Hydrodynamics

Class 12

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Dimensional Analysis

Drag on an object

diameter d, velocity U_∞, density ρ, dynamic viscosity μ, drag force F_x

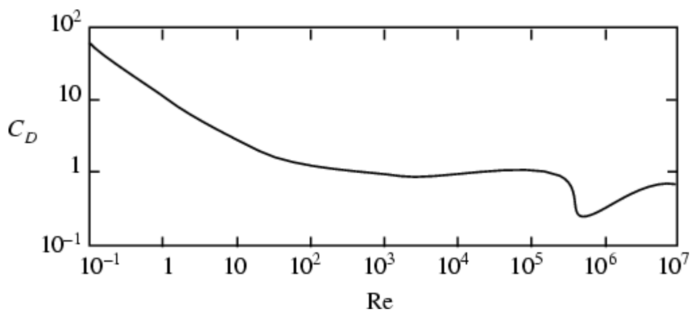
	F_x	ρ	μ	U_∞	d
M	1	1	1		1
L	1	−3	−1	1	
T	−2		−1	−1	

5 dimensional quantities − 3 dimensional units = 2 nondimensional
Among infinite number of possibilities, choose one unit containing F_x and another containing μ:

$$C_D \equiv \frac{F_x}{\rho U_\infty^2} \qquad R \equiv \frac{U_\infty d \rho}{\mu}$$

$$C_D = \mathcal{F}(R)$$

$$\frac{F_x}{\rho U_\infty^2} = \mathcal{F}\left(\frac{U_\infty d\rho}{\mu}\right)$$



Shock wave produced by nuclear explosion

Energy calculated by British hydrodynamicist G.I. Taylor using photos

Energy E, time t, radius r, atmospheric density ρ

	E	t	r	ρ
M	1			1
L	2		1	-3
T	-2	1		

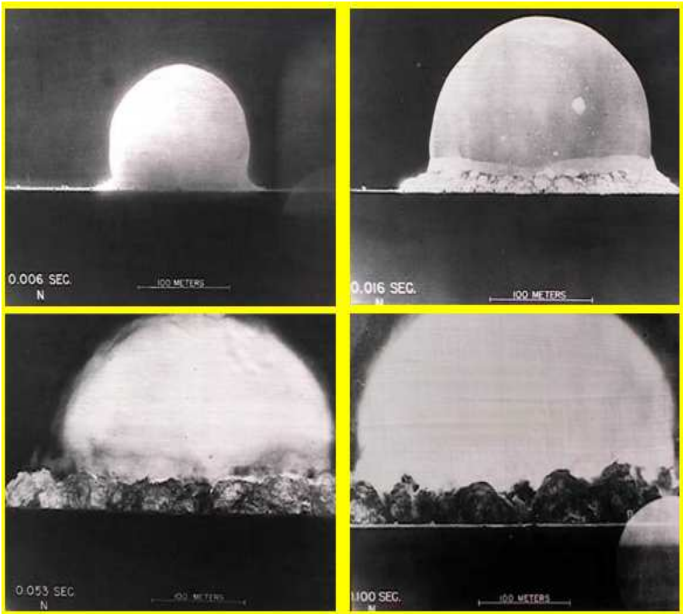
4 dimensional quantities $-$ 3 dimensional units = 1 nondimensional

$$\frac{[E]}{[\rho]} = \frac{ML^2}{T^2}\frac{L^3}{M} = \frac{L^5}{T^2}$$

$$\frac{E}{\rho r^5 t^2} = C \Longrightarrow Cr^5 = \frac{E}{\rho t^2} \Longrightarrow r = \left(\frac{E}{C\rho t^2}\right)^{1/5}$$

Taylor used *other arguments* **to conclude that** $C \sim 1$

$\Longrightarrow$ **E= 22 kilotons of TNT** **Actual value E=20 kilotons of TNT**


0.006 SEC.
N
100 METERS
0.016 SEC.
100 METERS
0.053 SEC
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100 METERS
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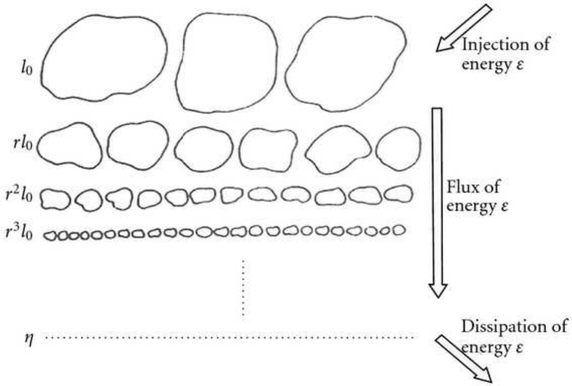
100 METERS

Turbulence

Richardson Cascade and Kolmogorov Theory



I. Injection

Agitate incompressible fluid at large velocity and length scales u_I **and** ℓ_I

The Reynolds number associated with this large-scale forcing is

$$R_I = \frac{u_I \ell_I}{\nu} \gg 1$$

Rate of energy/mass injection into the fluid:

$$[\epsilon] = \frac{\text{Energy}/\text{Time}}{\text{Mass}} = \frac{(ML^2/T^2)/T}{\text{M}} = \frac{L^2}{T^3}$$

