Laurette TUCKERMAN

laurette@pmmh.espci.fr

Hamiltonian Systems

Hamiltonian Systems

$$\mathcal{H}(p_1, \ldots p_N, q_1, \ldots q_N)$$

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

q_i : **positions** p_i : **momenta**

convention for number of degrees of freedom: N
(for dissipative systems, convention would be $2N$**)**

Conservation of volumes for $\dot{\mathrm{x}} = \mathrm{f}(x)$

$$
\begin{aligned}
V(t+dt) - V(t) &= dt \int_{\text{surface}} \mathrm{f} \cdot \mathrm{n}\, da \\
&= dt \int_{\text{volume}} \nabla \cdot \mathrm{f}\, dv \\
&= dt \int_{\text{volume}} \sum_i \frac{\partial \dot{x}_i}{\partial x_i}\, dv \\
&= dt \int_{\text{volume}} \sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) dv \\
&= dt \int_{\text{volume}} \sum_i \left(\frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} - \frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} \right) dv \\
&= 0
\end{aligned}
$$

Developed in 18th century for celestial mechanics, now used in plasma physics (e.g. fields in a Tokamak) quantum systems (e.g. quantum optics, Bose-Einstein condensation)

Fluid mechanics: 2D streamfunction $\psi(x, y)$ $\mathbf{u} = \mathbf{e_z} \times \nabla\psi$.
Motion of particle at (x, y):

$$\frac{dx}{dt} = u = -\frac{\partial\psi}{\partial y} \qquad \frac{dy}{dt} = v = \frac{\partial\psi}{\partial x}$$

Particles move along streamlines = curves of constant ψ

$\Longleftrightarrow$

In general, trajectories move along curves/surfaces of constant energy $\mathcal{H}$

Integrable systems

Hamiltonian system with N degrees of freedom is integrable if $\exists$ N functions $F_j(\mathrm{p},\mathrm{q})$ such that

$$\frac{dF_j}{dt} = 0 \qquad \textbf{and}$$

$$[F_j, F_k] \equiv \sum_i \left(\frac{\partial F_j}{\partial q_i}\frac{\partial F_k}{\partial p_i} - \frac{\partial F_j}{\partial p_i}\frac{\partial F_k}{\partial q_i} \right) = 0$$

Already have

$$\frac{d\mathcal{H}}{dt} = \sum_i \left(\frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i \right) = \sum_i \left(\frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} \right) = 0$$

so any system with one degree of freedom is integrable,

$$
\begin{aligned}
[F_j, \mathcal{H}] &= \sum_i \left(\frac{\partial F_j}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F_j}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} \right) \\
&= \sum_i \left(\frac{\partial F_j}{\partial q_i}\dot{q}_i + \frac{\partial F_j}{\partial p_i}\dot{p}_i \right) \\
&= \frac{dF_j}{dt}
\end{aligned}
$$

so

$$
\frac{dF_j}{dt} = 0 \implies [F_j, \mathcal{H}] = 0
$$

$N - 1$ functions F_j are needed for an N-degree-of-freedom system to be integrable.

For an integrable system, there exists a transformation

$$(\mathrm{I}, \theta) \leftarrow (\mathrm{p}, \mathrm{q})$$
$$\mathcal{H}'(\mathrm{I}, \theta) = \mathcal{H}(\mathrm{p}, \mathrm{q})$$

