Hydrodynamics

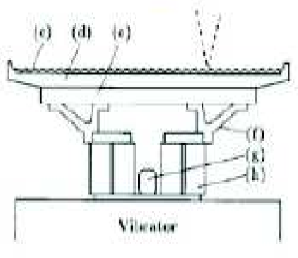
Class 11

Laurette TUCKERMAN

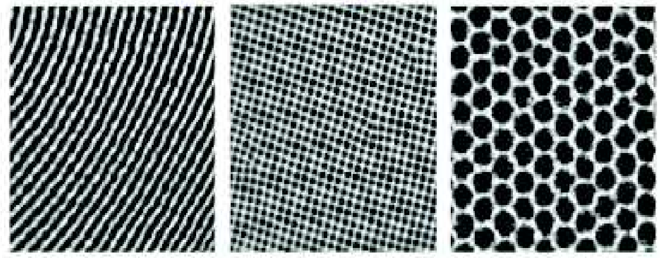
laurette@pmmh.espci.fr

Faraday instability

Faraday 1831

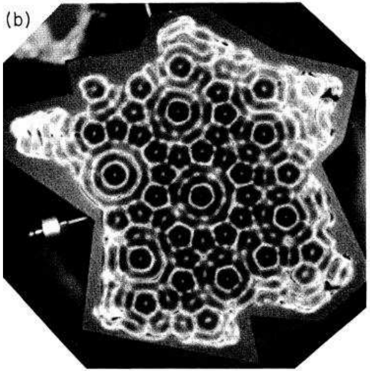


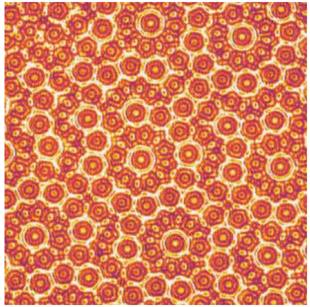
Crystalline patterns



Faraday (1831): Vertical vibration of fluid layer $\Longrightarrow$ stripes, squares, hexagons

In 1990s: first fluid-dynamical quasicrystals:



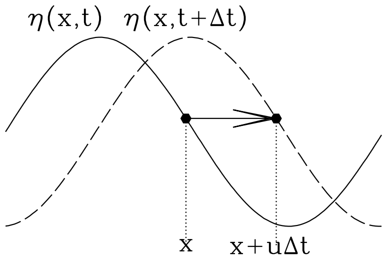


Edwards & Fauve
J. Fluid Mech. (1994)

Kudrolli, Pier & Gollub
Physica D (1998)

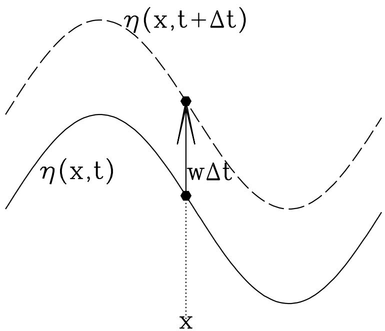
Effect of horizontal motion

$$\zeta(x + u\Delta t, t + \Delta t) = \zeta(x, t)$$
$$\zeta(x,t) + \frac{\partial \zeta}{\partial t}\Delta t + \frac{\partial \zeta}{\partial x} u \Delta t = \zeta(x,t)$$
$$\frac{\partial \zeta}{\partial t} = -\frac{\partial \zeta}{\partial x} u$$



Effect of vertical motion

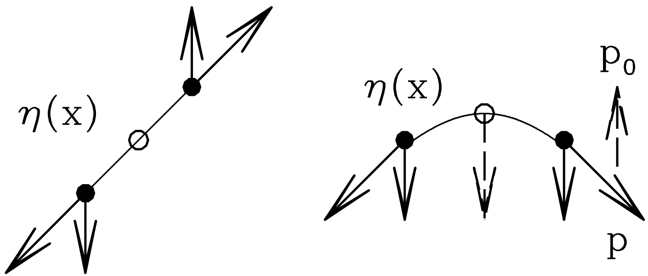
$$\zeta(x, t + \Delta t) = \zeta(x,t) + w\Delta t$$
$$\zeta(x,t) + \frac{\partial \zeta}{\partial t}\Delta t = \zeta(x,t) + w\Delta t$$
$$\frac{\partial \zeta}{\partial t} = w$$



Combined effect for $\boldsymbol{u} = \boldsymbol{\nabla}\phi$:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = w$$
$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} = \frac{\partial \phi}{\partial z}$$

Surface Tension



Tangential force along surface $\Longrightarrow$ normal force if slope varies.
$\zeta_{xx} < 0 \Longrightarrow F_z < 0$ **to be counterbalanced by** $p > p_0$**:**

$$p_0 - p = \sigma \frac{\partial^2 \zeta}{\partial x^2}$$

Bernouilli equation (ideal fluid) satisfied at surface:

$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 = \frac{p_0 - p}{\rho} - g\zeta$$

becomes:

$$\frac{\partial \phi}{dt} + \tfrac{1}{2}|\boldsymbol{\nabla}\phi|^2 = \frac{\sigma}{\rho}\frac{\partial^2 \zeta}{\partial x^2} - g\zeta$$

Bernouilli's equation at interface:

$$\partial_t \phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 = \frac{\sigma}{\rho}(\partial_x^2 + \partial_y^2)\zeta - \Phi$$

Oscillating frame of reference $\Longrightarrow$ "oscillating gravity"

$$G(t) = g - a\cos(\omega t)$$

Gravitational potential energy at interface:

$$\Phi = G(t)z = G(t)\zeta$$

Bernouilli's equation at interface:

$$\left[\partial_t \phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2\right]_{(x,y,z=\zeta(x,y))} = \left[\frac{\sigma}{\rho}(\partial_x^2 + \partial_y^2)\zeta - G(t)\zeta\right]_{(x,y)}$$

Interface $z = \zeta(x, y, t)$ moves according to:

$$\partial_t \zeta + \boldsymbol{u} \cdot \boldsymbol{\nabla}\zeta = w$$

Incompressibility:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \Delta\phi = 0$$

Base state:

$$\boldsymbol{u} = 0 \qquad \zeta = 0$$

For small perturbations:

$$\partial_t \zeta|_{(x,y)} + \cancel{\boldsymbol{u} \cdot \boldsymbol{\nabla} \zeta} = \partial_z \phi|_{(x,y,z=0+\cancel{\zeta(x,y)})}$$

$$\left[\partial_t \phi + \cancel{\frac{1}{2}|\boldsymbol{\nabla}\phi|^2}\right]_{(x,y,z=0+\cancel{\zeta(x,y)})} = \left[\frac{\sigma}{\rho}(\partial_x^2 + \partial_y^2)\zeta - G(t)\zeta\right]_{(x,y)}$$

Consider domain to be horizontally infinite (homogeneous) $\Longrightarrow$
solutions exponential/trigonometric in x $= (x, y)$
Seek bounded solutions $\Longrightarrow$ **trigonometric:** $\exp(i\boldsymbol{k} \cdot \boldsymbol{x}) = \exp(i(k_x x + k_y y))$

Height $\qquad \zeta(x, y, t) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\zeta}_k(t)$

Velocity $\qquad \phi(x, y, z, t) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\phi}_k(z, t)$

$$0 = \Delta\phi = (\partial_z^2 - k^2)\hat{\phi}_k \Longrightarrow \hat{\phi}_k \sim e^{\pm kz}$$

Assume infinite depth$(z \to -\infty) \Longrightarrow \hat{\phi}_k = e^{kz}\hat{\hat{\phi}}_k(t)$

Drop hats and subscript k

$$\partial_t \zeta = \partial_z \phi|_{z=0} = k\phi|_{z=0} \Longrightarrow \phi|_{z=0} = \partial_t \zeta / k$$

$$\partial_t \phi|_{z=0} = \frac{\sigma}{\rho}(\partial_x^2 + \partial_y^2)\zeta - G(t)\zeta$$

$$\partial_t \partial_t \zeta / k = \frac{\sigma}{\rho}(-k^2)\zeta - G(t)\zeta$$

$$\partial_t^2 \zeta = -k^3 \frac{\sigma}{\rho}\zeta - k(g - a\cos(\omega t))\zeta$$

Define $\omega_0^2 = \frac{\sigma}{\rho}k^3 + gk \qquad \hat{a} = \frac{ak}{\omega_0^2}$

$$\partial_t^2 \zeta = -\omega_0^2 (1 - \hat{a}\cos(\omega t))\zeta$$

$a = 0 \Longrightarrow$ **Gravity-capillary waves** $\Longrightarrow \zeta \sim e^{\pm i\omega_0 t}$

$a \neq 0 \Longrightarrow$ **Linear equation for** ζ **whose coefficients are periodic**

Floquet theory

Linear equations with constant coefficients:

$$a\ddot{x} + b\dot{x} + cx = 0 \implies x(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}$$

$$\textbf{where} \quad a\lambda^2 + b\lambda + c = 0$$

$$\dot{x} = cx \implies x(t) = e^{ct} x(0)$$

$$\sum_{n=0}^{N} c_n x^{(n)} = 0 \implies x(t) = \sum_{n=1}^{N} \alpha_n e^{\lambda_n t}$$

$$\textbf{where} \quad \sum_{n=0}^{N} c_n \lambda^n = 0$$

Generalize to linear equations with periodic coefficients:

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = 0 \implies x(t) = \alpha_1(t) e^{\lambda_1 t} + \alpha_2(t) e^{\lambda_2 t}$$

$$a(t), b(t), c(t) \ \textbf{have period}\ T \implies \alpha_1(t), \alpha_2(t) \ \textbf{have period}\ T$$