Energy injection takes place at scale ℓ_I **and does not depend on viscosity:**

$$\epsilon \sim \frac{u_I^3}{\ell_I}$$

Interpretation: **Energy/mass** $\sim u_I^2$ **Eddy turnover time** $\sim \ell_I/u_I$

$$\frac{\textbf{Energy/mass}}{\textbf{turnover time}} \sim \frac{u_I^2}{\ell_I/u_I} = \frac{u_I^3}{\ell_I}$$

II. Transfer: Inertial range

Energy transfer takes place at intermediate scales via triadic interactions $\boldsymbol{k} + \boldsymbol{k}' \to \boldsymbol{k} + \boldsymbol{k}'$ **via**

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}} = e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}$$

u_ℓ **determined only by** ϵ **and** ℓ **so**

$$\frac{L}{T} = \left(\frac{L^2}{T^3}\right)^{\alpha} L^{\beta}$$

$$u_\ell \sim \epsilon^{1/3}\ell^{1/3} = (\epsilon\ell)^{1/3}$$

Kolmogorov 1941 article: $-5/3$ law

Hypothesis of isotropy (directional invariance) for energy per unit mass:

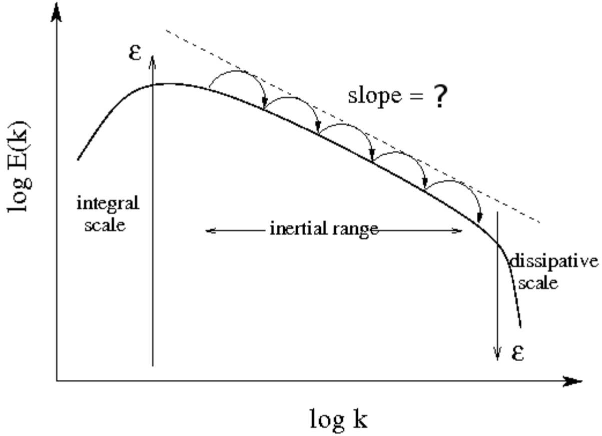
$$E_{\text{tot}} = \int d^3x \tilde{E}(\boldsymbol{x}) = \int d^3k \ \hat{E}(\boldsymbol{k}) = \int_0^\infty dk \left[\int_{\substack{\text{spherical}\\\text{surface}}} k^2 \hat{E}(k,\theta,\phi) \right] \equiv \int_0^\infty dk \ E(k)$$

$$[E_{\text{tot}}] = \frac{[L^2]}{[T^2]} \Longrightarrow [E(k)] = [L]\frac{[L^2]}{[T^2]} = \frac{[L^3]}{[T^2]}$$

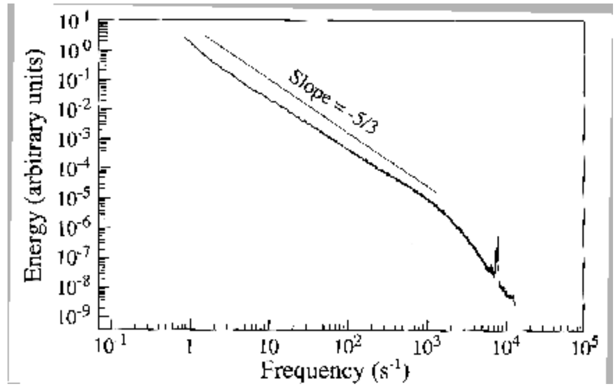
Hypothesis that flow at any intermediate length scale ℓ is described only by energy injection rate ϵ

$$\begin{array}{ccccc} \dfrac{[L^3]}{[T^2]} & = & \left(\dfrac{[L^2]}{[T^3]}\right)^{2/3} [L]^{3-4/3} = & \left(\dfrac{[L^2]}{[T^3]}\right)^{2/3} & [L]^{5/3} \\ E(k) & \sim & & \epsilon^{2/3} & k^{-5/3} \end{array}$$

Kolmogorov cascade



Schematic depiction of cascade



Data from ONERA wind tunnel
Anselmet, Gagne, Hopfinger, Antonia
J. Fluid Mech. 1984

Must resolve length scales over many orders of magnitude
$\Longrightarrow$ major challenge for experiment and numerical simulation

III. Dissipation

Dissipation takes place on scale ℓ_d for which

$$
\begin{aligned}
1 \;&\sim\; R_d \equiv \frac{\ell_d u_d}{\nu} \sim \frac{\ell_d\,(\epsilon\,\ell_d)^{1/3}}{\nu} \sim \frac{\ell_d^{4/3}\epsilon^{1/3}}{\nu} \\
\ell_d^{4/3} \;&\sim\; \frac{\nu}{\epsilon^{1/3}} \Longrightarrow \ell_d \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \\
u_d \;&\sim\; (\epsilon\,\ell_d)^{1/3} = \left(\epsilon\,\frac{\nu^{3/4}}{\epsilon^{1/4}}\right)^{1/3} = (\epsilon\nu)^{1/4} \\
t_d \;&\sim\; \frac{\ell_d}{u_d} = \frac{(\nu^3/\epsilon)^{1/4}}{(\epsilon\nu)^{1/4}} = \left(\frac{\nu}{\epsilon}\right)^{1/2}
\end{aligned}
$$