where, in fact,

$$\mathcal{H}'(\mathrm{I}, \theta) = \mathcal{H}'(\mathrm{I})$$

so that the dynamics in the (I, θ) variables are

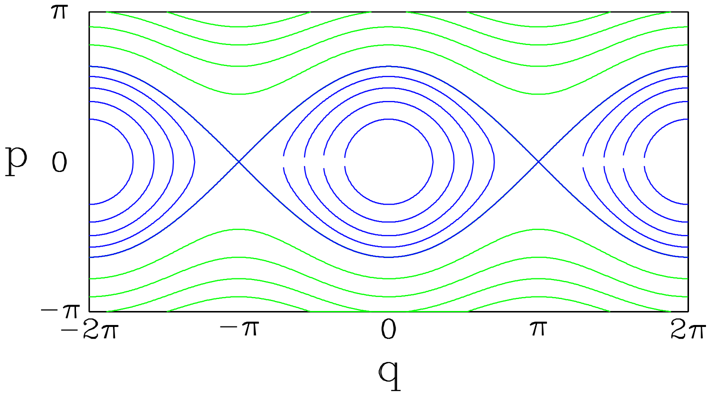
$$\dot{I}_i = -\frac{\partial \mathcal{H}'}{\partial \theta_i} = 0 \qquad \dot{\theta}_i = \frac{\partial \mathcal{H}'}{\partial I_i} = \omega_i(\mathrm{I})$$
$$I_i(t) = I_i(0) \qquad \theta(t) = \theta(0) + t\omega_i$$

Called twist map (I, θ) are called action-angle variables

Classic pendulum

$$\mathcal{H} = \frac{1}{2}p^2 - \cos q$$

q: angle (position), p: (velocity = momentum)



Phase portrait for the classic pendulum

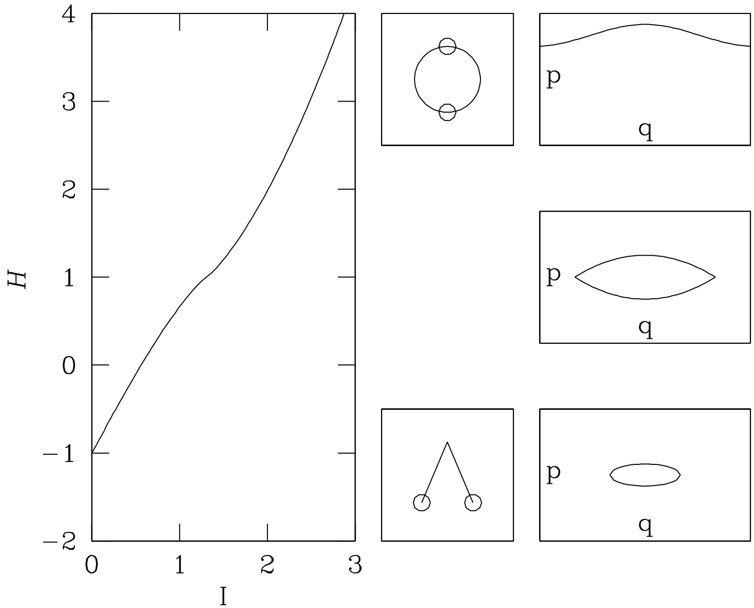
$$I \equiv \frac{1}{2\pi} \oint p dq$$

Integral taken over a closed trajectory, on which $\mathcal{H}$ has the constant value H. For the pendulum:

$$\begin{aligned} H &= \frac{1}{2}p^2 - \cos q \\ p^2 &= 2(H + \cos q) \\ p &= \sqrt{2(H + \cos q)} \\ I &= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2(H + \cos q)}\, dq \end{aligned}$$

$\Longrightarrow$ I (canonical action variable) as a function of value of H (can be inverted to define $\mathcal{H}'$ as function of I)

Also define θ (canonical angle variable) such that $\dot{\theta} = \omega$ is constant in time



Left: $\mathcal{H}(I)$. Middle: pendulum configuration for $\mathcal{H} > 1$ (repeated clockwise or counterclockwise rotations) and $\mathcal{H} < 1$ (small oscillations) Right: (q, p) trajectories, heteroclinic orbit at $\mathcal{H} = 1, \omega = d\mathcal{H}/dI = 0$.

