Hydrodynamics

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Mass Conservation

Mass density: $\rho(\boldsymbol{r}, t)$.

Total mass in fixed volume V:

$$M = \int_V \rho \, dV$$

Mass changes via flux flowing through boundary S of V:

$$\frac{dM}{dt} \equiv \frac{\partial}{\partial t} \int_V \rho \, dV = - \int_S \rho \boldsymbol{v} \cdot \boldsymbol{dS}.$$

Divergence theorem leads to:

$$\int_S \rho \boldsymbol{v} \cdot \boldsymbol{dS} = \int_V \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \, dV$$

Since V is arbitrary, integrands are equal:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0$$

Lagrangian Derivative

Physical laws apply to fluid particles/elements, not to positions in space.

Start with one dimension: $T(x,t)$ = temperature at (x,t)

Label particle by initial position ξ at $t=0$; its current position is $x(\xi,t)$.

Eulerian: function of (x,t) Lagrangian: function of (ξ,t)

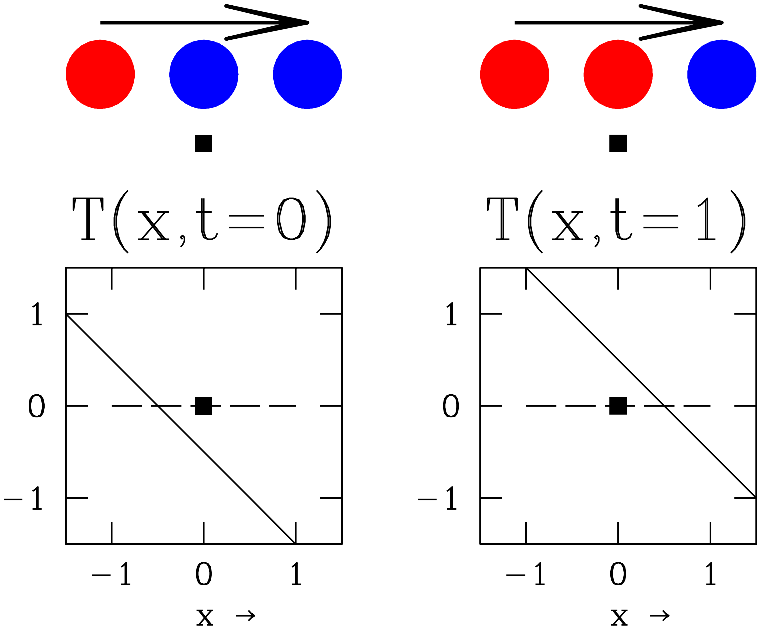
Temperature of particle is $T(x(\xi,t),t)$

Evolution in time of temperature of particle is

$$\frac{dT}{dt} = \left.\frac{\partial T}{\partial t}\right|_x + \left.\frac{\partial T}{\partial x}\right|_t \left.\frac{\partial x}{\partial t}\right|_\xi \qquad \left.\frac{\partial x}{\partial t}\right|_\xi = u(x(\xi,t),t)$$

Three dimensions: position $\boldsymbol{x} = (x,y,z)$, velocity $\boldsymbol{v} = (u,v,w)$

$$\frac{D}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$



$$\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x}$$

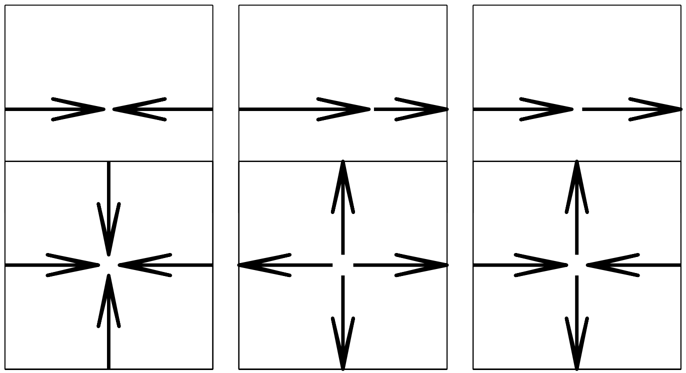
$$T(x,t=1) = T(x,t=0) - \underbrace{v}_{1}\underbrace{\frac{\partial T}{\partial x}}_{-1}\underbrace{[t]_0^1}_{1}$$