Floquet theory continued

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = 0 \Longrightarrow x(t) = \alpha_1(t)e^{\lambda_1 t} + \alpha_2(t)e^{\lambda_2 t}$$

$\alpha_1(t), \alpha_2(t)$	**Floquet functions**	
λ_1, λ_2	**Floquet exponents**	**growing solution if Real**$(\lambda_j) > 0$
$\mu_1 \equiv e^{\lambda_1 T}, \mu_2 \equiv e^{\lambda_2 T}$	**Floquet multipliers**	**growing solution if** $\lvert\mu_j\rvert > 1$

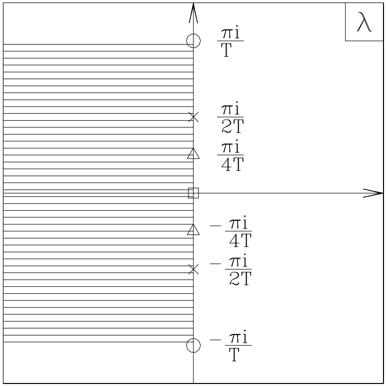
λ_1**,** λ_2 **not roots of polynomial** $\Longrightarrow$ **calculate numerically or asymptotically**

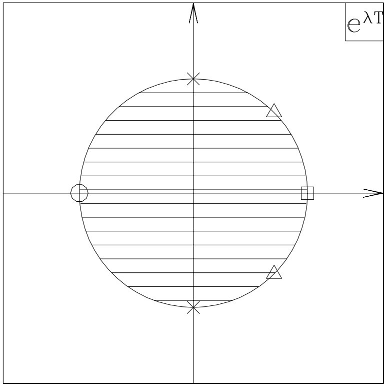
$$\dot{x} = c(t)x \quad \Longrightarrow \quad x(t) = e^{\lambda t}\alpha(t)$$

$$\sum_{n=0}^{N} c_n(t)x^{(n)} = 0 \quad \Longrightarrow \quad x(t) = \sum_{n=1}^{N} e^{\lambda_n t}\alpha_n(t)$$

Region of stability

for exponent λ **for multiplier** $e^{\lambda T}$





Imaginary part non-unique $\Longrightarrow$

choose Im$(\lambda) \in (-\pi i/T, \pi i/T] = (-i\omega/2, i\omega/2]$

$$\partial_t^2 \zeta = -\omega_0^2 (1 - a\cos(\omega t))\zeta$$

Temporal Floquet problem, with $T = 2\pi/\omega$

$$\zeta(t) = c_1 e^{\lambda_1 t} f_1(t \bmod T) + c_2 e^{\lambda_2 t} f_2(t \bmod T)$$

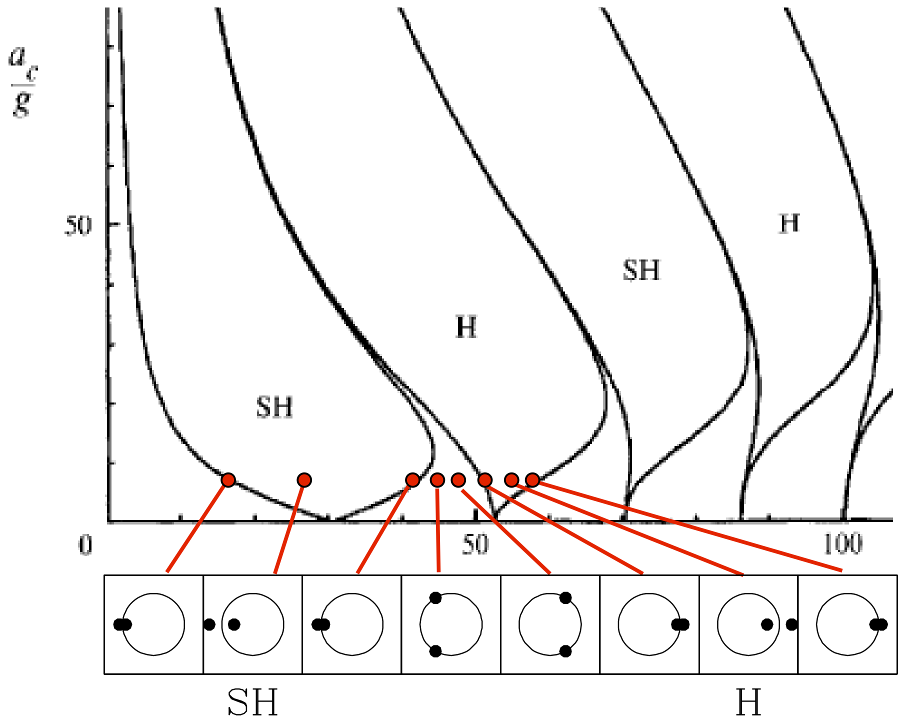
Two Floquet exponents λ_j **and Floquet functions** $f_j(t)$ **for each** k

