times\left(\frac{\psi(r, z)}{r}\hat{\boldsymbol{e}}_\phi\right) = \hat{\boldsymbol{e}}_{\boldsymbol{z}}\frac{1}{R}\frac{\partial \psi}{\partial R} - \hat{\boldsymbol{e}}_{\boldsymbol{R}}\frac{1}{R}\frac{\partial \psi}{\partial z}$$

b) $\boldsymbol{v}$ is independent of z

$$\boldsymbol{v} = \boldsymbol{\nabla}\psi(R, \phi) \times \hat{\boldsymbol{e}}_{\boldsymbol{z}} = \hat{\boldsymbol{e}}_{\boldsymbol{R}}\frac{1}{R}\frac{\partial \psi}{\partial \phi} - \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\frac{\partial \psi}{\partial R}$$

Flux between two streamlines



$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy = \boldsymbol{v} \cdot \boldsymbol{ds}$$

$$\psi_B - \psi_A = \int_{\psi_A}^{\psi_B} d\psi = \int_{(x_A,y_A)}^{(x_B,y_B)} \boldsymbol{v} \cdot \boldsymbol{ds}$$

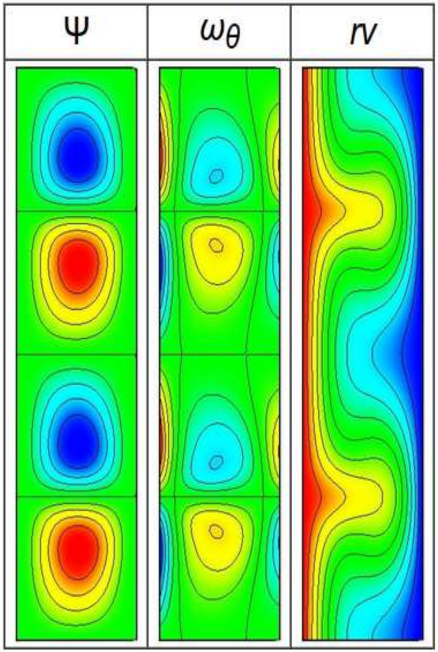
Direction of ψ contours $\Longrightarrow$ flow direction. Denseness $\Longrightarrow$ flow intensity

Streamline: tangent is parallel to $\boldsymbol{v}$
Pathline: path followed by fluid particles/elements
Streakline: fluid particles/elements passed through a given point in the past
All are identical for steady flows