ℓ_d, u_d, t_d are called Kolmogorov or dissipation length, velocity, and time

Increasing or decreasing ν changes dissipation length scale ℓ_d

Examples (from class notes of M.E. Brachet)

1) Turbulent disperson: motion of particles separated by ℓ

$$\frac{d\ell}{dt} \sim u_\ell \sim (\epsilon\ell)^{1/3}$$
$$\int \frac{d\ell}{\ell^{1/3}} \sim \epsilon^{1/3} \int dt$$
$$\ell^{2/3} - \ell_0^{2/3} \sim \epsilon^{1/3}(t - t_0)$$

At long times:

$$\ell \sim \epsilon^{1/2}(t - t_0)^{3/2}$$

Law established by Richardson in 1926 (before Kolmogorov 1941).

2) Terminal velocity of object with length scale ℓ:
frictional force = gravitational force = Mg
Laminar case: frictional force = $\rho\ell\nu u$

$$\rho\ell\nu u \sim Mg \Longrightarrow u_{\rm lam} \sim \frac{Mg}{\rho\ell\nu}$$

Turbulent case: frictional force is independent of ν

$$[\rho] = \frac{[M]}{[L^3]} \qquad [\ell] = [L] \qquad [u] = \frac{[L]}{[T]}$$

$$\left(\frac{[M]}{[L^3]}\right)^\alpha \left(\frac{[L]}{[T]}\right)^\beta [L]^\gamma = \frac{[ML]}{[T^2]}$$

$$\alpha = 1 \quad \beta = 2 \quad \gamma = 2$$

$$\rho u^2 \ell^2 = Mg \Longrightarrow u^2 = \frac{Mg}{\rho \ell^2} \Longrightarrow u_{\rm turb} = \left(\frac{Mg}{\rho \ell^2}\right)^{1/2}$$

Consider a falling man:

$$\rho_{\rm air} = 1kg/m^3 \qquad \nu_{\rm air} = 10^{-5} m^2/s \qquad M = 100kg \qquad g = 10m/s^2 \qquad \ell = 1m$$

$$u_{\rm lam} =\sim \frac{Mg}{\rho \ell \nu} = \frac{100kg \times 10m/s^2}{1kg/m^3 \times 1m \times 10^{-5} m^2/s} = 10^8 m/s$$

$$u_{\rm turb} \sim \left(\frac{Mg}{\rho\, \ell^2}\right)^{1/2} = \left(\frac{100kg \times 10m/s^2}{1kg/m^3 \times (1m)^2}\right)^{1/2} = 31m/s$$

He opens his parachute: $\ell = 10m$

$$u_{\rm turb} \sim \left(\frac{Mg}{\rho\, \ell^2}\right)^{1/2} = \left(\frac{100kg \times 10m/s^2}{1kg/m^3 \times (10m)^2}\right)^{1/2} = 3.1m/s$$

Computational Fluid Dynamics

$$\frac{R_I}{R_d} = \frac{u_I \ell_I}{u_d \ell_d} = \frac{(\epsilon \ell_I)^{1/3} \ell_I}{(\epsilon \ell_d)^{1/3} \ell_d} = \left(\frac{\ell_I}{\ell_d}\right)^{4/3}$$

To increase R by factor 10, must multiply ℓ by factor $10^{3/4} = 5.6$, must number of total gridpoints $\ell \times \ell \times \ell$ by factor $10^{9/4} = 178$.

	Reynolds	**Resolution**
Kaneda (Japan, 2003) largest supercomputer	10^5	$(4096)^3$
Earth's atmospheric boundary layer	300×10^5	$(300^{3/4} \times 4096)^3 = (72 \times 4096)^3$ $= (300\,000)^3$

Moore's law: size of largest possible computation doubles every 2 years

$$\begin{aligned}
2^n &= \left(\frac{300\,000}{4096}\right)^3 \\
n &= 3\,(\ln_2 300\,000 - \ln_2 4096) = 3(18.2 - 12) = 18.6 \\
18.6 \times 2 \textbf{ years} &= 37.2 \textbf{ years} \\
2003 + 37 &= 2040
\end{aligned}$$

Gain factor in R of 1.3 every 2 years.

