Hydrodynamics

Class 10

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References:

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Plane parallel flows

Plane parallel flow: $\mathbf{U} = U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}}$

By construction, incompressible and nonlinear term vanishes:

$$\begin{aligned}
(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U} &= (U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}}\cdot\boldsymbol{\nabla})U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}} = U(y)\partial_x U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}} = 0 \\
\boldsymbol{\nabla}\boldsymbol{\cdot}U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}} &= \partial_x U(y) = 0
\end{aligned}$$

Must satisfy Navier-Stokes equations:

$$0 = -\boldsymbol{\nabla}P + \frac{1}{R}\Delta\mathbf{U} = \begin{cases} -\partial_x P + \frac{1}{R}U''(y) & \Longrightarrow -P = g(y)x + h(y,z) \\ -\partial_y P & \Longrightarrow g(y) = G \text{ (constant)} \\ -\partial_z P & \Longrightarrow h(y,z) = H \text{ (constant)} \end{cases}$$

For ideal (inviscid) fluids, no viscous term $\Longrightarrow$ all $U(y)$ allowed

For viscous fluids, solve

$$\begin{aligned}
0 &= G + \frac{1}{R}U''(y) \\
\Longrightarrow U(y) &= -\frac{GR}{2}y^2 + ay + b
\end{aligned}$$

Poiseuille and Couette flow

$$U(y) = -\frac{GR}{2}y^2 + ay + b, \qquad -1 \le y \le +1$$

Poiseuille flow: pressure-driven

$$G \neq 0$$

Couette flow: shear-driven

$$U(y=1) \neq U(y=-1)$$

Can choose

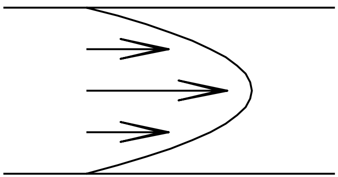
stationary plates $U(\pm 1) = 0$

$$U(y) = \frac{GR}{2}(1 - y^2)$$

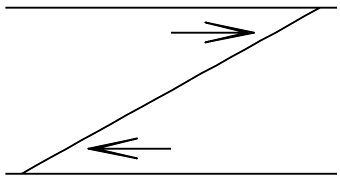
frame such that $U(\pm 1) = \pm 1$

$$U(y) = y$$

plane Poiseuille flow



plane Couette flow



Longstanding mystery: Poiseuille and Couette flow undergo sudden transition to three-dimensional turbulence that is not predicted by linear stability analysis.

x	main flow direction	streamwise
y	between the bounding plates	cross-channel
z		spanwise

Linear stability analysis of plane parallel flows

Perturbations $\boldsymbol{u} = (u, v, w)$. Linearize nonlinear term about $U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}}$:

$$
\begin{aligned}
(\mathbf{U} \cdot \boldsymbol{\nabla})\boldsymbol{u} &= (U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}} \cdot \boldsymbol{\nabla})\boldsymbol{u} = U(y)\partial_x \boldsymbol{u} \\
(\boldsymbol{u} \cdot \boldsymbol{\nabla})\mathbf{U} &= (\boldsymbol{u} \cdot \boldsymbol{\nabla})U(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}} = vU'(y)\hat{\boldsymbol{e}}_{\boldsymbol{x}}
\end{aligned}
$$

Linearized Navier-Stokes equations:

$$
\begin{aligned}
\partial_t \boldsymbol{u} + U\partial_x \boldsymbol{u} + vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}} &= -\boldsymbol{\nabla} p + \tfrac{1}{R}\Delta \boldsymbol{u} \\
\boldsymbol{\nabla \cdot u} &= 0
\end{aligned}
$$

Want to reduce (u, v, w, p) to $v = \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{u}$, $\eta \equiv \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$

$$\partial_t \boldsymbol{u} + U\partial_x \boldsymbol{u} + vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}} = -\boldsymbol{\nabla} p + \tfrac{1}{R}\Delta \boldsymbol{u}$$

Take divergence:

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\partial_t \boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{\cdot}(U\partial_x \boldsymbol{u}) + \boldsymbol{\nabla}\boldsymbol{\cdot}(vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}}) = \boldsymbol{\nabla}\boldsymbol{\cdot}(-\boldsymbol{\nabla} p) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{1}{R}\Delta \boldsymbol{u}\right)$$

Expand, using $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0$ and $\boldsymbol{\nabla} U = U'\hat{\boldsymbol{e}}_{\boldsymbol{y}}$:

$$\begin{aligned}
\boldsymbol{\nabla}\boldsymbol{\cdot}\partial_t \boldsymbol{u} &= \partial_t \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0 \\
\boldsymbol{\nabla}\boldsymbol{\cdot}(U\partial_x \boldsymbol{u}) &= U\boldsymbol{\nabla}\boldsymbol{\cdot}(\partial_x \boldsymbol{u}) + (\boldsymbol{\nabla} U)\cdot\partial_x \boldsymbol{u} \\
&= U\partial_x \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} + U'\hat{\boldsymbol{e}}_{\boldsymbol{y}}\cdot\partial_x \boldsymbol{u} = U'\partial_x v \\
\boldsymbol{\nabla}\boldsymbol{\cdot}(vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}}) &= \partial_x(vU') = U'\partial_x v \\
\boldsymbol{\nabla}\boldsymbol{\cdot}(-\boldsymbol{\nabla} p) &= -\Delta p \\
\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\tfrac{1}{R}\Delta \boldsymbol{u}\right) &= \tfrac{1}{R}\Delta\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{u} = 0
\end{aligned}$$

Obtain:

$$2U'\partial_x v = -\Delta p$$

$$\partial_t \boldsymbol{u} + U\partial_x \boldsymbol{u} + vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}} = -\boldsymbol{\nabla} p + \tfrac{1}{R}\Delta \boldsymbol{u}$$

Laplacian of y component:

$$\Delta\partial_t v + \Delta(U\partial_x v) = -\Delta\partial_y p + \tfrac{1}{R}\Delta^2 v$$