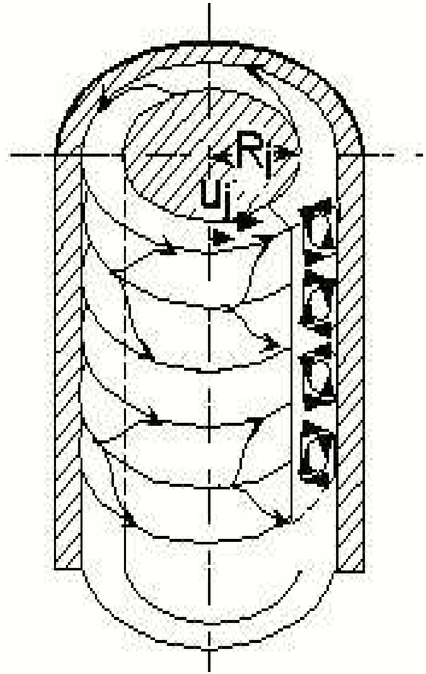
Taylor-Couette Flow

$$v_R = -\frac{1}{R}\frac{\partial \psi}{\partial z}$$

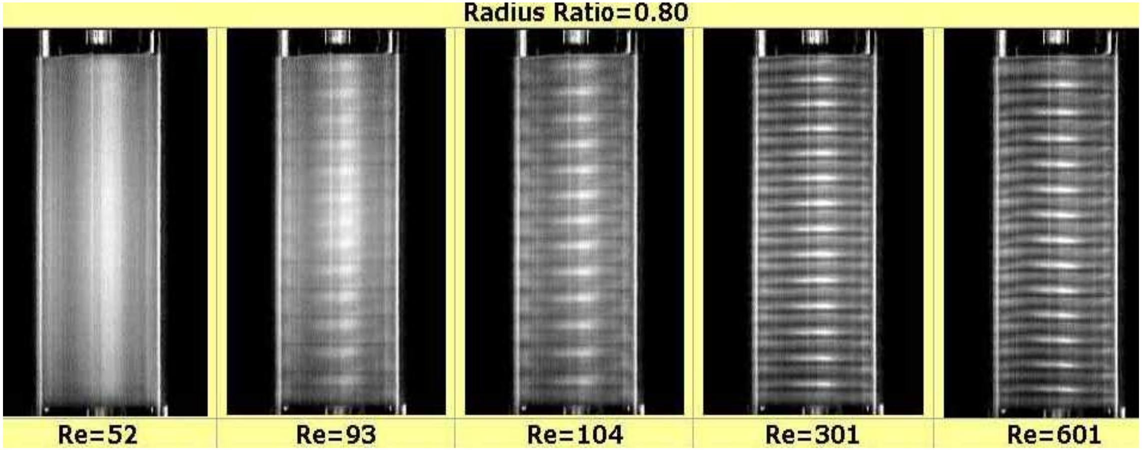
$$v_z = \frac{1}{R}\frac{\partial \psi}{\partial R}$$



v_ϕ



Taylor-Couette Flow



Vorticity

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$$

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \frac{1}{R}\begin{vmatrix} \hat{\boldsymbol{e}}_R & R\hat{\boldsymbol{e}}_\phi & \hat{\boldsymbol{e}}_z \\ \partial_R & \partial_\phi & \partial_z \\ v_R & Rv_\phi & v_z \end{vmatrix}$$

Solid body rotation:

$$\boldsymbol{v} = R\Omega\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \Longrightarrow \boldsymbol{\omega} = \hat{\boldsymbol{e}}_{\boldsymbol{z}}\frac{1}{R}\frac{\partial(Rv_\phi)}{\partial R} = \hat{\boldsymbol{e}}_{\boldsymbol{z}}\frac{1}{R}\frac{\partial(R^2\Omega)}{\partial R} = \hat{\boldsymbol{e}}_{\boldsymbol{z}}\frac{1}{R}2R\Omega = 2\Omega\hat{\boldsymbol{e}}_{\boldsymbol{z}}$$

Point vortex:

$$\boldsymbol{v} = \frac{\gamma}{R}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \Longrightarrow \boldsymbol{\omega} = \hat{\boldsymbol{e}}_{\boldsymbol{z}}\frac{1}{R}\frac{\partial(Rv_\phi)}{\partial R} = \hat{\boldsymbol{e}}_{\boldsymbol{z}}\frac{1}{R}\frac{\partial\gamma}{\partial R} = 0$$

Polar coordinates can’t be used at origin! ((R,ϕ) not defined)

$$\begin{aligned} \int_{R=0}^{\epsilon}\int_{\phi=0}^{2\pi} R\,dR\,d\phi\,\hat{\boldsymbol{e}}_{\boldsymbol{z}}\cdot\boldsymbol{\omega}(R) &= \int_{R=0}^{\epsilon}\int_{\phi=0}^{2\pi} R\,dR\,d\phi\,\hat{\boldsymbol{e}}_{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\boldsymbol{v} \\ \text{via Stokes' Theorem} &= \int_{\phi=0}^{2\pi} Rd\phi\,\hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\cdot\boldsymbol{v}\big|_{R=\epsilon} = 2\pi\gamma \\ \textbf{Circulation }\Gamma &= 2\pi\gamma \quad \text{for } \textit{any} \text{ radius } \epsilon \end{aligned}$$