Two-dimensional turbulence

Slightly different phenomenology:
both forward and inverse energy cascade
(both smaller and larger structures are generated)

Enstrophy cascade

Statistical Studies

Systematic and random variations in time and space

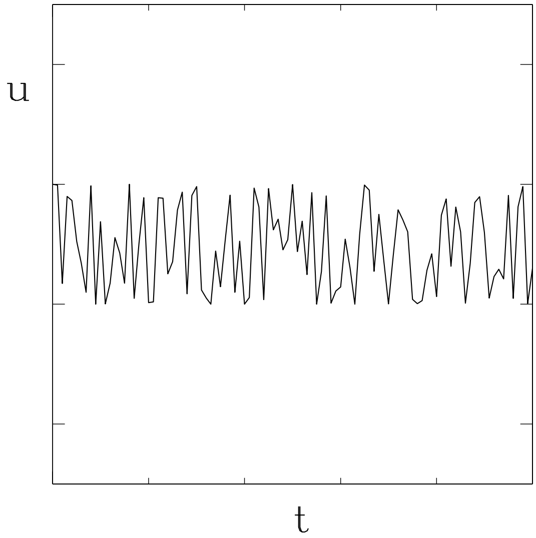
Repeat experiment $i = 1, 2, \ldots N$ **times. Ensemble averages:**

$$\textbf{mean} = \bar{\boldsymbol{u}}(\boldsymbol{x}, t) \equiv \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{u}_i(\boldsymbol{x}, t)$$

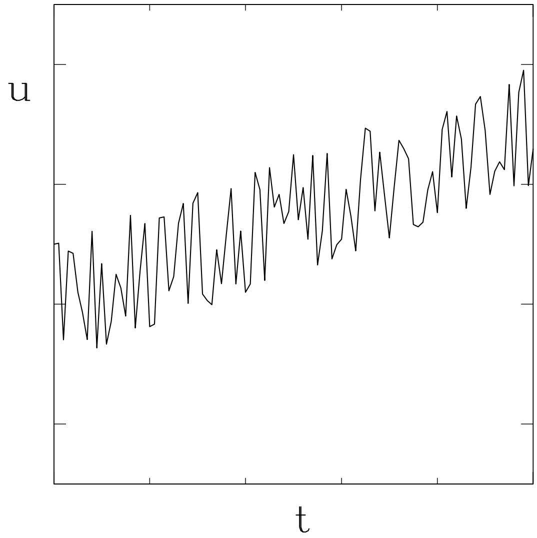
$$\textbf{variance} = \overline{\boldsymbol{u}^2}$$

$$\textbf{root-mean-square} = \overline{\boldsymbol{u}^2}^{1/2}$$

$$\textbf{standard deviation} = \overline{(\boldsymbol{u} - \bar{\boldsymbol{u}})^2}^{1/2}$$



statistically stationary

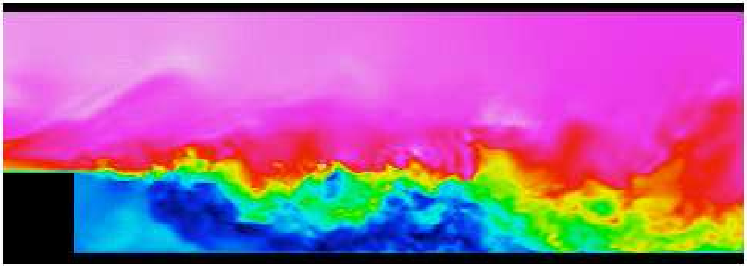


not statistically stationary

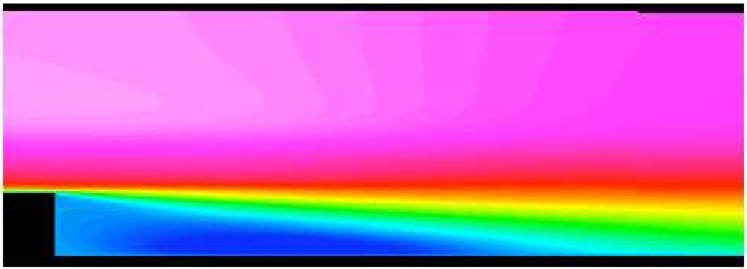
For statistically stationary turbulence, ensemble averages are not functions of time: can substitute temporal averages.

For statistically homogeneous turbulence, ensemble averages are not functions of space: can substitute spatial averages.

Flow behind backward facing step



instantaneous flow



mean flow

Take ensemble or time average, not spatial average

Reynolds Average Navier-Stokes (RANS)

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}' \qquad \overline{\boldsymbol{u}'} = 0 \qquad \bar{\bar{\boldsymbol{u}}} = \bar{\boldsymbol{u}}$$

$$\partial_t(\bar{\boldsymbol{u}} + \boldsymbol{u}') + ((\bar{\boldsymbol{u}} + \boldsymbol{u}') \cdot \boldsymbol{\nabla})(\bar{\boldsymbol{u}} + \boldsymbol{u}') = -\boldsymbol{\nabla}(\bar{p} + p')/\rho + \nu\Delta(\bar{\boldsymbol{u}} + \boldsymbol{u}')$$

$$\boldsymbol{\nabla} \cdot (\bar{\boldsymbol{u}} + \boldsymbol{u}') = 0$$

Apply average, using $\overline{\boldsymbol{u}'} = 0$:

$$\partial_t\bar{\boldsymbol{u}} + \overline{((\bar{\boldsymbol{u}} + \boldsymbol{u}') \cdot \boldsymbol{\nabla})(\bar{\boldsymbol{u}} + \boldsymbol{u}')} = -\boldsymbol{\nabla}\bar{p}/\rho + \nu\Delta\bar{\boldsymbol{u}} \qquad \boldsymbol{\nabla} \cdot \bar{\boldsymbol{u}} = 0$$