Mass Conservation Revisited

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

Divergence $\Longleftrightarrow$ source or sink



$\boldsymbol{\nabla} \cdot \boldsymbol{v} \neq 0$ $\boldsymbol{\nabla} \cdot \boldsymbol{v} \neq 0$ $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$

$$\partial_t \rho = \boldsymbol{\nabla} \rho = 0 \Longleftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0.$$

Water flowing out of faucet

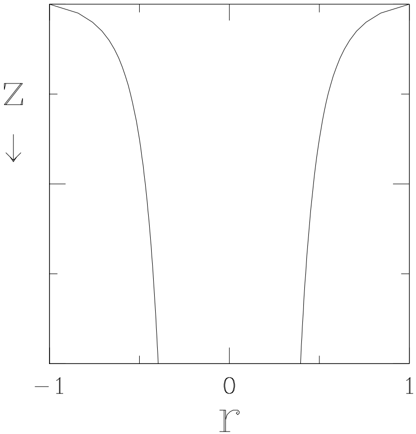
Top of faucet $z = 0$: $\quad A = A_0, \quad v = v_0, \quad t = 0$
At distance downwards z: $A = A(z), v = v(z(t)) = v_0 + gt \Longrightarrow t = (v - v_0)/g$

$$\begin{aligned} z(t) &= v_0 t + gt^2/2 = v_0(v - v_0)/g + g((v - v_0)/g)^2/2 \\ &= \left[v_0(v - v_0) + (v - v_0)^2/2\right]/g \\ &= \left[v_0 v - v_0^2 + v^2/2 + v_0^2/2 - v v_0\right]/g = (v^2 - v_0^2)/(2g) \\ v^2 &= v_0^2 + 2gz \end{aligned}$$

Mass conservation for steady flow:

$$\int_S \rho \boldsymbol{v} \cdot \boldsymbol{dA} = \rho[v_0 A_0 - v(z)A(z)] = 0 \Longrightarrow A = A_0 v_0 / v$$

$$A = \frac{A_0 v_0}{\sqrt{v_0^2 + 2gz}} \Longrightarrow R = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{A_0 v_0}{\pi \sqrt{v_0^2 + 2gz}}}$$



$$
\begin{aligned}
\frac{1}{R}\frac{\partial(Rv_R)}{\partial R} &+ \frac{\partial v_z}{\partial z} = 0 \\
\frac{1}{R}\frac{\partial(Rv_R)}{\partial R} &= -\frac{\partial v_z}{\partial z} = -\frac{\partial}{\partial z}[v_0^2 + 2gz]^{1/2} \\
&= -g[v_0^2 + 2gz]^{-1/2} \\
\frac{\partial(Rv_R)}{\partial R} &= -gR[v_0^2 + 2gz]^{-1/2} \\
Rv_R &= -\frac{gR^2}{2}[v_0^2 + 2gz]^{-1/2} + f(z) \\
v_R &= -\frac{gR}{2}[v_0^2 + 2gz]^{-1/2} + \frac{f(z)}{R}
\end{aligned}
$$

Forces

Newton's 2nd law:

$$\rho\, dV \frac{D\boldsymbol{v}}{Dt} = \rho\, dV \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\, \boldsymbol{v}\right] = \boldsymbol{F}$$

Pressure exerts force only if not spatially uniform

One dimension: pressure force in x direction on slab of dimensions dx, dy, dz

$$[P(x - dx/2, y, z, t) - P(x + dx/2, y, z, t)]\, dy\, dz = -\frac{\partial P}{\partial x} dV$$

Three dimensions: force is $-\boldsymbol{\nabla} P\, dV$.