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \frac{1}{R}\begin{vmatrix} \hat{\boldsymbol{e}}_R & R\hat{\boldsymbol{e}}_\phi & \hat{\boldsymbol{e}}_z \\ \partial_R & \partial_\phi & \partial_z \\ v_R & Rv_\phi & v_z \end{vmatrix}$$

Taylor-Couette flow

$$\boldsymbol{v}_{TC} = \left(aR + \frac{b}{R}\right)\hat{\boldsymbol{e}}_\phi$$

Sum of solid body rotation (a) and point vortex (b)

$$\boldsymbol{\nabla}\times\boldsymbol{v}_{TC} = \frac{1}{R}\hat{\boldsymbol{e}}_{\boldsymbol{z}}\partial_R(Rv_\phi) = \frac{1}{R}\hat{\boldsymbol{e}}_{\boldsymbol{z}}\partial_R(aR^2+b) = \frac{1}{R}\hat{\boldsymbol{e}}_{\boldsymbol{z}}2aR = 2a\hat{\boldsymbol{e}}_{\boldsymbol{z}}$$

For incompressible axisymmetric flow in (R, z) plane (meridional)

$$\boldsymbol{v} = \boldsymbol{\nabla}\times\left(\frac{\psi(R,z)}{R}\hat{\boldsymbol{e}}_\phi\right)$$

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{v} &= \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left(\frac{\psi(R,z)}{R}\hat{\boldsymbol{e}}_\phi\right) = \underbrace{\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\left(\frac{\psi(R,z)}{R}\hat{\boldsymbol{e}}_\phi\right)}_{=0} - \boldsymbol{\nabla}^2\left(\frac{\psi(R,z)}{R}\hat{\boldsymbol{e}}_\phi\right) \\ &= -\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial z^2}\right)\hat{\boldsymbol{e}}_\phi\end{aligned}$$

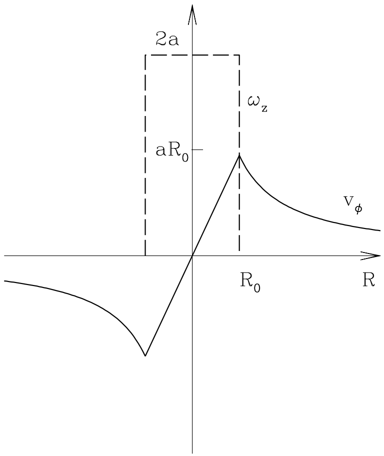
Rankine vortex: model for tornado or cyclone

Solid body rotation surrounded by point vortex:

$$\boldsymbol{v} = v_\phi(R) = \begin{cases} aR & \text{for} \quad R < R_0 \\ b/R & \text{for} \quad R < R_0 \end{cases}$$

For continuity at R_0, require $aR_0 = b/R_0 \Longrightarrow b = aR_0^2$

$$\boldsymbol{\omega} = \hat{\boldsymbol{e}}_{\boldsymbol{z}} \frac{1}{R} \frac{\partial (R v_\phi)}{\partial R} = \hat{\boldsymbol{e}}_{\boldsymbol{z}} \begin{cases} 2a & \text{for} \quad R < R_0 \\ 0 & \text{for} \quad R < R_0 \end{cases}$$

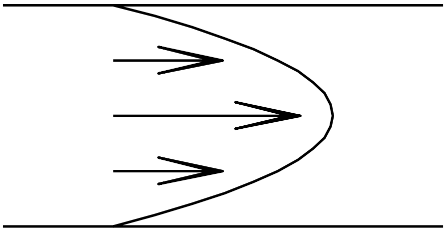


2D incompressible flows: $\boldsymbol{v} = \boldsymbol{\nabla}\times(\psi(x,y)\hat{\boldsymbol{e}}_z)$

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\psi\hat{\boldsymbol{e}}_z) = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot(\psi(x,y)\hat{\boldsymbol{e}}_z) - \boldsymbol{\nabla}^2(\psi(x,y)\hat{\boldsymbol{e}}_z) = -\hat{\boldsymbol{e}}_z\nabla^2\psi$$