$$\begin{aligned}
\overline{(\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}}} &= (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} \\
\overline{(\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\boldsymbol{u}'} &= (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\overline{\boldsymbol{u}'} = 0 \\
\overline{(\boldsymbol{u}' \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}}} &= (\overline{\boldsymbol{u}'} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} = 0 \\
\overline{(\boldsymbol{u}' \cdot \boldsymbol{\nabla})\boldsymbol{u}'} &= \textbf{new term}
\end{aligned}$$

$$\partial_t\bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} + \underbrace{\overline{(\boldsymbol{u}' \cdot \boldsymbol{\nabla})\boldsymbol{u}'}}_{\textbf{closure problem}} = -\boldsymbol{\nabla}\bar{p}/\rho + \nu\Delta\bar{\boldsymbol{u}}$$

$$(\boldsymbol{u}' \cdot \boldsymbol{\nabla})\boldsymbol{u}' = u'_j \partial_j u'_i \hat{\boldsymbol{e}}_i = \partial_j (u'_i u'_j)\hat{\boldsymbol{e}}_i - u'_i \underbrace{\partial_j u'_j}_{\boldsymbol{\nabla} \boldsymbol{\cdot} u' = 0} \hat{\boldsymbol{e}}_i = \partial_j (u'_i u'_j)\hat{\boldsymbol{e}}_i$$

Reynolds stress tensor:

$$\tau'_{ij} \equiv -\rho \overline{u'_i u'_j}$$

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \tau' = -\rho \partial_j \tau'_{ij} \hat{\boldsymbol{e}}_i = \rho \partial_j \overline{(u'_i u'_j)} \hat{\boldsymbol{e}}_i$$

Recall usual stress tensor:

$$\tau_{ij} = -p\delta_{ij} + \rho\nu(\partial_i u_j + \partial_j u_i)$$

$$\begin{aligned} \partial_t \bar{\boldsymbol{u}} + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{\boldsymbol{u}} &= \frac{1}{\rho} \boldsymbol{\nabla} \boldsymbol{\cdot} (\tau + \tau') \\ &= -\boldsymbol{\nabla} \bar{p}/\rho + \nu \Delta \bar{\boldsymbol{u}} + \boldsymbol{\nabla} \boldsymbol{\cdot} \tau'/\rho \end{aligned}$$

Need to estimate τ'

Prandtl and Taylor: Eddy Viscosity and Mixing Length

Consider mean flow $\bar{u}(y)\hat{e}_x$ **so that** $\tau_{ij} = -p\delta_{ij} + \rho\nu(\partial_i u_j + \partial_j u_i)$

The only non-zero off-diagonal component is

$$\tau_{xy} = \rho\nu \frac{d\bar{u}}{dy}$$

In a gas $\nu \sim a\lambda$

where a **is r.m.s. speed of molecular motion and** λ **is mean-free-path, i.e. average distance traveled by molecule between collisions.**

By analogy, **take Reynolds stress tensor to be**

$$\tau'_{xy} \equiv -\rho\,\overline{u'v'} = \rho\nu_{\mathrm{t}} \frac{d\bar{u}}{dy}$$

estimating $\nu_{\mathrm{t}} \sim u_0 \ell_m$

turbulent viscosity ν_{t} **based on** *typical velocity* u_0 **and**
mixing length ℓ_m **traveled in** y **before transferring momentum**

Turbulent Planar Jet

$$\begin{aligned}(\bar{u}\partial_x + \bar{v}\partial_y)\,\bar{u} &= -\frac{1}{\rho}\partial_x\bar{p} + \nu(\partial_x^2 + \partial_y^2)\bar{u} - \partial_x\overline{(u'u')} - \partial_y\overline{(u'v')} \\ (\bar{u}\partial_x + \bar{v}\partial_y)\,\bar{v} &= -\frac{1}{\rho}\partial_y\bar{p} + \nu(\partial_x^2 + \partial_y^2)\bar{v} - \partial_x\overline{(u'v')} - \partial_y\overline{(v'v')}\end{aligned}$$

As done for boundary layer theory, hypothesize

$$\bar{u} \sim u_0(x) \gg \bar{v} \sim v_0 \qquad \partial_y \sim \delta(x) \gg \partial_x \sim 1$$

Neglect molecular viscosity (no walls) and remove pressure gradient $d\bar{p}/dx$

Mixing length/turbulent viscosity:

$$\nu_{\mathrm{t}} = u_0(x)\delta(x)$$

$$\begin{aligned}\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} &= -\frac{\partial}{\partial y}\overline{(u'v')} = -\frac{\partial}{\partial y}\left(\nu_{\mathrm{t}}(x)\frac{\partial\bar{u}}{\partial y}\right) = -\frac{\partial}{\partial y}\left(u_0(x)\delta(x)\frac{\partial\bar{u}}{\partial y}\right) \\ &= -u_0(x)\delta(x)\frac{\partial^2\bar{u}}{\partial y^2}\end{aligned}$$