Expand nonlinear term, using $\boldsymbol{\nabla} U = U'\hat{\boldsymbol{e}}_{\boldsymbol{y}}$ and $\Delta U = U''$:

$$\begin{aligned}
\Delta(U\partial_x v) &= U\Delta(\partial_x v) + 2(\boldsymbol{\nabla} U)\cdot\boldsymbol{\nabla}(\partial_x v) + (\Delta U)(\partial_x v) \\
&= U\partial_x\Delta v + 2U'\partial_{xy} v + U''\partial_x v
\end{aligned}$$

$$\begin{aligned}
\text{Recall:}\quad -\Delta p &= 2U'\partial_x v \\
-\Delta\partial_y p &= \partial_y(2U'\partial_x v) = 2U'\partial_{xy} v + 2U''\partial_x v
\end{aligned}$$

$$\begin{aligned}
\Delta\partial_t v + U\partial_x\Delta v + \cancel{2U'\partial_{xy} v + U''\partial_x v} &= \cancel{2U'\partial_{xy} v} + \cancel{2}U''\partial_x v + \frac{1}{R}\Delta^2 v \\
(\partial_t + U\partial_x)\Delta v &= U''\partial_x v + \tfrac{1}{R}\Delta^2 v
\end{aligned}$$

$\Longrightarrow$ Equation containing only v

$$\partial_t \boldsymbol{u} + U\partial_x \boldsymbol{u} + vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}} = -\boldsymbol{\nabla} p + \tfrac{1}{R}\Delta \boldsymbol{u}$$

Take y component of curl:

$$\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times (\partial_t \boldsymbol{u} + U\partial_x \boldsymbol{u} + vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}}) = \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times \left(-\boldsymbol{\nabla} p + \tfrac{1}{R}\Delta \boldsymbol{u}\right)$$

Expand, using $\eta \equiv \hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times \boldsymbol{u} = \partial_z u - \partial_x w$:

$$\begin{aligned}
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times \partial_t \boldsymbol{u} &= \partial_t \eta \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times (U\partial_x \boldsymbol{u}) &= \partial_z(U\partial_x u) - \partial_x(U\partial_x w) = U\partial_{xz} u - U\partial_{xx} w = U\partial_x \eta \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times (vU'\hat{\boldsymbol{e}}_{\boldsymbol{x}}) &= \partial_z(vU') = U'\partial_z v \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times (-\boldsymbol{\nabla} p) &= 0 \\
\hat{\boldsymbol{e}}_{\boldsymbol{y}} \cdot \boldsymbol{\nabla}\times \left(\frac{1}{R}\Delta \boldsymbol{u}\right) &= \frac{1}{R}(\partial_z \Delta u - \partial_x \Delta w) = \frac{1}{R}(\Delta \partial_z u - \Delta \partial_x w) = \frac{1}{R}\Delta \eta
\end{aligned}$$

This gives a second equation, which couples η and v:

$$(\partial_t + U\partial_x)\eta + U'\partial_z v = \tfrac{1}{R}\Delta \eta$$

$$
\begin{aligned}
(\partial_t + U\partial_x)\Delta v &= U''\partial_x v + \tfrac{1}{R}\Delta^2 v \\
(\partial_t + U\partial_x)\eta + U'\partial_z v &= \tfrac{1}{R}\Delta\eta
\end{aligned}
$$

Equations are 4^{th} order for v, 2^{nd} order for η

In x, z, assume periodic boundary conditions (works for any order)

In y, apply $u = v = w = 0$ at $y_\pm \Longrightarrow$ 6 BCs. Transform:

$$
\begin{aligned}
v &= 0 \\
\partial_x u = \partial_z w &= 0 \quad \Longrightarrow \partial_y v = 0 \\
\partial_z u = \partial_x w &= 0 \quad \Longrightarrow \eta = 0
\end{aligned}
$$

$$
\begin{aligned}
(\partial_t + U\partial_x)\Delta v &= U''\partial_x v + \tfrac{1}{R}\Delta^2 v \\
(\partial_t + U\partial_x)\eta + U'\partial_z v &= \tfrac{1}{R}\Delta\eta
\end{aligned}
$$

Homogeneous in x, z, t so dependence is exponential or trigonometric (if bounded).
Not homogeneous in y since: –boundary conditions at $y = y_\pm$
–base state U depends on y

$$
\Longrightarrow \begin{cases} v(x,y,z,t) = \hat{v}(y)e^{i(\alpha(x-ct)+\beta z)} \\ \eta(x,y,z,t) = \hat{\eta}(y)e^{i(\alpha(x-ct)+\beta z)} \end{cases}
$$

α, β real, but c can be complex. $-ic = c_i - ic_r$
$c_i =$ growth rate: perturbations grow if $c_i > 0$, decay if $c_i < 0$, neutral if $c_i = 0$.
$c_r =$ phase speed: a peak moves at speed c_r.

Define $D \equiv d/dy$ and $k^2 \equiv \alpha^2 + \beta^2$:

$$
\begin{aligned}
(-i\alpha c + Ui\alpha)(D^2 - k^2)\hat{v} &= U''i\alpha\hat{v} + \tfrac{1}{R}(D^2 - k^2)^2\hat{v} \\
-i\alpha c\hat{\eta} + Ui\alpha\hat{\eta} + U'i\beta\hat{v} &= \tfrac{1}{R}(D^2 - k^2)\hat{\eta}
\end{aligned}
$$

$$
\begin{aligned}
(-i\alpha c + Ui\alpha)(D^2 - k^2)\hat{v} &= U'' i\alpha \hat{v} + \tfrac{1}{R}(D^2 - k^2)^2 \hat{v} \\
-i\alpha c\hat{\eta} + Ui\alpha\hat{\eta} + U' i\beta \hat{v} &= \tfrac{1}{R}(D^2 - k^2)\hat{\eta}
\end{aligned}
$$

Dividing by $i\alpha$:

$$
\begin{aligned}
(U - c)(D^2 - k^2)\hat{v} &= U''\hat{v} + \tfrac{1}{Ri\alpha}(D^2 - k^2)^2\hat{v} && \text{Orr-Sommerfeld equation (1907-8)} \\
(U - c)\hat{\eta} + U'\tfrac{\beta}{\alpha}\hat{v} &= \tfrac{1}{Ri\alpha}(D^2 - k^2)\hat{\eta} && \text{Squire's equation (1933)}
\end{aligned}
$$

Eigenvalue problem with eigenvalues c and eigenvectors $\hat{v}$, $\hat{\eta}$.