Shear flows: $\boldsymbol{v} = v_x(y)\hat{\boldsymbol{e}}_x \quad \Longrightarrow \boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v} = \hat{\boldsymbol{e}}_z(\partial_x v_y - \partial_y v_x) = -v_x'(y)\hat{\boldsymbol{e}}_z$

Examples:

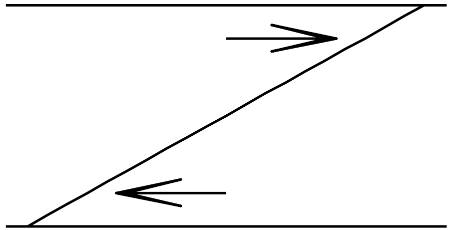


plane Poiseuille flow

$v_x(y) = 1 - y^2$

$\psi = y - \frac{y^3}{3}$

$\omega_z = 2y$



plane Couette flow

$v_x(y) = y$

$\psi = \frac{y^2}{2}$

$\omega_z = -1$

Irrotational flows: $\boldsymbol{\nabla}\times\boldsymbol{v} = 0 \Longrightarrow$ Can show that $\boldsymbol{v} = \boldsymbol{\nabla}\varphi$ $\quad$ (φ = potential)

Irrotational and incompressible flows:

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi = \nabla^2\varphi = 0 \qquad\qquad \boldsymbol{\nabla}\times\boldsymbol{v} = -\hat{\boldsymbol{e}}_{\boldsymbol{z}}\nabla^2\psi = 0$$

$$\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi = v_x\partial_x\psi + v_y\partial_y\psi = \partial_y\psi\partial_x\psi - \partial_x\partial_y\psi = 0$$

φ and ψ are perpendicular harmonic functions

$$\begin{aligned}
\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{v} &= \epsilon^{ijk}\hat{\boldsymbol{e}}_i v_j(\boldsymbol{\nabla}\times\boldsymbol{v})_k \\
&= \epsilon^{ijk}\hat{\boldsymbol{e}}_i v_j\epsilon^{klm}\partial_l v_m \\
&= \epsilon^{kij}\epsilon^{klm}\hat{\boldsymbol{e}}_i v_j\partial_l v_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\hat{\boldsymbol{e}}_i v_j\partial_l v_m \\
&= \hat{\boldsymbol{e}}_i v_j\partial_i v_j - \hat{\boldsymbol{e}}_i v_j\partial_j v_i \\
&= (\hat{\boldsymbol{e}}_i\partial_i)\left(\tfrac{1}{2}v_j v_j\right) - (v_j\partial_j)(\hat{\boldsymbol{e}}_i v_i) \\
&= \boldsymbol{\nabla}\left(\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}\right) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}
\end{aligned}$$

Bernoulli Equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2\right) - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi$$

If ρ is constant, then

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2\right) - \boldsymbol{v}\times\boldsymbol{\omega} = -\boldsymbol{\nabla}\frac{P}{\rho} - \boldsymbol{\nabla}\Phi$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \frac{P}{\rho} + \Phi\right) = \boldsymbol{v}\times\boldsymbol{\omega}$$

If ρ is not constant, but $P = P(\rho)$, i.e. *barotropic flow*, then *enthalpy*

$$\mathcal{H} = \int \frac{dP}{\rho}$$

is well-defined and can be substituted for P/ρ

Bernoulli Equation, continued

Bernouilli equation:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \frac{P}{\rho} + \Phi\right) = \boldsymbol{v}\times\boldsymbol{\omega}$$

$$\underbrace{\frac{\partial \boldsymbol{v}}{\partial t}}_{\text{vanishes if steady}} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \frac{P}{\rho} + \Phi\right) = \underbrace{\boldsymbol{v}\times\boldsymbol{\omega}}_{\text{vanishes if irrotational}}$$