Wall-bounded shear flow

Near wall, velocity is independent of far-stream velocity and geometry
Define *friction velocity***:**

$$u_* \equiv \sqrt{\nu \left. \frac{d\bar{u}}{dy} \right|_{y=0}}$$

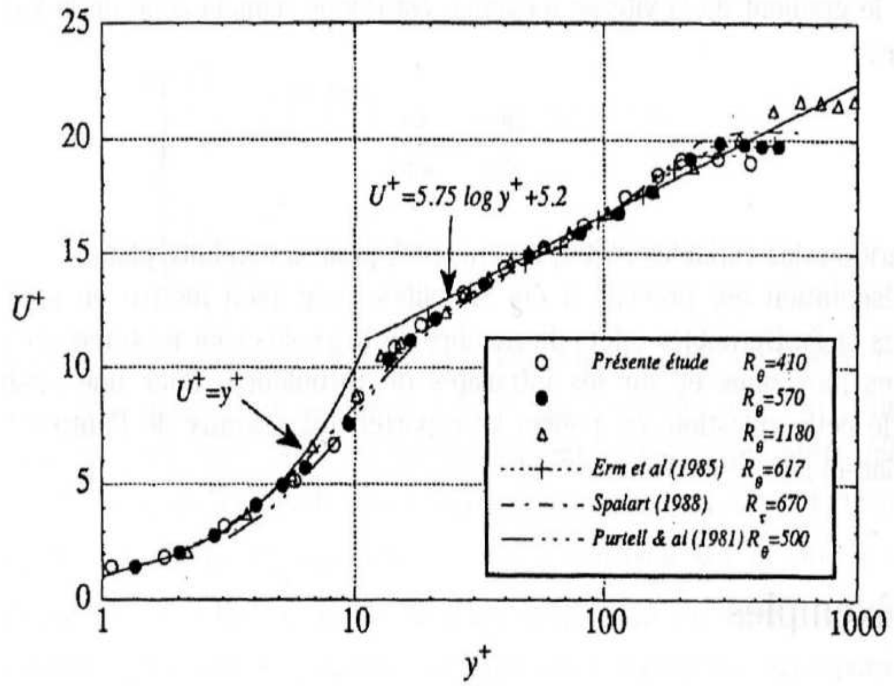
$\bar{u}$ **depends on friction velocity** u_***, viscosity** ν**, and distance from wall** y
4 dimensional $-$ **2 units = 2 dimensionless** $\Longrightarrow$ *Law of the Wall***:**

$$\frac{\bar{u}}{u_*} = f\left(\frac{yu_*}{\nu}\right) = (\text{ very close to wall }) = \frac{yu_*}{\nu} \Longleftrightarrow u_+ = y_+$$

Further away from wall, $d\bar{u}/dy$ **depends on** u_* **and** y **(not** ν**):**
3 dimensional $-$ **2 units = 1 dimensionless** $\Longrightarrow$ *Logarithmic Law***:**

$$\frac{y}{u_*}\frac{d\bar{u}}{dy} = \frac{1}{k} \Longrightarrow \frac{d\bar{u}}{dy} = \frac{u_*}{ky}$$

$$\bar{u} = \frac{u_*}{k}\ln y + C \Longrightarrow \frac{\bar{u}}{u_*} = \frac{\ln y}{k} + C$$



from Laadhari et al.

Large Eddy Simulation (LES) for numerical simulations

$$\begin{aligned}\frac{\partial \bar{u}_i}{\partial t} + (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\bar{u}_i &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{u}_i}{\partial x_j} - \overline{(u_i' u_j')}\right) \\ &\approx -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{u}_i}{\partial x_j} + \nu_{\mathrm{t}} \frac{\partial \bar{u}_i}{\partial x_j}\right)\end{aligned}$$

Smagorinsky Model (1963):

$$\nu_{\mathrm{t}} \sim (\Delta x)^2 \left[\sum_{i,j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)^2\right]^{1/2}$$

Other engineering turbulence models include other fields and equations:

$k - \epsilon$ **model**

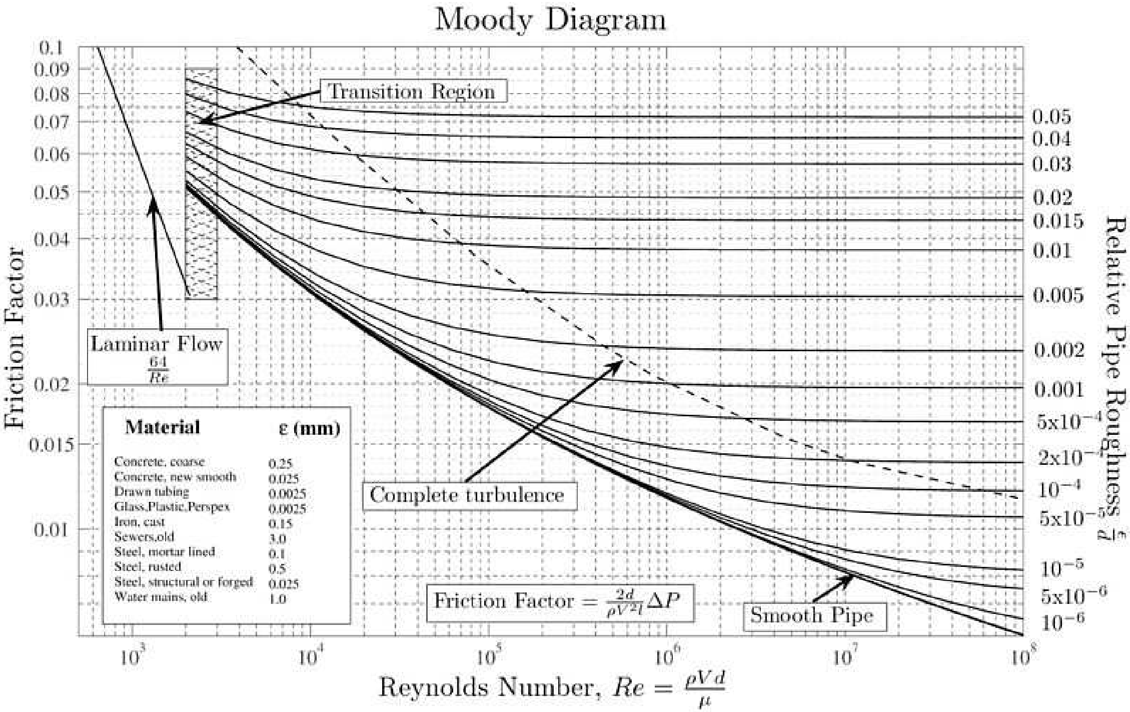
$k - \omega$ **model**

Other models:

Eddy-Damped-Quasi-Normal-Markovian (EDQNM)

Wavelet filtering

Transition to turbulence in a pipe: Moody diagram



Understanding is mainly empirical

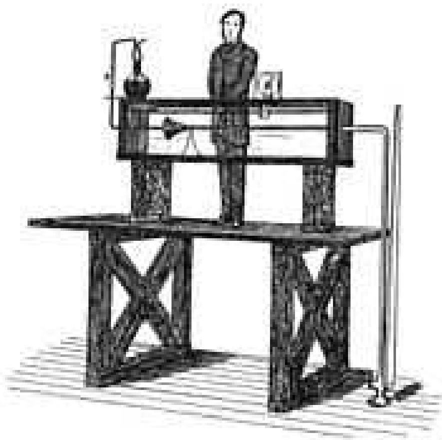
Reynolds’ 1883 experiment in Manchester

PHILOSOPHICAL
TRANSACTIONS:

An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels

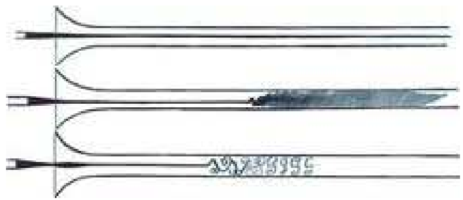
Osborne Reynolds

Phil. Trans. R. Soc. Lond. 1883 **174**, doi: 10.1098/rstl.1883.0029



Reynolds’ assistant

Osborne Reynolds



three outcomes

Peixinho and Mullin's 2006 experiment in Manchester

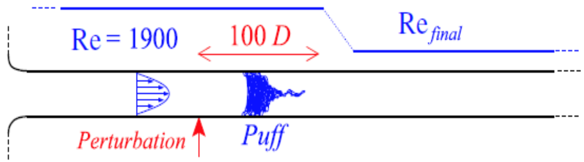
PRL **96,** 094501 (2006) PHYSICAL REVIEW LETTERS week ending 10 MARCH 2006

Decay of Turbulence in Pipe Flow

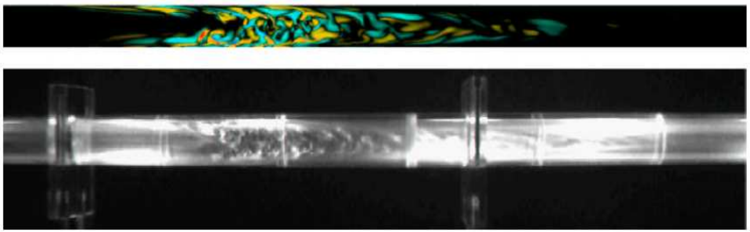
J. Peixinho and T. Mullin
School of Physics and Astronomy, The University of Manchester, Manchester M13 9PL, United Kingdom
(Received 15 November 2005; published 8 March 2006)