Matrix form:

$$
\begin{bmatrix} \left[(U - c) - \tfrac{1}{Ri\alpha}(D^2 - k^2)\right](D^2 - k^2) - U'' & 0 \\ U'\tfrac{\beta}{\alpha} & (U - c) - \tfrac{1}{Ri\alpha}(D^2 - k^2) \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix}
$$

$$
\equiv \begin{bmatrix} \mathcal{L}_{OS} & 0 \\ B & \mathcal{L}_{SQ} \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Upper triangular matrix: $\begin{bmatrix} \mathcal{L}_{OS} & 0 \\ B & \mathcal{L}_{SQ} \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{\eta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Two families of eigenvalues/vectors:

Orr-Sommerfeld modes	Squire modes
$\mathcal{L}_{OS}\hat{v} = 0, \quad \hat{v} \neq 0$	$\hat{v} = 0$
$\mathcal{L}_{SQ}\hat{\eta} = -B\hat{v}$	$\mathcal{L}_{SQ}\hat{\eta} = 0, \quad \hat{\eta} \neq 0$

Squire modes neutral for ideal fluids ($R = \infty$), damped for viscous fluids:

$$\begin{aligned} 0 &= \int dy\ \hat{\eta}^* \mathcal{L}_{SQ}\hat{\eta} = \int dy\ \hat{\eta}^* \left(U - c - \tfrac{1}{Ri\alpha}(D^2 - k^2)\right)\hat{\eta} \\ &= \int dy\ U|\hat{\eta}|^2 - c\int dy\ |\hat{\eta}|^2 - \tfrac{1}{Ri\alpha}\int dy\ \hat{\eta}^* D^2\hat{\eta} + \tfrac{k^2}{Ri\alpha}\int dy\ |\hat{\eta}|^2 \end{aligned}$$

Integration by parts: $\displaystyle\int_{-1}^{+1} dy\ \hat{\eta}^* D^2\hat{\eta} = \cancel{\hat{\eta}^* D\hat{\eta}]_{y_-}^{y_+}} - \int_{y_-}^{y_+} dy\ |D\hat{\eta}|^2$

Imaginary part: $\displaystyle c_i \int dy\ |\hat{\eta}|^2 = -\tfrac{1}{R\alpha}\int dy\ |D\hat{\eta}|^2 - \tfrac{k^2}{R\alpha}\int dy\ |\hat{\eta}|^2 \leq 0$

so need Orr-Sommerfeld equation and modes for instability

Squire's Transformation (1933)

$$(U-c)(D^2-k^2)\hat{v} = U''\hat{v} + \frac{1}{Ri\alpha}(D^2-k^2)^2\hat{v}$$

$$\text{Define: } \begin{cases} \tilde{\alpha}^2 & \equiv & k^2 = \alpha^2 + \beta^2 \geq \alpha^2 \\ \tilde{\beta} & \equiv & 0 \Longrightarrow \text{ } z\text{-independent perturbation} \\ \tilde{R} & \equiv & R\alpha/\tilde{\alpha} \leq R \end{cases}$$

$$(U-c)(D^2-\tilde{\alpha}^2)\hat{v} = U''\hat{v} + \frac{1}{\tilde{R}i\tilde{\alpha}}(D^2-\tilde{\alpha}^2)^2\hat{v}$$

O-S equation with $\alpha \uparrow, \beta \downarrow, R \downarrow, c$ unchanged, $\hat{v}$ unchanged

Viscous fluid: Find α, β with lowest R for which $c_i(\alpha, \beta, R) > 0$
Squire's Theorem: lowest R is achieved for $\beta = 0$
Ideal fluid: O.S. eq. with $\alpha, \beta > 0 \Longleftrightarrow$ O.S. eq. with $\tilde{\alpha}, \tilde{\beta} = 0$

Successive simplifications:
4 PDEs $\Longrightarrow$ 2 PDEs $\Longrightarrow$ 2 ODEs $(c, \alpha, \beta) \Longrightarrow$ 1 ODE $\Longrightarrow$ 1 ODE (c, α)

Can prove rigourously that Poiseuille and Couette flow are linearly stable in the Reynolds numbers range in which transition to turbulence occurs, both experimentally and numerically. At this time, there is no definite resolution to this dilemma, but much research.

Ideal fluids: Rayleigh Equation

$$0 = \left[(U-c)(D^2-k^2) - U''\right]\hat{v}$$

Rayleigh's inflection point theorem (1880, 1887): $c_i \neq 0 \implies U(y)$ has an inflection point.

Since $c_i \neq 0$, can divide by $-(U-c)$:

$$0 = \int_{y_-}^{y_+} dy\ \hat{v}^* \left[(-D^2+k^2) + \frac{U''}{U-c}\right]\hat{v}$$

Integration by parts: $\int\ dy\ \hat{v}^*(-D^2)\hat{v} = \int\ dy\ |D\hat{v}|^2$

$$\frac{U''}{U-c} = \frac{U''(U-c_r+ic_i)}{(U-c_r-ic_i)(U-c_r+ic_i)} = \frac{U''(U-c_r)}{|U-c|^2} + \frac{ic_iU''}{|U-c|^2}$$

$$0 = \underbrace{\int_{y_-}^{y_+} dy\ \left[|D\hat{v}|^2 + k^2|\hat{v}|^2 + \frac{U''(U-c_r)}{|U-c|^2}|\hat{v}|^2\right]}_{\mathcal{I}_r} + ic_i \underbrace{\int_{y_-}^{y_+} dy\ \frac{U''}{|U-c|^2}|\hat{v}|^2}_{\mathcal{I}_i}$$

Both real and imaginary parts are zero

$c_i \neq 0 \implies \mathcal{I}_i = 0 \implies U''$ changes sign over $[y_-, y_+] \implies U$ has inflection point y_S

If $c_i \neq 0$ (instability of U), then $U(y)$ has an inflection point.
If $U(y)$ does not have an inflection point, then $c_i = 0$ (stability of U).