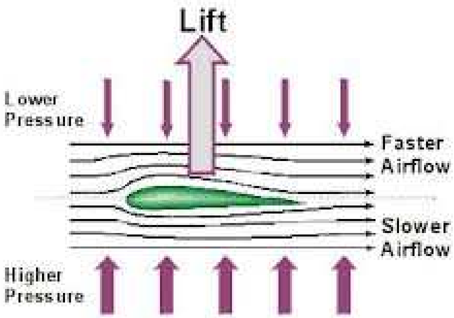
$$\boldsymbol{v}\cdot\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \frac{P}{\rho} + \Phi\right) = \boldsymbol{v}\cdot(\boldsymbol{v}\times\boldsymbol{\omega}) = 0$$

If steady:

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \frac{P}{\rho} + \Phi\right) = 0$$

i.e. $\frac{1}{2}v^2 + \frac{P}{\rho} + \Phi$ is constant along each streamline.
If flow is uniform somewhere (e.g. far away), then constant everywhere.

Aerodynamics



Above wing, flow is faster and pressure is lower, since $\frac{1}{2}v^2 + \frac{P}{\rho} + \Phi$ is constant along each streamline $\Longrightarrow$ lift

Evolution of Vorticity

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2\right) - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi$$

Take curl, assume constant density or $P = P(\rho)$ else include *baroclinic* term $\boldsymbol{\nabla}\times((\boldsymbol{\nabla}P)/\rho) = \boldsymbol{\nabla}(1/\rho)\times\boldsymbol{\nabla}P$

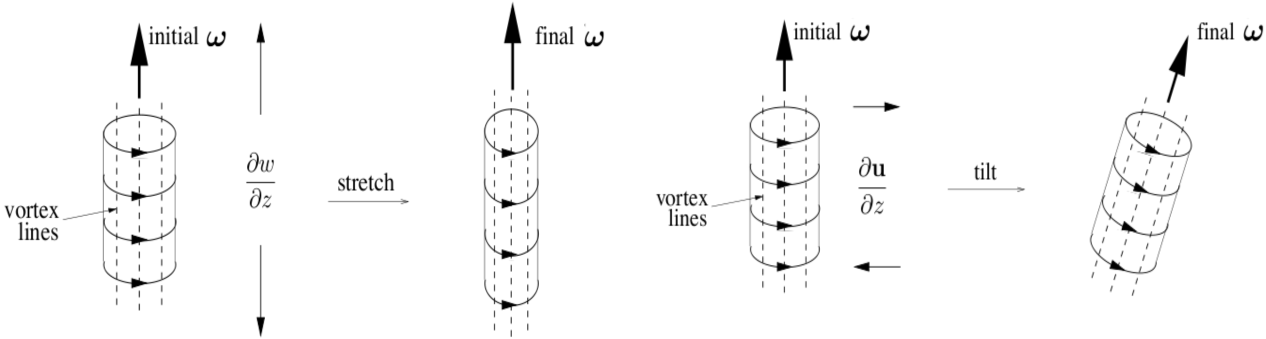
$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0$$

$$\begin{aligned}
\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) &= \epsilon^{ijk}\hat{\boldsymbol{e}}_i\partial_j(\boldsymbol{v}\times\boldsymbol{\omega})_k = \epsilon^{ijk}\hat{\boldsymbol{e}}_i\partial_j\epsilon^{klm}(v_l\omega_m) \\
&= \epsilon^{kij}\epsilon^{klm}\hat{\boldsymbol{e}}_i\partial_j(v_l\omega_m) = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})\hat{\boldsymbol{e}}_i\partial_j(v_l\omega_m) \\
&= \delta^{il}\delta^{jm}\hat{\boldsymbol{e}}_i\partial_j(v_l\omega_m) - \delta^{im}\delta^{jl}\hat{\boldsymbol{e}}_i\partial_j(v_l\omega_m) \\
&= \hat{\boldsymbol{e}}_i\partial_j(v_i\omega_j) - \hat{\boldsymbol{e}}_i\partial_j(v_j\omega_i) \\
&= \hat{\boldsymbol{e}}_i\omega_j\partial_j v_i + \hat{\boldsymbol{e}}_i v_i\partial_j\omega_j - \hat{\boldsymbol{e}}_i v_j\partial_j\omega_i - \hat{\boldsymbol{e}}_i\omega_i\partial_j v_j \\
&= (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} - \underbrace{\boldsymbol{\omega}(\boldsymbol{\nabla}\cdot\boldsymbol{v})}_{\text{0 if incompressible}} \\
&= (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} \\
\frac{\partial \boldsymbol{\omega}}{\partial t} &+ (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}
\end{aligned}$$