Gravity:

$$-\rho g \boldsymbol{e}_z = -\rho \boldsymbol{\nabla} \Phi \Longrightarrow \Phi = gz$$

Case of incompressible liquid: ρ constant and uniform.

Force balance on volume element dV resulting from pressure and gravity:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi$$

Curvilinear Coordinates

Cartesian: $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{\nabla})(v_x\boldsymbol{e_x} + v_y\boldsymbol{e_y} + v_z\boldsymbol{e_z})$

Cylindrical: $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\nabla}(v_R\boldsymbol{e_R} + v_\phi\boldsymbol{e_\phi} + v_z\boldsymbol{e_z})$

Since

$$\boldsymbol{e_R} = (\cos\phi, \sin\phi, 0) \qquad \boldsymbol{e_\phi} = (-\sin\phi, \cos\phi, 0) \qquad \boldsymbol{e_z} = (0, 0, 1)$$

we have

$$\frac{\partial \boldsymbol{e_R}}{\partial\phi} = \boldsymbol{e_\phi} \qquad \frac{\partial \boldsymbol{e_\phi}}{\partial\phi} = -\boldsymbol{e_R}$$

so

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\nabla}(v_R\boldsymbol{e_R} + v_\phi\boldsymbol{e_\phi} + v_z\boldsymbol{e_z}) = \begin{cases} \left((\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R - \dfrac{v_\phi^2}{R}\right)\boldsymbol{e_R} \\ \left((\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\phi + \dfrac{v_R v_\phi}{R}\right)\boldsymbol{e_\phi} \\ (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_z\boldsymbol{e_z} \end{cases}$$

Rotating Frames

Constant angular velocity $\Omega \boldsymbol{e_z}$
Measure velocities relative to rotating frame $\Longleftrightarrow$ add fictitious accelerations:

$$\underbrace{-2\Omega \boldsymbol{e_z} \times \boldsymbol{v}}_{\text{Coriolis}} + \underbrace{R\Omega^2 \boldsymbol{e_R}}_{\text{centrifugal}}$$

Taylor-Proudman Theorem

$$R\Omega^2 \boldsymbol{e_R} = \boldsymbol{\nabla} \frac{R^2\Omega^2}{2}$$

$$\rho \text{ constant} \implies \frac{\boldsymbol{\nabla} P}{\rho} = \boldsymbol{\nabla}\left(\frac{P}{\rho}\right)$$

Suppose $\partial_t \boldsymbol{v} \approx 0$ and $|\boldsymbol{v}| \ll R\Omega$ so $|(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}| \ll |2\Omega \boldsymbol{e_z} \times \boldsymbol{v}|$ and other forces negligeable

$$\underbrace{\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}}_{\approx 0} = -2\Omega \boldsymbol{e_z} \times \boldsymbol{v} + \boldsymbol{\nabla}\left(-\frac{P}{\rho} + \frac{R^2\Omega^2}{2} - \Phi\right)$$

Geostrophic Equilibrium:

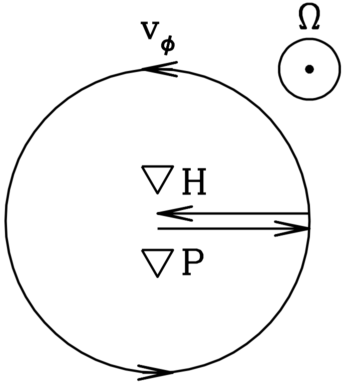
$$\begin{aligned}
\boldsymbol{\nabla} H = \boldsymbol{\nabla}\left(-\frac{P}{\rho}+\frac{R^2\Omega^2}{2}-\Phi\right) &= 2\Omega \boldsymbol{e_z}\times\boldsymbol{v} \\
\boldsymbol{\nabla}\times\boldsymbol{\nabla}\left(-\frac{P}{\rho}+\frac{R^2\Omega^2}{2}-\Phi\right) &= \boldsymbol{\nabla}\times 2\Omega \boldsymbol{e_z}\times\boldsymbol{v} \\
0 &= 2\Omega\boldsymbol{\nabla}\times(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x}) \\
&= 2\Omega\left\{\begin{array}{l}
\boldsymbol{e_x}\left[\partial_y(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x})_z-\partial_z(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x})_y\right] \\
\boldsymbol{e_y}\left[\partial_z(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x})_x-\partial_x(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x})_z\right] \\
\boldsymbol{e_z}\left[\partial_x(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x})_y-\partial_y(v_x\boldsymbol{e_y}-v_y\boldsymbol{e_x})_x\right]
\end{array}\right. \\
&= 2\Omega\left\{\begin{array}{c}
\boldsymbol{e_x}\left[-\partial_z v_x\right] \\
\boldsymbol{e_y}\left[\partial_z(-v_y)\right] \\
\boldsymbol{e_z}\left[\partial_x v_x-\partial_y(-v_y)\right]
\end{array}\right. = -2\Omega\left\{\begin{array}{l}
\boldsymbol{e_x}\partial_z v_x \\
\boldsymbol{e_y}\partial_z v_y \\
\boldsymbol{e_z}\partial_z v_z
\end{array}\right.
\end{aligned}$$

Strong rotation $\Longrightarrow$ $\boldsymbol{v}$ independent of z

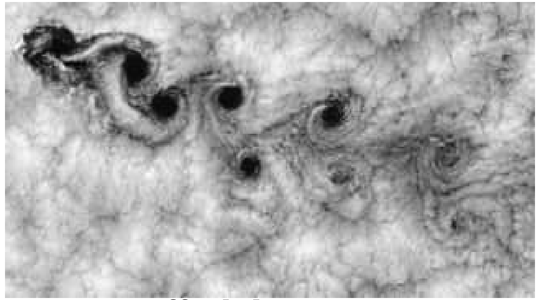
$v_z = 0$ at bottom $\Longrightarrow$ v_z at all heights

Flow in atmosphere is approximately horizontal and independent of height

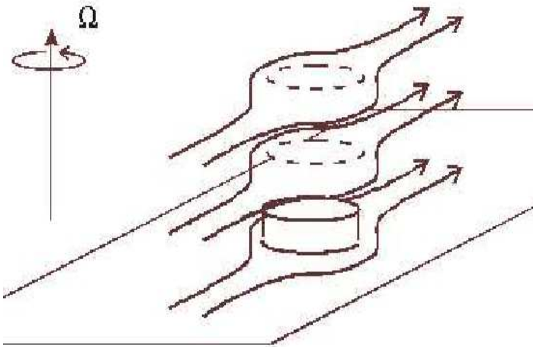
Low-lying obstacle $\Longrightarrow$ fluid avoids entire tall column above obstacle



Geostrophic equilibrium



Off Chilean coast
past Juan Fernandez islands



Taylor column in rotating flow

Suppose $\boldsymbol{v} = \boldsymbol{e}_\phi v_\phi(r)$

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \begin{cases} \left((\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R - \dfrac{v_\phi^2}{R}\right)\boldsymbol{e_R} \\ \left((\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\phi + \dfrac{v_R v_\phi}{R}\right)\boldsymbol{e_\phi} \\ (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_z\boldsymbol{e_z} \end{cases} = \begin{cases} -\dfrac{v_\phi^2}{R}\boldsymbol{e_R} \\ 0\,\boldsymbol{e_\phi} \\ 0\,\boldsymbol{e_z} \end{cases}$$

$$\begin{aligned} \frac{v_\phi^2}{R} &= \partial_R\left(\frac{P}{\rho} - \frac{R^2\Omega^2}{2} + gz\right) = \partial_R f(R) \\ 0 &= \partial_z\left(\frac{P}{\rho} - \frac{R^2\Omega^2}{2} + gz\right) \Longrightarrow \frac{P}{\rho} - \frac{R^2\Omega^2}{2} + gz = f(R) \end{aligned}$$