Fjortoft's Theorem (1950):
if $c_i \neq 0$, then $U''(y)(U(y) - U(y_S))$ is negative over a portion of $[y_-, y_+]$.

$$0 = \mathcal{I}_r = \int_{y_-}^{y_+} dy \, \left[|D\hat{v}|^2 + k^2|\hat{v}|^2 + \frac{U''(U - c_r)}{|U - c|^2} |\hat{v}|^2 \right]$$

$$\int_{y_-}^{y_+} dy \, \frac{U''(U - c_r)}{|U - c|^2} |\hat{v}|^2 = - \int_{y_-}^{y_+} dy \, \left[|D\hat{v}|^2 + k^2|\hat{v}|^2 \right] < 0$$

Define $U_S \equiv U(y_S)$ and multiply $\mathcal{I}_i$ by $(c_r - U_S)/c_i$:

$$\int_{y_-}^{y_+} dy \, \frac{U''(c_r - U_S)}{|U - c|^2} |\hat{v}|^2 = 0$$

Add:

$$\int_{y_-}^{y_+} dy \, \frac{U''(U - c_r + c_r - U_S)}{|U - c|^2} |\hat{v}|^2 = - \int_{y_-}^{y_+} dy \, \left[|D\hat{v}|^2 + k^2|\hat{v}|^2 \right] < 0$$

Therefore:

$$\int_{y_-}^{y_+} dy \, \frac{U''(U - U_S)}{|U - c|^2} |\hat{v}|^2 < 0 \Longrightarrow U''(y)(U - U_S) < 0 \text{ over portion of interval}$$

Rayleigh and Fjortoft apply to inviscid fluids and only demonstrate stability

Rayleigh's inflection point theorem:

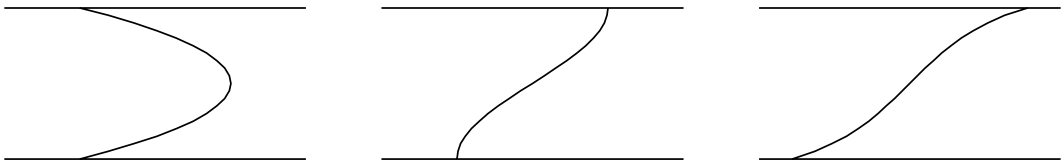
$c_i \neq 0 \Longrightarrow U(y)$ has an inflection point

$U(y)$ has no inflection point $\Longrightarrow c_i = 0 \Longrightarrow U(y)$ stable

Fjortoft's Theorem:

$c_i \neq 0$ and $U''(y_S) = 0 \Longrightarrow U''(y)(U(y) - U_S) < 0$ somewhere in $[y_-, y_+]$

$U''(y)(U(y) - U_S) > 0$ over all $[y_-, y_+] \Longrightarrow c_i = 0 \Longrightarrow U(y)$ stable



No inflection point	Has inflection point	Has inflection point
	$U''(y_S) = 0$ $U > U_S$ where $U'' < 0$ $U < U_S$ where $U'' > 0$ $U''(U - U_S) < 0$	$U''(y_S) = 0$ $U > U_S$ where $U'' > 0$ $U < U_S$ where $U'' < 0$ $U''(U - U_S) > 0$
Rayleigh $\Longrightarrow$ stable	?	Fjortoft $\Longrightarrow$ stable

Howard's semicircle theorem (1961)

Unstable eigenvalues $c = c_r + ic_i$ of Rayleigh equation obey:

$$\left(c_r - \frac{1}{2}(U_{\max} + U_{\min})\right)^2 + c_i^2 \leq \frac{1}{2}(U_{\max} - U_{\min})^2$$

$\Longrightarrow$ unstable c located inside circle whose diameter connects $U_{\max}$ and $U_{\min}$.

Kelvin-Helmholtz Instability

Caused by velocity gradient. Begin with piecewise-constant profile on $[-\infty, +\infty]$

$$U = \begin{cases} U_+ \text{ for } y > 0 \\ U_- \text{ for } y < 0 \end{cases}$$

Must specify jump conditions at each discontinuity of U:

1) Interface remains well-defined: $\left[\!\left[\dfrac{\hat{v}}{U-c} \right]\!\right] = 0$

2) Continuity of normal stress (pressure): $[\![((U-c)D - U')\hat{v}]\!] = 0$

Rayleigh equation: $0 = \left[(U-c)(D^2-k^2) - U''\right]\hat{v}$

$$\begin{cases} y > 0 \qquad (D^2-k^2)\hat{v} = 0 \quad \& \quad \hat{v}(+\infty) = 0 \implies \hat{v} = Ae^{-ky} \\ y < 0 \qquad (D^2-k^2)\hat{v} = 0 \quad \& \quad \hat{v}(-\infty) = 0 \implies \hat{v} = Be^{ky} \end{cases}$$

Apply jump conditions at $y = 0$:

$$\begin{aligned} 0 &= \frac{\hat{v}}{U-c}(0^+) - \frac{\hat{v}}{U-c}(0^-) = \frac{A}{U_+-c} - \frac{B}{U_- - c} \\ 0 &= ((U-c)D - U')\hat{v}(0^+) - ((U-c)D - U')\hat{v}(0^-) \\ &= (U_+-c)(-kA) - (U_- - c)kB \end{aligned}$$

or

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{U_+ - c} & \frac{-1}{U_- - c} \\ (U_+ - c) & (U_- - c) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

A non-trivial solution exists if the determinant is zero:

$$\begin{aligned}
0 &= \frac{U_- - c}{U_+ - c} + \frac{U_+ - c}{U_- - c} \\
0 &= (U_- - c)^2 + (U_+ - c)^2 = 2\left(c^2 - (U_+ + U_-)c + (U_+^2 + U_-^2)/2\right) \\
c &= \frac{1}{2}\left[U_+ + U_- \pm \sqrt{(U_+ + U_-)^2 - 2(U_+^2 + U_-^2)}\right] \\
&= \frac{U_+ + U_-}{2} \pm \frac{1}{2}\sqrt{-(U_+ - U_-)^2} \\
&\equiv \bar{U} \pm i\frac{\Delta U}{2} \\
v(x, y, t) &= \hat{v}(y) e^{ik(x-ct)} = \left\{ \begin{array}{c} Ae^{-ky} \\ Be^{ky} \end{array} \right\} e^{ik(x-\bar{U}t)} e^{\pm t\Delta U/2}
\end{aligned}$$