$(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} \Longleftrightarrow$ vortex stretching (in direction of $\boldsymbol{\omega}$) and tilting (other directions)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v}$$

$(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} \Longleftrightarrow$ vortex stretching (in direction of $\boldsymbol{\omega}$) and tilting (other directions)



Tensor: multidimensional generalization of vector

Deformation-rate tensor:

$$e_{ij} \equiv \frac{\partial v_i}{\partial x_j}$$

Strain rate or shear rate:

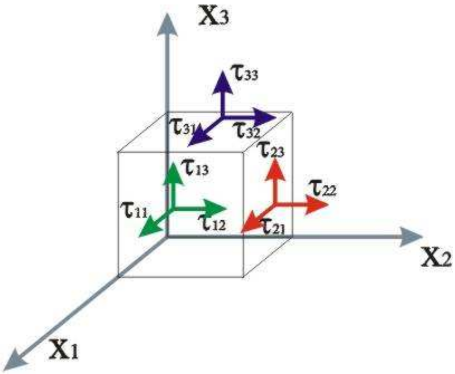
$$\sigma_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}(e_{ij} + e_{ji})$$

Reynolds stress tensor (used in study of turbulence):

$$R_{ij} \equiv v_i v_j \qquad \boldsymbol{R} = \hat{\boldsymbol{e}}_i v_i v_j \hat{\boldsymbol{e}}_j$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{R} &= \partial_i(\hat{\boldsymbol{e}}_i v_i v_j \hat{\boldsymbol{e}}_j)_i = \partial_i(v_i v_j)\hat{\boldsymbol{e}}_j = v_i \partial_i v_j \hat{\boldsymbol{e}}_j + v_j \hat{\boldsymbol{e}}_j \underbrace{\partial_i v_i}_{0 \text{ if incompressible}} \\ &= v_i \partial_i v_j \hat{\boldsymbol{e}}_j = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \end{aligned}$$

Force on a surface: stress tensor $\boldsymbol{\tau}$



$\boldsymbol{\tau}_{ij}$: two directions
normal of surface on which force acts (i)
direction of force produced (j)

Force per area in direction 1 on surface with normal in direction $\boldsymbol{n}$:

$$n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\tau} \cdot \hat{\boldsymbol{e}}_1$$

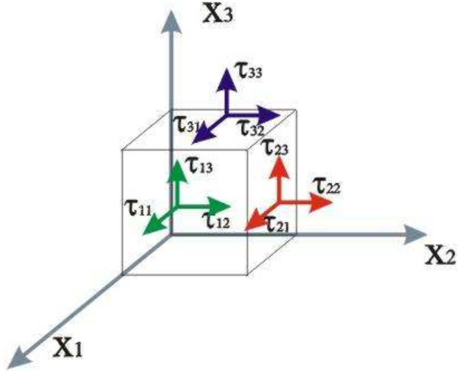
Force per area in direction $\boldsymbol{m}$ on surface with normal in direction 1:

$$\tau_{11}m_1 + \tau_{12}m_2 + \tau_{13}m_3 = \hat{\boldsymbol{e}}_1 \cdot \boldsymbol{\tau} \cdot \hat{\boldsymbol{m}}$$