Real$(\lambda) \geq 0$ **for some** $j, k \Longrightarrow$ **flat surface unstable** $\Longrightarrow$

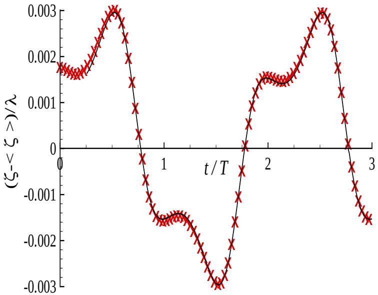
Faraday waves with spatial wavenumber k **and temporal frequency Im**(λ)

Im(λ)	$e^{\lambda T}$	**waves**	**period**	
0	**1**	**harmonic**	$T = 2\pi/\omega$	**(same as forcing)**
$\omega/2$	-1	**subharmonic**	$2T = 4\pi/\omega$	**(twice forcing period)**

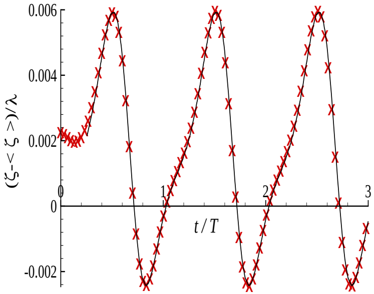
Instability Tongues



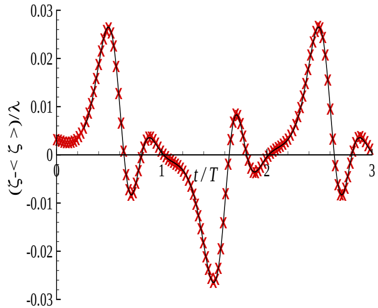
Floquet functions



within tongue 1 /2
subharmonic
$\mu = -1$



within tongue 2 /2
harmonic
$\mu = +1$



within tongue 3 /2
subharmonic
$\mu = -1$

Inclusion of viscosity

$$\begin{aligned}
\rho\partial_t \boldsymbol{u} &= -\boldsymbol{\nabla} p + \mu\Delta\boldsymbol{u} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \\
\hat{\boldsymbol{e}}_{\boldsymbol{z}} \cdot \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\rho\partial_t\boldsymbol{u} &= -\hat{\boldsymbol{e}}_{\boldsymbol{z}} \cdot \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\nabla} p + \hat{\boldsymbol{e}}_{\boldsymbol{z}} \cdot \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mu\Delta\boldsymbol{u} \\
-\rho\partial_t\Delta w &= -\mu\Delta^2 w
\end{aligned}$$

Assuming $\boldsymbol{\nabla}\cdot v = 0$, then

$$\tau_{ij} = -p\delta_{ij} + \mu(\partial_{x_j} U_i + \partial_{x_i} U_j)$$

As before, for linear stability analysis, evaluate at $z = 0$ using flat interface with normal in z direction. Continuity of tangential stress $\Longrightarrow$

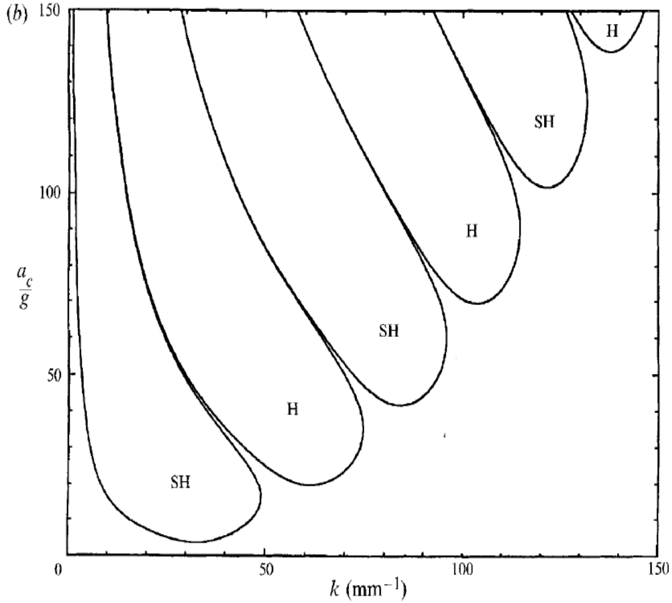
$$0 = \tau_{xz} = \mu(\partial_x w + \partial_z u) \qquad\qquad 0 = \tau_{yz} = \mu(\partial_y w + \partial_z v)$$

$$0 = \partial_x\tau_{xz} + \partial_y\tau_{yz} = \mu(\partial_x^2 w + \partial_y^2 w + \partial_{xz} u + \partial_{yz} v) = \mu(\partial_x^2 + \partial_y^2 - \partial_z^2) w$$