Existence of growing perturbation $\Longrightarrow U(y)$ is unstable

Discontinuous piecewise-constant profile $\Longrightarrow c$ is independent of k

Piecewise linear profile which is continuous but not smooth

$$\begin{array}{c|c|c|l|l|}
y > +\delta & U = U_+ & U' = 0 & \hat{v}_+ \equiv A_+ e^{-ky} & D\hat{v}_+ = -kA_+ e^{-ky} \\
-\delta < y < +\delta & U = \bar{U} + \frac{\Delta U}{2\delta} y & U' = \frac{\Delta U}{2\delta} & \hat{v}_0 \equiv A_0 e^{-ky} + B_0 e^{ky} & D\hat{v}_0 = -kA_0 e^{-ky} + kB_0 e^{ky} \\
y < -\delta & U = U_- & U' = 0 & \hat{v}_- \equiv B_- e^{ky} & D\hat{v}_- = kB_- e^{ky}
\end{array}$$

If U is continuous then continuity of $\hat{v}/(U-c)$ implies continuity of $\hat{v}$

$$\begin{aligned}
0 &= \hat{v}_0(\delta) - \hat{v}_+(\delta) = A_0 e^{-k\delta} + B_0 e^{k\delta} - A_+ e^{-k\delta} \Longrightarrow A_+ e^{-k\delta} = A_0 e^{-k\delta} + B_0 e^{k\delta} \\
0 &= \hat{v}_0(-\delta) - \hat{v}_-(-\delta) = A_0 e^{k\delta} + B_0 e^{-k\delta} - B_- e^{-k\delta} \Longrightarrow B_- e^{-k\delta} = B_0 e^{-k\delta} + A_0 e^{k\delta} \\
0 &= [(U-c)D - U']\hat{v}_0(\delta) - [(U-c)D - U']\hat{v}_+(\delta) \\
&= (U-c)[D\hat{v}_0(\delta) - D\hat{v}_+(\delta)] - [(U'\hat{v}_+)(\delta) - (U'\hat{v}_0)(\delta)] \\
&= (U_+ - c)[-kA_0 e^{-k\delta} + kB_0 e^{k\delta} - (-kA_+ e^{-k\delta})] + \frac{\Delta U}{2\delta}(A_0 e^{-k\delta} + B_0 e^{k\delta}) \\
&= (U_+ - c)[-kA_0 e^{-k\delta} + kB_0 e^{k\delta} - (-k(A_0 e^{-k\delta} + B_0 e^{k\delta})] + \frac{\Delta U}{2\delta}(A_0 e^{-k\delta} + B_0 e^{k\delta}) \\
0 &= [(U-c)D - U']\hat{v}_0(-\delta) - [(U-c)D - U']\hat{v}_-(-\delta) \\
&= (U-c)[D\hat{v}_0(-\delta) - D\hat{v}_-(-\delta)] - [(U'\hat{v}_-)(-\delta) - (U'\hat{v}_0)(-\delta)] \\
&= (U_- - c)[-kA_0 e^{k\delta} + kB_0 e^{-k\delta} - (kB_- e^{-k\delta})] + \frac{\Delta U}{2\delta}(A_0 e^{k\delta} + B_0 e^{-k\delta}) \\
&= (U_- - c)[-kA_0 e^{k\delta} + kB_0 e^{-k\delta} - (k(B_0 e^{-k\delta} + A_0 e^{k\delta}) + \frac{\Delta U}{2\delta}(A_0 e^{k\delta} + B_0 e^{-k\delta})
\end{aligned}$$

$$
\begin{aligned}
0 &= (U_+ - c)2kB_0e^{k\delta} + \frac{\Delta U}{2\delta}(A_0e^{-k\delta} + B_0e^{k\delta}) \\
0 &= (U_- - c)(-2k)A_0e^{k\delta} + \frac{\Delta U}{2\delta}(A_0e^{k\delta} + B_0e^{-k\delta})
\end{aligned}
$$

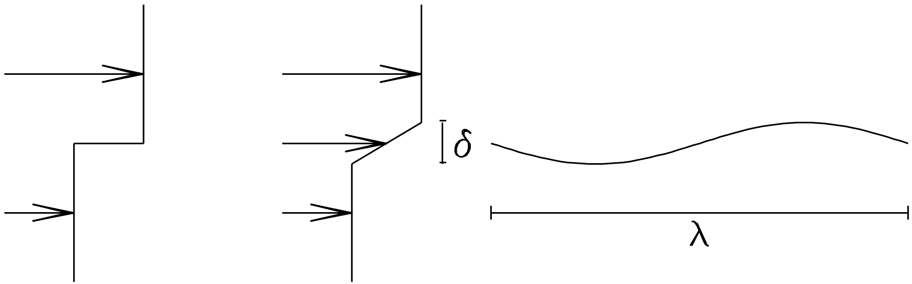
$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\Delta U}{2\delta}e^{-k\delta} & ((U_+ - c)(-2k) + \frac{\Delta U}{2\delta})e^{k\delta} \\ ((U_- - c)2k + \frac{\Delta U}{2\delta})e^{k\delta} & \frac{\Delta U}{2\delta}e^{-k\delta} \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}
$$

Has solution if determinant is zero $\Longrightarrow$

$$
c = \bar{U} \pm \frac{\Delta U}{4k\delta}\sqrt{(1 - 2k\delta)^2 - e^{-4k\delta}}
$$

c depends on k and can be real (neutral) or complex (growing or damped)
Boundary between two regimes is found by solving numerically

$$
(1 - 2\hat{k})^2 = e^{-2\hat{k}} \qquad \Longrightarrow \qquad \hat{k} = 0.64
$$