Another derivation

Consider a trajectory

$$x(t)\mathbf{e}_x + y(t)\mathbf{e}_y = (x(t), y(t)) = x(t) + iy(t) = R(t)e^{i\phi(t)}$$

To translate from Cartesian to polar coordinates, we write:

$$\begin{aligned}
\partial_t(Re^{i\phi}) &= (\dot{R} + Ri\dot{\phi})e^{i\phi} = (v_R + iv_\phi)e^{i\phi} \\
\partial_t^2(Re^{i\phi}) &= (\ddot{R} + \dot{R}i\dot{\phi} + Ri\ddot{\phi})e^{i\phi} + (\dot{R} + Ri\dot{\phi})i\dot{\phi}e^{i\phi} \\
&= \left[(\ddot{R} - R\dot{\phi}^2) + i(R\ddot{\phi} + 2\dot{R}\dot{\phi})\right] e^{i\phi} = (a_R + ia_\phi)e^{i\phi}
\end{aligned}$$

$$a_R = \ddot{R} - R\dot{\phi}^2 = \partial_t \dot{R} - \frac{(R\dot{\phi})^2}{R} \qquad \Longrightarrow \quad \partial_t v_R = a_R + \frac{v_\phi^2}{R}$$

$$a_\phi = R\ddot{\phi} + 2\dot{R}\dot{\phi} = \begin{cases} R\ddot{\phi} + \dot{R}\dot{\phi} + \dot{R}\dot{\phi} = \partial_t(R\dot{\phi}) + \dfrac{\dot{R}(R\dot{\phi})}{R} & \Longrightarrow \quad \partial_t v_\phi = a_\phi - \dfrac{v_R v_\phi}{R} \\ \dfrac{R^2\ddot{\phi} + 2R\dot{R}\dot{\phi}}{R} = \dfrac{\partial_t(R^2\dot{\phi})}{R} & \Longrightarrow \quad \partial_t(Rv_\phi) = Ra_\phi \end{cases}$$

Polar coordinates introduce extra terms even in a non-rotating frame.

Rotating frame: define ϕ' and $v_{\phi'}$:

$$\phi' \equiv \phi - \Omega t \implies \dot{\phi}' = \dot{\phi} - \Omega$$
$$R\dot{\phi}' = R\dot{\phi} - \Omega R \implies v_{\phi'} = v_\phi - \Omega R \Longrightarrow a_{\phi'} = a_\phi$$

Rewrite equations in terms of (r, ϕ') and $(v_r, v_{\phi'})$:

$$\partial_t v_R - a_R = \frac{v_\phi^2}{R} = \frac{(v_{\phi'} + \Omega R)^2}{R} = \underbrace{\frac{v_{\phi'}^2}{R}}_{\text{polar coords}} + \underbrace{\Omega^2 R}_{\text{centrifugal}} + \underbrace{2\Omega v_{\phi'}}_{\text{Coriolis}}$$

$$\partial_t v_{\phi'} - a_{\phi'} = \partial_t (v_\phi - \Omega R) - a_{\phi'} = \partial_t v_\phi - \Omega\,(\partial_t R) - a_{\phi'} = \frac{-v_R v_\phi}{R} - \Omega v_R$$
$$= \frac{-v_R (v_{\phi'} + \Omega R)}{R} - \Omega v_R = \underbrace{\frac{-v_R v_{\phi'}}{R}}_{\text{polar coords}} - \underbrace{2\Omega v_R}_{\text{Coriolis}}$$

Rotating frame introduces additional centrifugal and Coriolis (fictitious) forces.