Normal stress is not zero at interface: counterbalanced by surface tension

$$-\sigma(\partial_x^2 + \partial_z^2)\zeta = \tau_{zz} = -(p - \rho G(t)\zeta) + 2\mu\partial_z w|_{z=0}$$

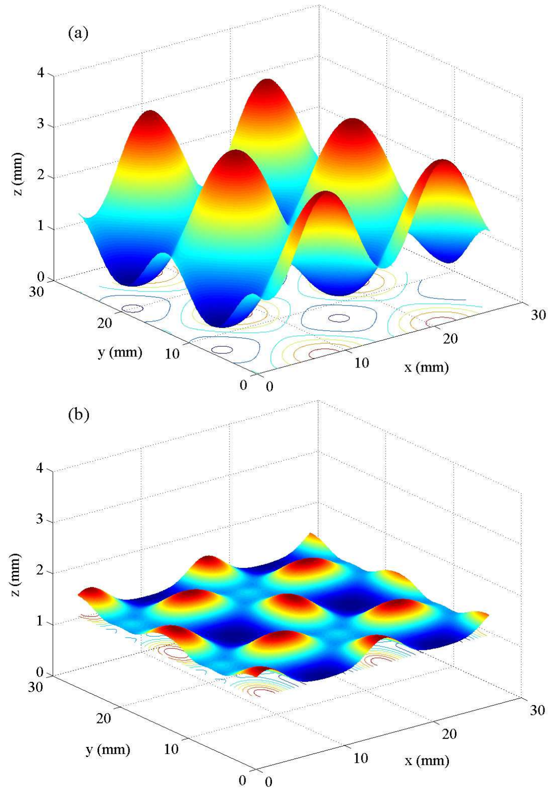
Instability tongues for viscous fluids



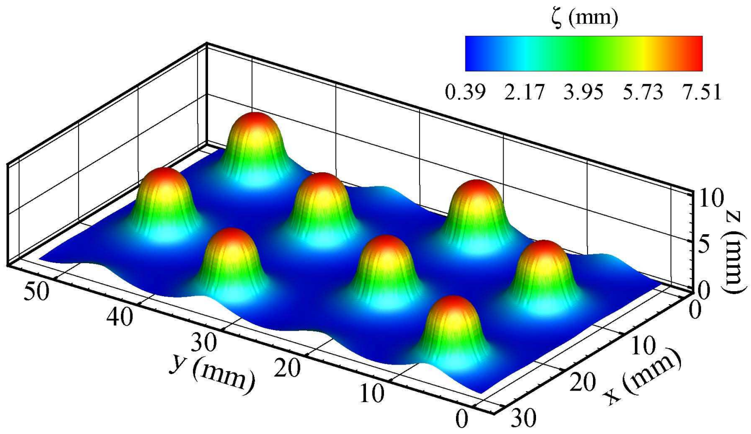
Thresholds are finite instead of zero. Tongues are rounded

Minima of tongues rise with frequency (1/2, 2/2, 3/2, . . . tongues)