Generally, force per area in direction $\hat{\boldsymbol{n}}$ on surface with normal $\hat{\boldsymbol{m}}$:

$$\begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

Forces on surfaces $\Longrightarrow$ forces on volumes



Sum surfaces to obtain force on volume (dx_1, dx_2, dx_3) in direction 1 =

$$
\begin{aligned}
&\tau_{11}(x_1 + dx_1, x_2, x_3)dx_2dx_3 - \tau_{11}(x_1, x_2, x_3)dx_2dx_3 \\
&+\tau_{21}(x_1, x_2 + dx_2, x_3)dx_1dx_3 - \tau_{21}(x_1, x_2, x_3)dx_1dx_3 \\
&+\tau_{31}(x_1, x_2, x_3 + dx_3)dx_2dx_3 - \tau_{31}(x_1, x_2, x_3)dx_2dx_3 \\
&= \frac{\partial \tau_{11}}{\partial x_1}dx_1dx_2dx_3 + \frac{\partial \tau_{21}}{\partial x_2}dx_1dx_2dx_3 + \frac{\partial \tau_{31}}{\partial x_3}dx_1dx_2dx_3 = \frac{\partial \tau_{i1}}{\partial x_i}dV
\end{aligned}
$$

Sum forces in three directions: $\hat{\boldsymbol{e}}_j \frac{\partial \tau_{ij}}{\partial x_i}dV = \boldsymbol{\nabla} \cdot \boldsymbol{\tau}\, dV$

Determining $\boldsymbol{\tau}$: constitutive relation

Fourier's law is an example of a constitutive relation:
Heat flux by conduction is proportional to negative gradient of temperature T

$$\boldsymbol{q} = -\kappa \boldsymbol{\nabla} T$$

Seek an analogous relation to use in momentum equations. Assumptions:
1) For fluid at rest, stress is isotropic and contains only normal components:

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij} \qquad \text{where } \sigma = 0 \text{ if } \boldsymbol{v} = 0$$

2) $\boldsymbol{\sigma}$ is linearly related to deformation-rate tensor $\boldsymbol{e}$:
this defines a *Newtonian fluid*:

$$\sigma_{ij} = C_{ijkl} e_{kl} \qquad \Longrightarrow \qquad \text{81 elements } C_{ijkl} \text{ !}$$

3) Solid-body rotation does not give rise to shear stresses:

$$e_{ij} = \frac{\partial v_i}{\partial x_j} = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \underbrace{\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)}_{\text{rotation}} \quad \Longrightarrow \boldsymbol{\sigma} \text{ symmetric}$$

4) Independent of orientation of coordinate system (isotropy):

$$\sigma_{ij} = \mu(e_{ij} + e_{ji}) + \lambda\delta_{ij}e_{kk} = \mu(e_{ij} + e_{ji}) + \lambda\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v}$$

where μ is called *dynamic viscosity* and λ the *second viscosity coefficient*

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\tau} &= \partial_i\hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_i\tau_{ij}\hat{\boldsymbol{e}}_j) = \partial_i\tau_{ij}\hat{\boldsymbol{e}}_j \\
&= \partial_i\left(-p\delta_{ij} + \mu(e_{ij} + e_{ji}) + \lambda\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)\hat{\boldsymbol{e}}_j \\
\partial_i(-p\delta_{ij} + \lambda\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{v})\hat{\boldsymbol{e}}_j &= \hat{\boldsymbol{e}}_i\partial_i(-p + \lambda\boldsymbol{\nabla}\cdot\boldsymbol{v}) = -\boldsymbol{\nabla}p + \boldsymbol{\nabla}(\lambda\boldsymbol{\nabla}\cdot\boldsymbol{v}) \\
\frac{\partial}{\partial x_i}(\mu(e_{ij} + e_{ji}))\hat{\boldsymbol{e}}_j &= \frac{\partial}{\partial x_i}\left(\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)\hat{\boldsymbol{e}}_j \\
&= \left(\mu\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) + \frac{\partial \mu}{\partial x_i}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\right)\hat{\boldsymbol{e}}_j \\
&= \mu\boldsymbol{\nabla}^2\boldsymbol{v} + \mu\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{\nabla}\mu)\cdot(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T)
\end{aligned}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v} + \boldsymbol{\nabla}(\lambda\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \mu\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{\nabla}\mu)\cdot(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^T)$$