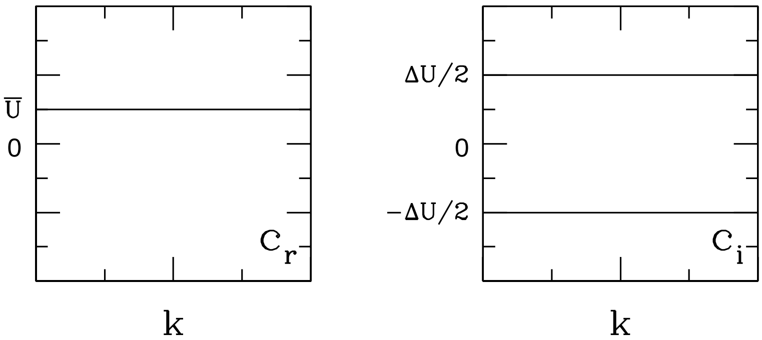
Unstable perturbations:

$$\hat{k} = k\delta < 0.64$$
$$\frac{2\pi\delta}{\lambda} < 0.64$$
$$\lambda > \frac{2\pi}{0.64}\,\delta \approx 10\,\delta$$

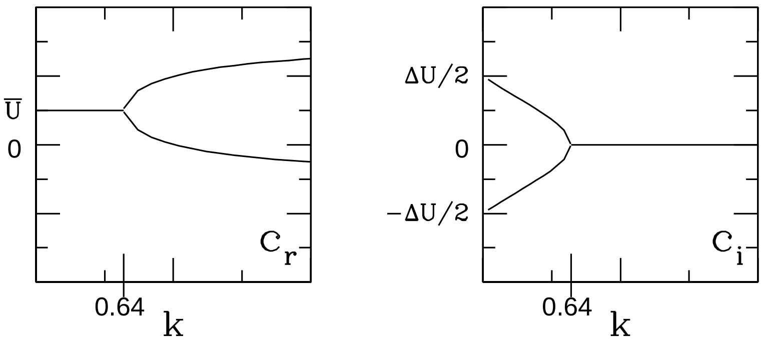
The piecewise-linear profile is unstable to perturbations whose wavelength λ is more than 10 times the width δ of the interface.

As $\delta \to 0$, recover the previous result of the discontinuous profile.

Discontinuous piecewise-constant profile



Continuous piecewise-linear profile



Beyond eigenvalues

Poiseuille and Couette flow: instability / transition to turbulence

Flow	R_T	R_L
plane Poiseuille	1000	5772
plane Couette	300	∞
pipe Poiseuille	2000	∞

Pipe Poiseuille flow is $U(r)\hat{\boldsymbol{e}}_{\boldsymbol{z}} = (1 - r^2)\hat{\boldsymbol{e}}_{\boldsymbol{z}}$.

R_L is linear instability threshold

R_T is R at which experiments and simulations show transition to turbulence

Turbulence is 3D, unlike 2D ($\beta = 0$) perturbations shown by Squire's Theorem to be the most unstable.

Many ideas have been proposed to liberate hydrodynamic stability theorem from the tyranny of Squire's Theorem and eigenvalues.

Many articles are published each year in this very active field.

Energy theory

Take $\mathbf{U}\cdot$ of Navier-Stokes equations:

$$\frac{\partial}{\partial t}\frac{1}{2}\mathbf{U}\cdot\mathbf{U} = \mathbf{U}\cdot\frac{\partial \mathbf{U}}{\partial t} = -\mathbf{U}\cdot[(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U}] - \mathbf{U}\cdot\boldsymbol{\nabla}P + \frac{1}{R}\mathbf{U}\cdot\Delta\mathbf{U}$$

Integrate over volume with periodic or zero velocity through boundaries.
The nonlinear term is conservative:

$$\int dV\ \mathbf{U}\cdot[(\mathbf{U}\cdot\boldsymbol{\nabla})\mathbf{U}] = \int dV\ \boldsymbol{\nabla}\cdot\left(\mathbf{U}\frac{U^2}{2}\right) - \frac{U^2}{2}\cancel{\boldsymbol{\nabla}\cdot\mathbf{U}} = \int dA\ \hat{\mathbf{n}}\cdot\mathbf{U}\frac{U^2}{2} = 0$$

We can also calculate an analogous calculation on the *nonlinear* equations which govern the growth of a non-infinitesimal perturbation $\boldsymbol{u}$ of the steady flow $\mathbf{U}$.

$$\boldsymbol{u}\cdot\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\mathbf{U} + \boldsymbol{u}\cdot(\mathbf{U}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}p + \frac{1}{R}\boldsymbol{u}\cdot\Delta\boldsymbol{u}$$

$$\boldsymbol{u}\cdot\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\mathbf{U}+\boldsymbol{u}\cdot(\mathbf{U}\cdot\boldsymbol{\nabla})\boldsymbol{u}+\boldsymbol{u}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}=-\boldsymbol{u}\cdot\boldsymbol{\nabla}p+\frac{1}{R}\boldsymbol{u}\cdot\Delta\boldsymbol{u}$$

$$\begin{aligned}
\boldsymbol{u}\cdot\partial_t\boldsymbol{u} &= \partial_t\left(\frac{|u|^2}{2}\right)\\
\boldsymbol{u}\cdot[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}] &= \boldsymbol{\nabla\cdot}\left(\boldsymbol{u}\frac{|u|^2}{2}\right)-\frac{|u|^2}{2}\boldsymbol{\nabla\cdot u}\\
\boldsymbol{u}\cdot[(\mathbf{U}\cdot\boldsymbol{\nabla})\boldsymbol{u}] &= \boldsymbol{\nabla\cdot}\left(\mathbf{U}\frac{|u|^2}{2}\right)-\frac{|u|^2}{2}\boldsymbol{\nabla\cdot}\mathbf{U}\\
-\boldsymbol{u}\cdot\boldsymbol{\nabla}p &= -\boldsymbol{\nabla\cdot}(p\,\boldsymbol{u})+p\,\boldsymbol{\nabla\cdot u}\\
\boldsymbol{u}\cdot\Delta\boldsymbol{u} &= \boldsymbol{\nabla\cdot\nabla}\left(\frac{|u|^2}{2}\right)-|\boldsymbol{\nabla u}|^2
\end{aligned}$$

Integrate over volume, use Gauss's theorem, incompressibility, and boundary conditions to eliminate all the terms above which are divergences.