Square patterns in Faraday instability

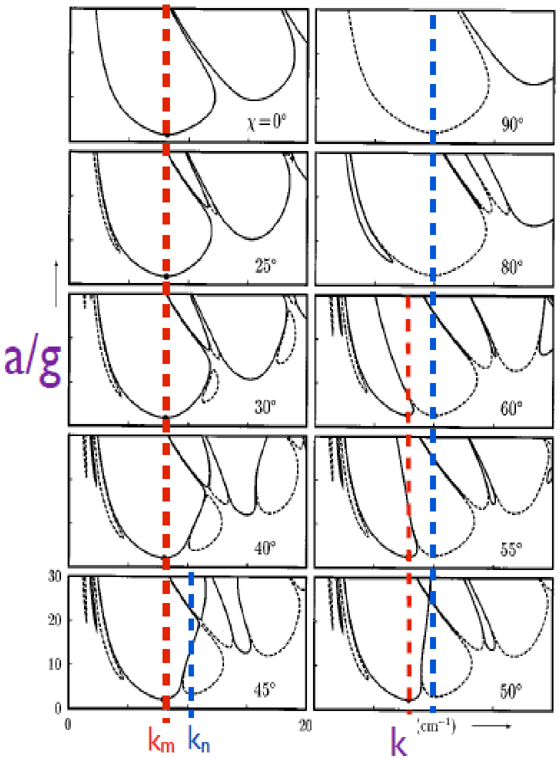


Hexagonal patterns in Faraday instability



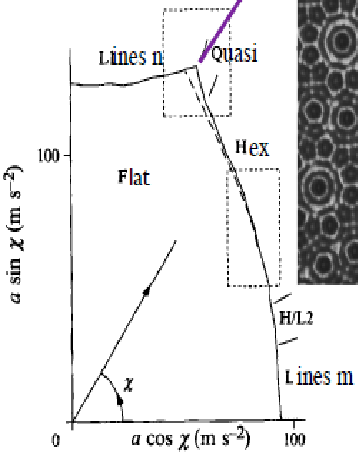
Two-frequency forcing

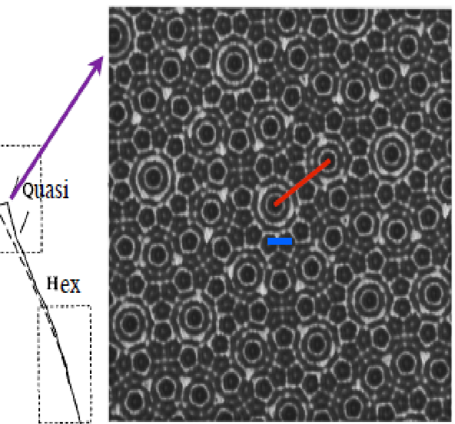
$$f(t) = a[\cos(\chi)\boxed{\cos(m\omega t)} + \sin(\chi)\boxed{\cos(l\omega t + \phi)}]$$



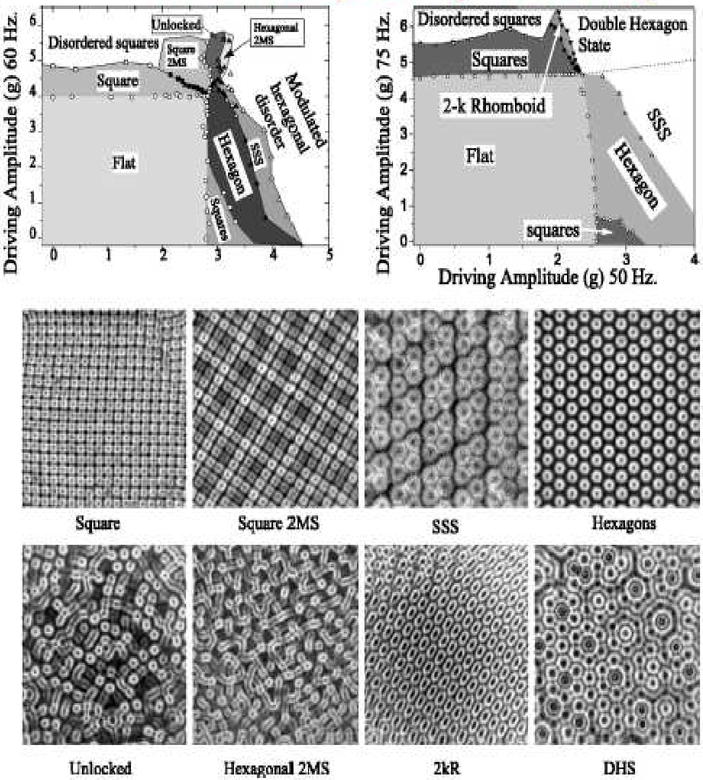
(m,n)=(4,5)

bicritical point:
two length scales





Two-frequency forcing

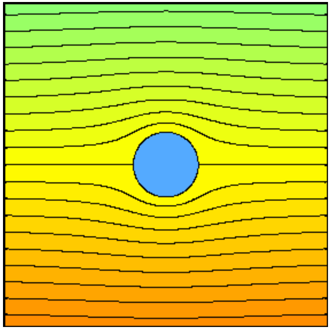


(m,n)=(2,3)

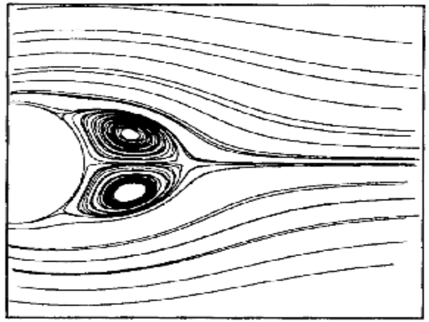
Arbell & Fineberg, (2002)

Cylinder wake

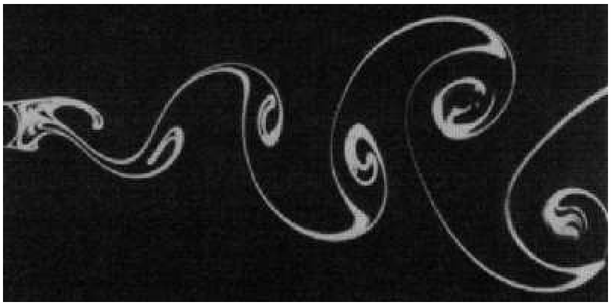
Ideal flow



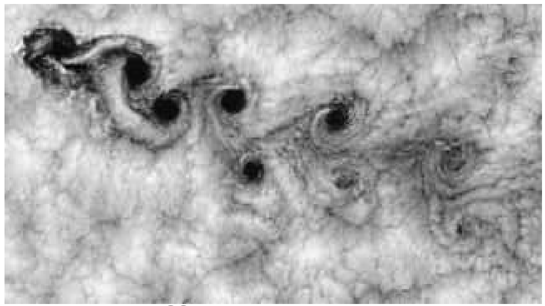
with downstream recirculation zone



Von Kármán vortex street ($\mathrm{Re} \geq 46$)