If ρ is constant ($\boldsymbol{\nabla}\cdot v = 0$) and μ is constant ($\boldsymbol{\nabla}\mu = 0$) then

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau} = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v}$$

Navier-Stokes Equations

Incompressible and Newtonian fluids: $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ and

$$\rho\frac{D\boldsymbol{v}}{Dt} = \rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{v}$$

$$\frac{D\boldsymbol{v}}{Dt} = \left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}^2\boldsymbol{v}$$

Kinematic viscosity = $\nu \equiv \mu/\rho$

Euler equations: $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ and

$$\frac{D\boldsymbol{v}}{Dt} = \left(\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) = -\frac{1}{\rho}\boldsymbol{\nabla}p$$

without viscosity, fluid layers can slide over one another $\Longrightarrow$ flow not unique

Plane shear flows:

$$\boldsymbol{v} = v(y)\hat{\boldsymbol{e}}_x \Longrightarrow (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = 0 \text{ and } \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$$

$$\boldsymbol{\nabla} p = \mu \boldsymbol{\nabla}^2 \boldsymbol{v} \Longrightarrow \begin{cases} \partial_x p = \mu v''(y) \\ \left.\begin{array}{c} \partial_y p = 0 \\ \partial_z p = 0 \end{array}\right\} \Longrightarrow p = p(x) \end{cases}$$

$$\underbrace{\partial_x p}_{\text{function of } x} = \underbrace{\mu v''}_{\text{function of } y}$$

$$\partial_x p = -G \Longrightarrow p = Gx + p_0$$

$$\mu v'' = -G \Longrightarrow v = -\frac{G}{2\mu}y^2 + ay + b$$

Plane Couette flow: $G = 0$ and $v(\pm 1) = \pm 1$:

$$\left.\begin{array}{r} 1 = v(1) = a + b \\ -1 = v(-1) = -a + b \end{array}\right\} \Longrightarrow a = 1 \text{ and } b = 0 \Longrightarrow v = y$$

Plane Poiseuille flow: $G \neq 0$ and $v(\pm 1) = 0$:

$$\left.\begin{array}{l} 0 = v(1) = -\frac{G}{2\mu} + a + b \\ 0 = v(-1) = -\frac{G}{2\mu} - a + b \end{array}\right\} \Longrightarrow a = 0 \text{ and } b = \frac{G}{2\mu} \Longrightarrow v = \frac{G}{2\mu}(1 - y^2)$$

Taylor-Couette flow

$$\boldsymbol{v} = v(R)\hat{\boldsymbol{e}}_\phi \Longrightarrow \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \text{ and } (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{v^2}{R}\hat{\boldsymbol{e}}_R$$

$$\left.\begin{aligned} -\rho\frac{v^2}{R} &= -\partial_R p \\ 0 &= -\partial_\phi p + \mu\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R}Rv \\ 0 &= -\partial_z p \end{aligned}\right\} \Longrightarrow p = f(R)$$

$$\begin{aligned} \mu\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R}Rv &= 0 \\ \frac{1}{R}\frac{\partial}{\partial R}Rv &= a \\ \frac{\partial}{\partial R}Rv &= aR \\ Rv &= \frac{a}{2}R^2 + b \\ v &= \frac{a}{2}R + \frac{b}{R} \end{aligned}$$

a, b determined by boundary conditions $v(R_i) = \Omega_i R_i$ and $\partial_R p = \rho v^2/R$