$\Longrightarrow$ Reynolds-Orr equation:

$$\underbrace{\frac{d}{dt}\int dV\left(\frac{|u|^2}{2}\right)}_{\frac{dE(\boldsymbol{u})}{dt}=}=\underbrace{-\int dV\,\boldsymbol{u}\cdot[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\mathbf{U}]}_{\mathcal{P}(\boldsymbol{u})}\underbrace{-\frac{1}{R}\int dV\,|\boldsymbol{\nabla u}|^2}_{-\frac{1}{R}\mathcal{D}(\boldsymbol{u})}$$

$$\underbrace{\frac{d}{dt}\int dV \left(\frac{|u|^2}{2}\right)}_{\frac{dE(\boldsymbol{u})}{dt}=} = \underbrace{-\int dV\, \boldsymbol{u}\cdot[(\boldsymbol{u}\cdot\boldsymbol{\nabla})\mathbf{U}]}_{\mathcal{P}(\boldsymbol{u})} \underbrace{-\frac{1}{R}\int dV\, |\boldsymbol{\nabla}\boldsymbol{u}|^2}_{-\frac{1}{R}\mathcal{D}(\boldsymbol{u})}$$

$\mathcal{D}(\boldsymbol{u})/R$ is energy lost by $\boldsymbol{u}$ to viscous dissipation.
How is $\boldsymbol{u}$ supplied with energy? Base flow $\mathbf{U}$ is maintained (for example, by providing pressure gradient for Poiseuille flow or by moving the bounding plates for Couette flow).
$\mathcal{P}(\boldsymbol{u})$ measures energy given by base flow $\mathbf{U}$ to perturbation $\boldsymbol{u}$. The evolution of the energy of $\boldsymbol{u}$ comes from terms in N-S equations which are linear in $\boldsymbol{u}$: viscous dissipation and the nonlinear term linearized about $\mathbf{U}$. We see that

$$\frac{1}{E(\boldsymbol{u})}\frac{dE(\boldsymbol{u})}{dt}$$

is independent of the amplitude of $\boldsymbol{u}$, which is a consequence of the fact that the evolution of the energy of $\boldsymbol{u}$ comes from terms which are linear in $\boldsymbol{u}$.

Joseph (1976) defined an critical energy Reynolds number R_E such that
– For $R < R_E$, $\dot{E}(u) < 0$ for all $\boldsymbol{u}$.
– For $R > R_E$, there exists a perturbation $\boldsymbol{u}$ such that $\dot{E}(\boldsymbol{u}) > 0$.

Can show that:

$$\frac{1}{R_E} = \max_{\boldsymbol{u}} \left(-\frac{\mathcal{P}(\boldsymbol{u})}{\mathcal{D}(\boldsymbol{u})} \right)$$

Maximum is realized for perturbations $\boldsymbol{u}$ with $\beta \neq 0$:

Flow	R_E	β
plane Poiseuille	49.7	2.05
plane Couette	20.7	1.56

Transient growth

(Butler & Farrell, 1992, Trefethen et al., 1993, Schmid & Henningson, 1994)
Consider the model problem:

$$\frac{d}{dt}\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} -1/R & 0 \\ 1 & -2/R \end{bmatrix}\begin{bmatrix} v \\ \eta \end{bmatrix}$$

Matrix is upper triangular $\Longrightarrow$ eigenvalues are $-1/R$ et $-2/R$, both negative.
The corresponding eigenvectors are:

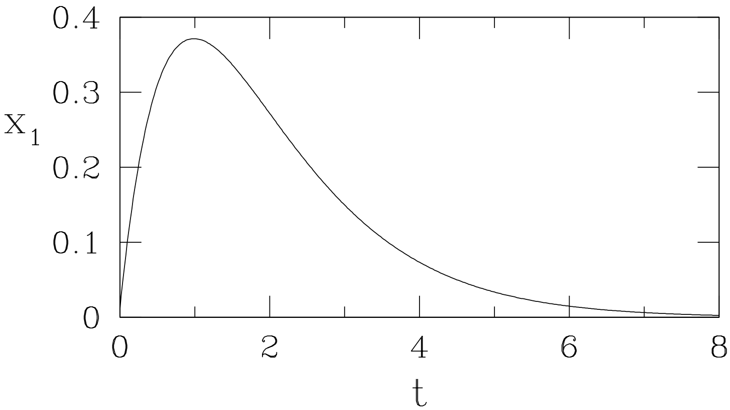
$$\lambda_1 = -\frac{1}{R} \ : \quad \begin{bmatrix} 1 \\ R \end{bmatrix} \qquad \lambda_2 = -\frac{2}{R} \ : \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The variables evolve as

$$\begin{bmatrix} v \\ \eta \end{bmatrix} = v_0 \begin{bmatrix} 1 \\ R \end{bmatrix} e^{-t/R} + (\eta_0 - v_0 R)\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t/R}$$

Expand for t small:

$$\begin{aligned} \eta &= \eta_0 e^{-2t/R} + R v_0 (e^{-t/R} - e^{-2t/R}) \\ &= \eta_0 e^{-2t/R} + R v_0 (1 - \frac{t}{R} + \cdots - 1 + \frac{2t}{R} - \cdots) \\ &= \eta_0 e^{-2t/R} + v_0 t \end{aligned}$$



The difference between two *decreasing* exponentials can lead to algebraic *growth* over short times, called *transient growth*, as is seen in Jordan blocks.

The matrix approaches a Jordan block as $R \to \infty$. This tendency can also be seen via the scalar product between the eigenvectors:

$$\begin{bmatrix} 1 \\ R \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = R = \sqrt{1+R^2}\cos\phi$$

where ϕ is the angle between the two eigenvectors.
$\phi \to 0 \Longrightarrow$ eigenvectors become parallel as $R \to \infty$.