Laboratory experiment (Taneda, 1982)



Off Chilean coast past Juan Fernandez islands

Stability analysis of von Kármán vortex street

2D limit cycle $\boldsymbol{U}_{2D}(x, y, t \bmod T)$ **obeys:**

$$\partial_t \boldsymbol{U}_{2D} = -(\boldsymbol{U}_{2D} \cdot \nabla)\boldsymbol{U}_{2D} - \nabla P_{2D} + \frac{1}{Re}\Delta \boldsymbol{U}_{2D}$$

Add 3D perturbation

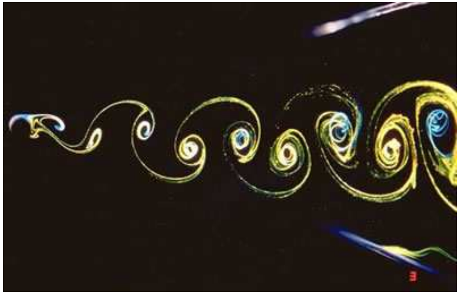
$$\begin{aligned}\partial_t(\cancel{\boldsymbol{U}_{2D}} + \boldsymbol{u}_{3D}) = & \cancel{-(\boldsymbol{U}_{2D}(t) \cdot \nabla)\boldsymbol{u}_{2D}} - (\boldsymbol{U}_{2D}(t) \cdot \nabla)\boldsymbol{u}_{3D} \\ & -(\boldsymbol{u}_{3D} \cdot \nabla)\boldsymbol{U}_{2D}(t) - (\boldsymbol{u}_{3D} \cdot \nabla)\boldsymbol{u}_{3D} \\ & -\nabla(\cancel{P_{2D}} + p_{3D}) + \frac{1}{Re}\Delta(\cancel{\boldsymbol{U}_{2D}} + \boldsymbol{u}_{3D})\end{aligned}$$

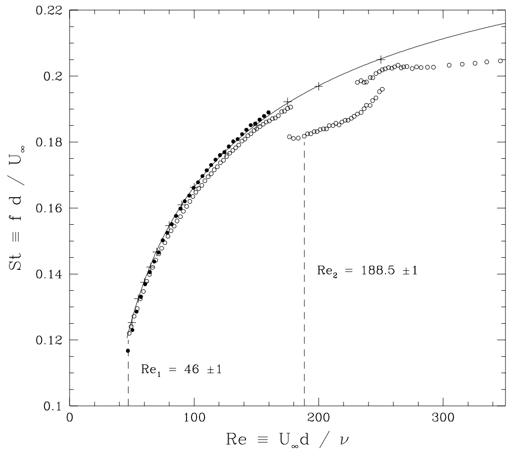
Subtract 2D equation from 3D equation and neglect quadratic terms to obtain equation governing perturbation $\boldsymbol{u}_{3D}$**:**

$$\partial_t \boldsymbol{u}_{3D} = -(\boldsymbol{U}_{2D}(t) \cdot \nabla)\boldsymbol{u}_{3D} - (\boldsymbol{u}_{3D} \cdot \nabla)\boldsymbol{U}_{2D}(t) - \nabla p_{3D} + \frac{1}{Re}\Delta \boldsymbol{u}_{3D}$$

Linear equation which is homogeneous in z **and periodic in** t

von Kármán vortex street: $Re = U_\infty d/\nu \geq 46$





spatially:
two-dimensional (x, y)
(homogeneous in z**)**

temporally:
periodic, $St = fd/U_\infty$
appears spontaneously

$$\boldsymbol{U}_{2D}(x, y, t \bmod T)$$

Infinitesimal perturbation $\boldsymbol{u}_{3D}$ obeys linear equation:

$$\partial_t \boldsymbol{u}_{3D} = -(\boldsymbol{U}_{2D}(t) \cdot \nabla)\boldsymbol{u}_{3D} - (\boldsymbol{u}_{3D} \cdot \nabla)\boldsymbol{U}_{2D}(t) - \nabla p_{3D} + \frac{1}{Re}\Delta \boldsymbol{u}_{3D}$$