Hydrostatic pressure

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -2\Omega \boldsymbol{e_z} \times \boldsymbol{v} - \boldsymbol{\nabla}\left(\frac{P}{\rho} - \frac{R^2\Omega^2}{2} + gz\right)$$

Suppose $\boldsymbol{v} = 0$, $\Omega = 0$, $\Phi = gz$:

$$0 = -\boldsymbol{\nabla}\left(\frac{P}{\rho} + gz\right) \Longrightarrow P = -\rho g z + P_0$$

Suppose $\boldsymbol{v} = 0$, $\Omega \neq 0$, $\Phi = 0$:

$$0 = -\boldsymbol{\nabla}\left(\frac{P}{\rho} - \frac{R^2\Omega^2}{2}\right) \Longrightarrow P = \rho\frac{R^2\Omega^2}{2} + P_0$$

Both: $$0 = -\boldsymbol{\nabla}\left(\frac{P}{\rho} - \frac{R^2\Omega^2}{2} + gz\right) \Longrightarrow P = -\rho g z + \rho\frac{R^2\Omega^2}{2} + P_0$$

At a free surface, P is constant:

$$P_{\text{surf}} = -\rho g z_{\text{surf}} + \rho\frac{R^2\Omega^2}{2} + P_0 \Longrightarrow z_{\text{surf}} = \frac{R^2\Omega^2}{2g} + \frac{P_0 - P_{\text{surf}}}{\rho g}$$

Parabolic surface (Newton's bucket)

Vector Identities

Cartesian coordinates: i, j, k represent any of x, y, z

Sum over any index which appears twice:

$$\begin{aligned}
\boldsymbol{v} &= v_i \boldsymbol{e}_i = \sum_i v_i \boldsymbol{e}_i = v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z \\
\boldsymbol{A} \cdot \boldsymbol{B} &= A_i B_i \\
\boldsymbol{\nabla} f &= \boldsymbol{e}_i \partial_i f \\
\boldsymbol{\nabla} \cdot \boldsymbol{f} &= \partial_i f_i \\
(\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} &= (v_i \partial_i) v_j \boldsymbol{e}_j
\end{aligned}$$

Component j of equation of motion:

$$\rho[\partial_t + (v_i \partial_i)] v_j = -\partial_j P - \rho \partial_j \Phi$$

$$\begin{aligned}
\delta_{ij} = 1 &\quad \text{if } i = j, \quad 0 \text{ otherwise} \\
\epsilon^{ijk} = 1 &\quad \text{if } ijk = 123, 231 \text{ or } 312 \text{ (even permutation)} \\
\epsilon^{ijk} = -1 &\quad \text{if } ijk = 321, 132 \text{ or } 213 \text{ (odd permutation)} \\
\epsilon^{ijk} = 0 &\quad \text{if } i = j \text{ or } i = k \text{ or } j = k
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{A} &= \epsilon^{ijk} \boldsymbol{e}_i \partial_j A_k \\
\boldsymbol{A} \times \boldsymbol{B} &= \epsilon^{ijk} \boldsymbol{e}_i A_j B_k \\
\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) &= A_i (B \times C)_i = A_i \epsilon^{ijk} B_j C_k \\
\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) &= \epsilon^{ijk} \boldsymbol{e}_i A_j (B \times C)_k = \epsilon^{ijk} \boldsymbol{e}_i A_j \epsilon^{klm} B_l C_m = \epsilon^{kij} \epsilon^{klm} \boldsymbol{e}_i A_j B_l C_m
\end{aligned}$$

A useful identity:

$$\epsilon^{kij} \epsilon^{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}.$$

So

$$\begin{aligned}
\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) &= \epsilon^{kij} \epsilon^{klm} \boldsymbol{e}_i A_j B_l C_m \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \boldsymbol{e}_i A_j B_l C_m \\
&= \boldsymbol{e}_i A_j B_i C_j - \boldsymbol{e}_i A_j B_j C_i \\
&= \boldsymbol{B} (\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C} (\boldsymbol{A} \cdot \boldsymbol{B})
\end{aligned}$$