Jorge Peixinho

Tom Mullin

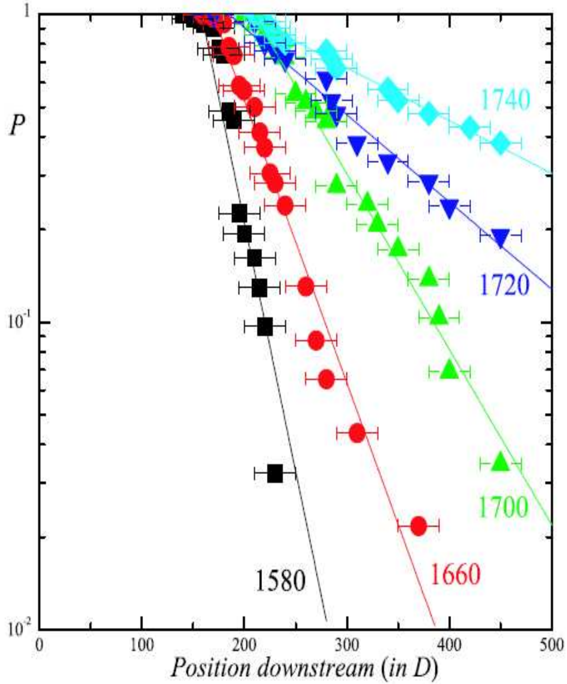


experimental protocol

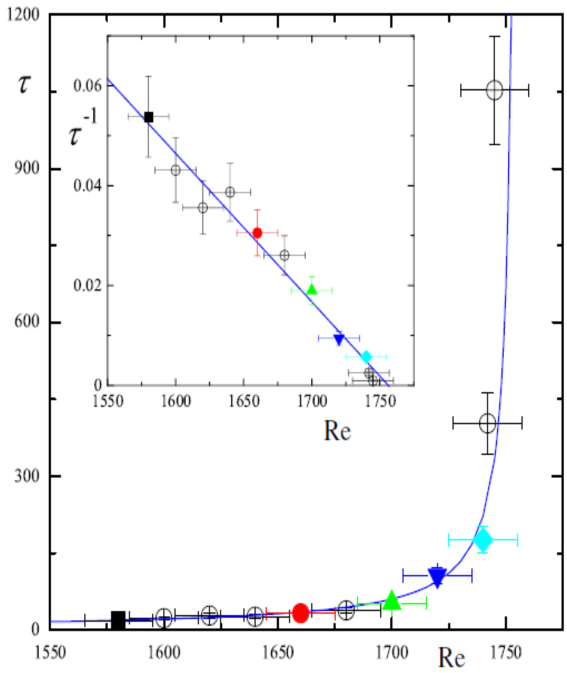


Numerical and experimental turbulent puff

Probability of turbulent survival slope increases with R

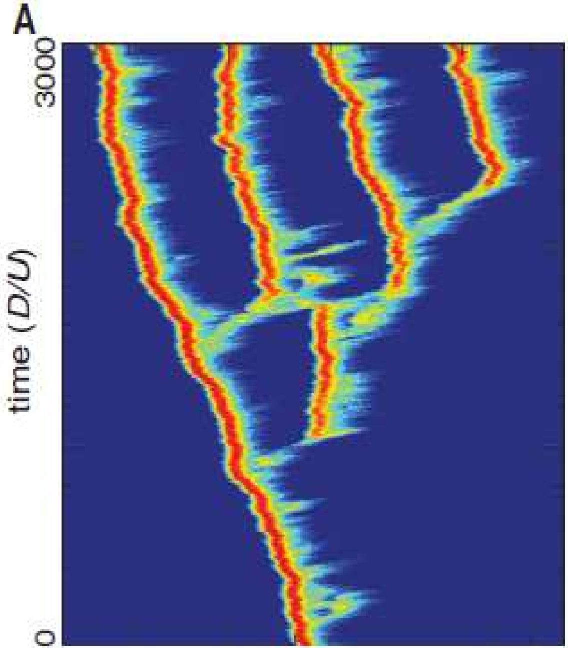


Time-scale for turbulent survival seems to diverge at finite R?

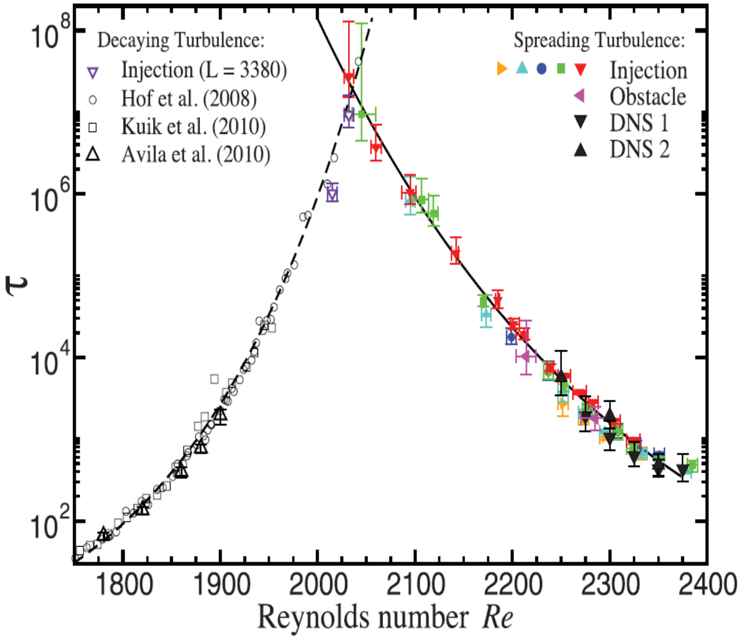


$$Re_c = 1750????$$

Avila, Moxey, de Lozar, Avila, Barkley, Hof, *Science, 2012***:**
competition with puff splitting



Avila, Moxey, de Lozar, Avila, Barkley, Hof, *Science, 2012***:**
Crossover between splitting time and lifetime curves is threshold:
R=2040.



Wall-bounded shear flows near transition contain structures: longitudinal streaks and vortices