Squire's Theorm shows lowest R for growing eigenmodes found for $\beta = 0$.
Not so for eigenvectors showing the largest transient growth.

$$\text{Define amplification factor: } G \equiv \max_{\alpha,\beta,t,\hat{v}_0,\hat{\eta}_0} \frac{E(t)}{E(0)}$$

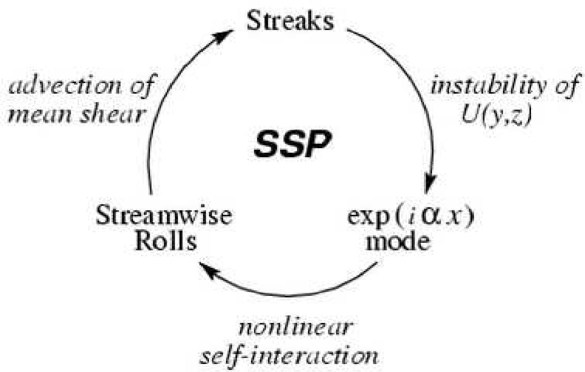
Flow	G	$t_{\max}$	$\alpha_{\max}$	$\beta_{\max}$
plane Poiseuille	$0.20\ R^2 \times 10^{-3}$	$0.076\ R$	0	2.04
plane Couette	$1.18\ R^2 \times 10^{-3}$	$0.117\ R$	$35/R$	1.6
pipe Poiseuille	$0.07\ R^2 \times 10^{-3}$	$0.048\ R$	0	1
Blasius boundary layer	$1.18\ R^2 \times 10^{-3}$	$0.778\ R$	0	0.65

$G = 200$ for plane Poiseuille flow at $R = 1000$
$G = 100$ for plane Couette flow at $R = 300$
Near where transition to turbulence occurs

Perturbation which maximizes G has $\alpha = 0$ or $\alpha \to 0$.
Plane Couette flow: $\beta \approx \pi/2 \Longrightarrow \lambda_z/2 = \pi/\beta \approx 2$, distance between plates.

Perturbations with largest transient growth ("optimal perturbations") are longitudinal vortices, like convective rolls, aligned with base flow (x direction). These are observed experimentally and numerically.
Suggestive, but role of transient growth not demonstrated

Self-Sustaining Process



Fabian Waleffe, *Physics of Fluids*, **9**, 1997

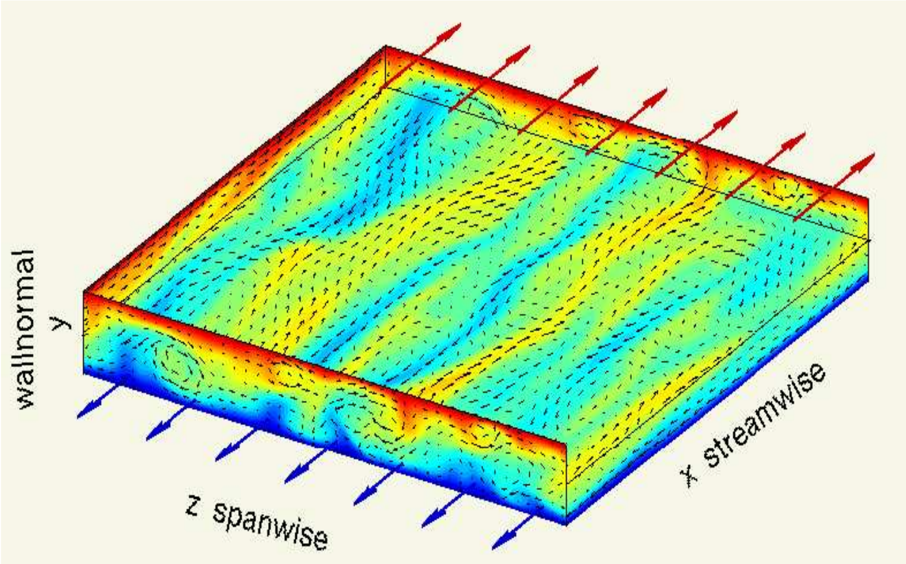
Based on numerical simulations of turbulence at low R, Waleffe (1990,1995) proposed a nonlinear 3D theory for transitional turbulence:

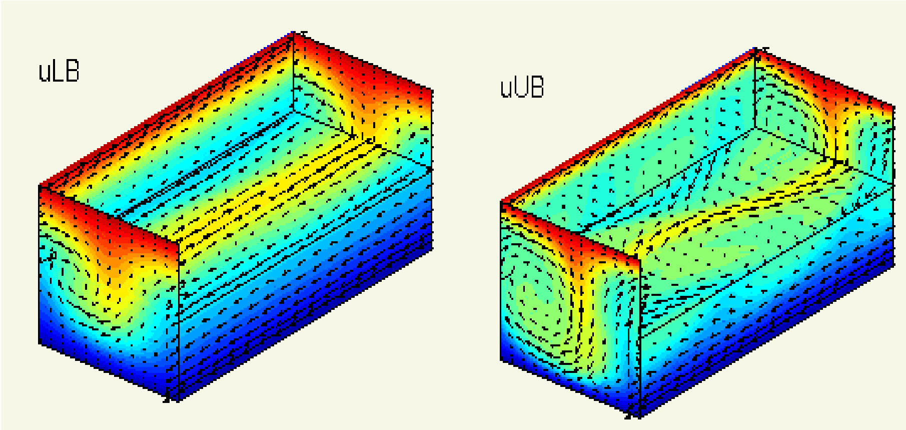
–Streamwise rolls sustain "streaks" (i.e. spanwise (z) modulation of the streamwise velocity)

–Streaks suffer a wake-like instability due to the spanwise inflections that leads to the onset of a streamwise ondulation,

–Nonlinear self-interaction of that streamwise ondulation regenerates the streamwise rolls.

Unstable steady states and travelling waves





Unstable steady states of plane Couette flow at $R = 400$, computed by Gibson and Cvitanovic (2008).

Until 1990, the only solutions known for plane Poiseuille and Couette flow and for pipe flow were the basic laminar solution and, in the case of plane Poiseuille flow, the two-dimensional Tollmien-Schlichting waves that bifurcate at $Re_L = 5772$. However, starting in 1990, large numbers of unstable solutions solutions of wall-bounded shear flows, such as plane Couette and pipe Poiseuille flow have been discovered computationally. It is hypothesized that weak turbulence can be understood as chaotic trajectories that visit in turn the vicinities of the various unstable branches In order to explain weak turbulence in wall-bounded shear flows, researchers focus on the unstable manifolds and time-dependent trajectories which connect the branches. The non-trivial solutions mostly consist of wavy longitudinal vortices ($\alpha, \beta \neq 0$) and are created via saddle-node bifurcations. The Reynolds-number threshold for weak turbulence in wall-bounded shear flows is sometimes thought to be related to the lowest of these saddle-nodes.