Equation is linear and

- **homogeneous in z. Seek solutions which are bounded in z, hence periodic**

$$\boldsymbol{u}_{3D}(x, y, z, t) \sim e^{i\beta z}$$

- **with coefficients which are periodic in t with period T: Floquet form**

$$\boldsymbol{u}_{3D}(x, y, z, t) \sim e^{\lambda t} f(t \bmod T)$$

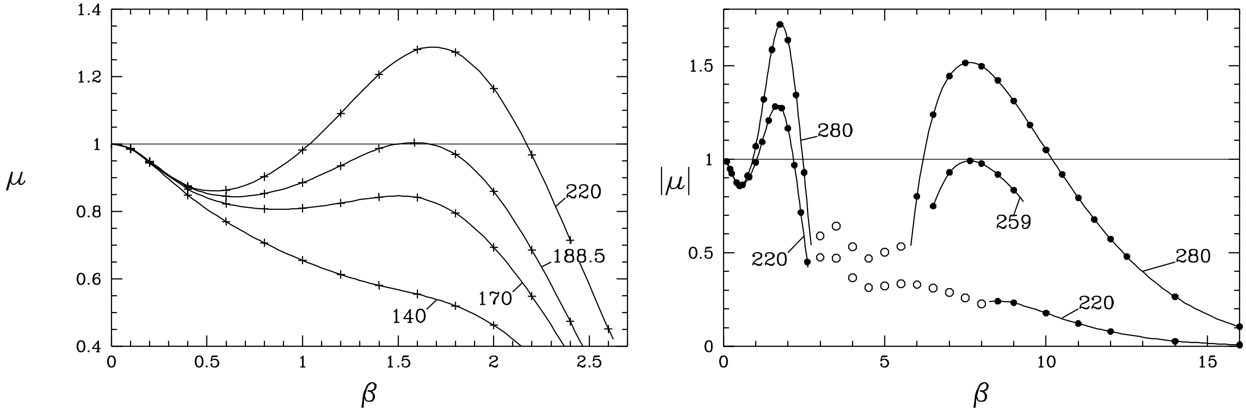
Therefore

$$\boldsymbol{u}_{3D}(x, y, z, t) \sim e^{i\beta z} e^{\lambda_\beta t} \mathbf{f}_\beta(x, y, t \bmod T)$$

Fix β, calculate λ_β and $\mu_\beta \equiv e^{\lambda_\beta T}$. Real part of $\lambda_\beta > 0 \iff |\mu_\beta| > 1$

(For each β value, there are actually many eigenvalues λ_β.
Select λ_β with largest real part.)

From Barkley & Henderson, J. Fluid Mech. (1996)



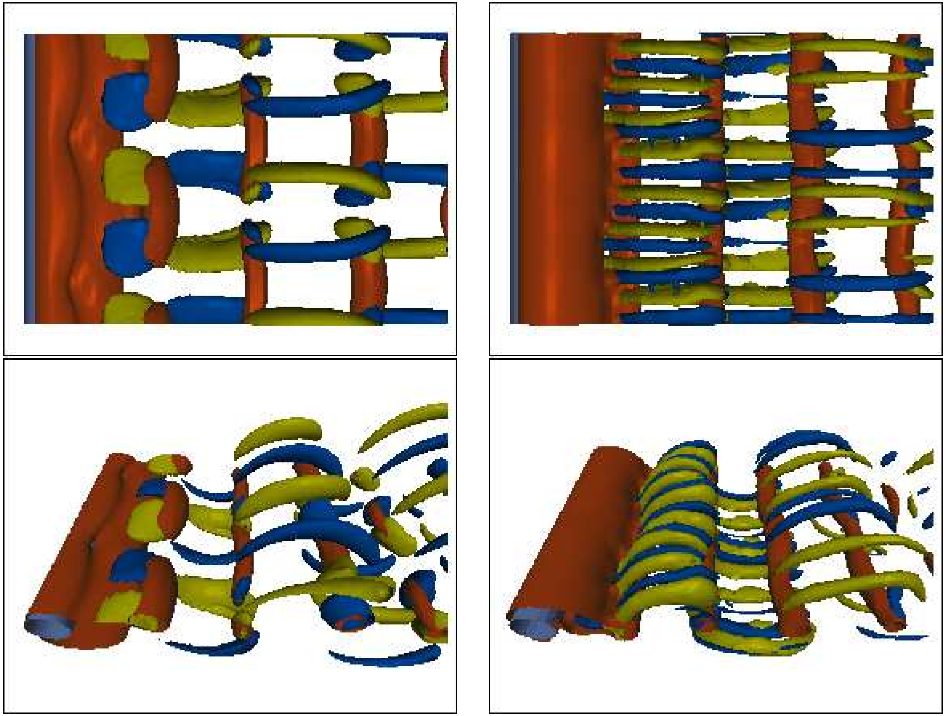
mode A: $\mathrm{Re_c} = 188.5$

$\beta_\mathrm{c} = 1.585 \Longrightarrow 2\pi/\beta_\mathrm{c} \approx 4$

mode B: $\mathrm{Re_c} = 259$

$\beta_\mathrm{c} = 7.64 \Longrightarrow 2\pi/\beta_\mathrm{c} \approx 1$

mode A at $\mathrm{Re} = 210$ **mode B at** $\mathrm{Re} = 250$



From M.C. Thompson, Monash University, Australia
(http://mec-mail.eng.monash.edu.au/∼mct/mct/docs/cylinder.html)