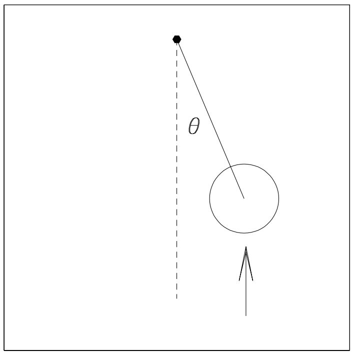
Non-integrable perturbations

All Hamitonian systems with $N = 1$ are integrable

Simplest non-integrable systems are of form:

$\mathcal{H}(p_1, p_2, q_1, q_2)$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $N = 2$

$\mathcal{H}(p, q, t)$ $\quad\quad\quad\quad\quad\quad$ **sometimes called** $N = 1.5$



Rotor in horizontal plane gets a "kick" with period $\tau = 2\pi$

$$\mathcal{H}(v,\theta,t) = \frac{v^2}{2} + \epsilon \cos\theta \sum_n \delta(t - n\tau)$$

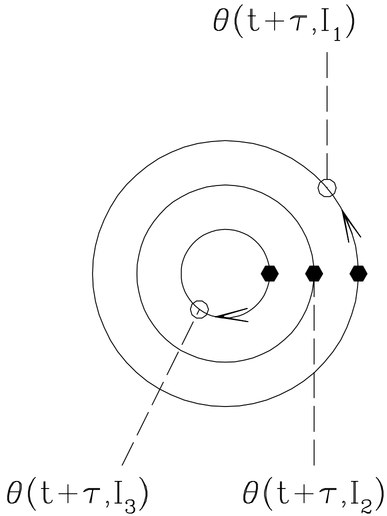
$$\begin{array}{l|l} \dot{\theta} = \frac{\partial \mathcal{H}}{\partial v} = v & \theta_{n+1} - \theta_n = v_n \mod 2\pi \\ \dot{v} = -\frac{\partial \mathcal{H}}{\partial \theta} = \epsilon \sin\theta \sum_n \delta(t - n\tau) & v_{n+1} - v_n = \epsilon \sin\theta_{n+1} \end{array}$$

$$\mathcal{H} = \mathcal{H}_0(\theta, v) + \epsilon\,\mathcal{H}_1(\theta, v, t)$$

$$\mathcal{H}_0 = \frac{v^2}{2}$$

$\mathcal{H}_0 = v^2/2$ is integrable and already in action-angle variables $\Longrightarrow$ phase space = set of concentric curves, each with its own (constant) angular velocity.

Unperturbed Twist Map



Points on circle I_j rotate with velocity I_j

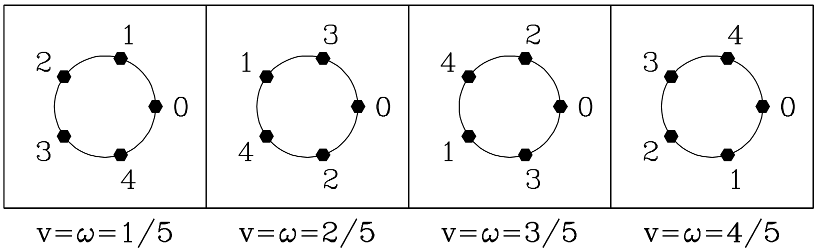
Here $I \sim p \sim v \sim \omega \sim r$

Define the Poincaré or first return map arising from $\mathcal{H}_0$:

$$T_0(I,\theta) = (I,\theta)(t = 2\pi) = (I, (\theta + 2\pi I) \bmod 2\pi)$$

Each circle is invariant under T_0, but its individual points are not necessarily invariant.

Circle $v = 0$ circle consists of fixed points, circle $v = 1/2$ consists of 2-cycles, circle $v = 1/3$ consists of 3-cycles.



Five-cycles of map T_0

Define $n^{\rm th}$ iterate of T_0.

$$T_0^n(I,\theta) = (I,\theta)(t=2\pi n) = (I,(\theta+2\pi n\, I) \mod 2\pi)$$

If $v = I = \omega(I) = m/n$, then

$$\begin{aligned} T_0^n(I,\theta) &= \left(I, \left(\theta + 2\pi\, n\,\frac{m}{n}\right) \mod 2\pi\right) \\ &= (I, (\theta + 2\pi\, m) \mod 2\pi) = (I,\theta) \end{aligned}$$

so circles $I = m/n$ consist of fixed points of T_0^n.
Re-introduce the perturbation:

$$\mathcal{H}_\epsilon \equiv \mathcal{H}_0 + \epsilon\, \mathcal{H}_1$$

and the corresponding maps

$$T_\epsilon^n(I,\theta) \equiv (I,\theta)(t = 2\pi\, n)$$

where I and θ evolve according to Hamiltonian $\mathcal{H}_\epsilon$

Poincaré-Birkhoff Theorem

Action of map T_ϵ^n on curve $I = \frac{m}{n}$:

–The image of (I, θ) under T_ϵ^n is

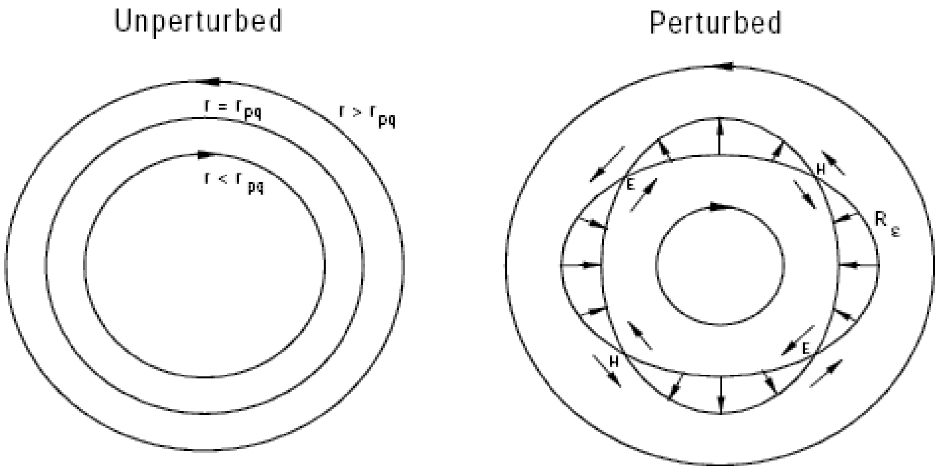
$$T_\epsilon^n(I, \theta) = (I', \theta)$$

The radius changes but not the angle.

–Curves (I, θ) and (I', θ) intersect each other a multiple of $2n$ times, creating alternating hyperbolic and elliptic points.

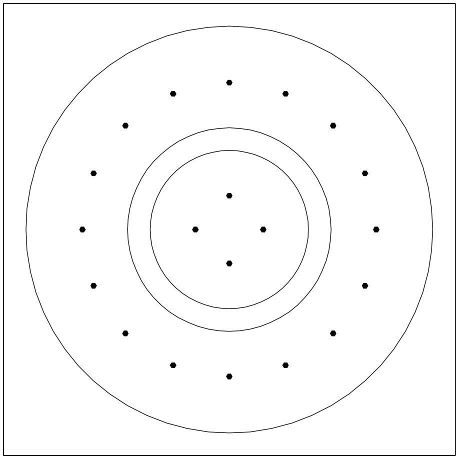
–Area inside (I', θ) is the same as that inside (I, θ).

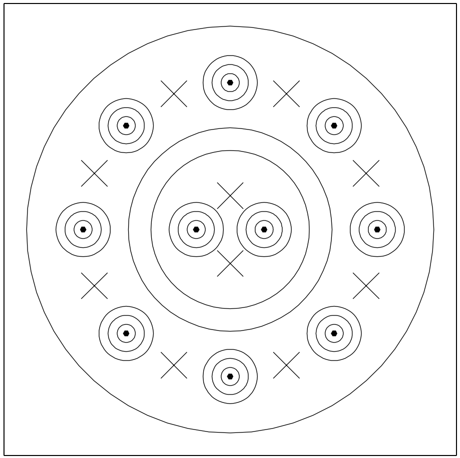
Dynamics of T^q near a circle with rational $I = p/q$



Left: Points on intermediate circle are fixed, those on outer (inner) circle rotate counterclockwise (clockwise)
Right: P-B theorem implies curves (I, θ) and $T_\epsilon^n(I, \theta) = (I', \theta)$ intersect at alternating set of elliptic and hyperbolic fixed points. Angular flow is counterclockwise outside and clockwise inside and radial flow alternates inwards and outwards.

Each new elliptic point is now surrounded by invariant circles, some of which have rational winding numbers.
Poincaré-Birkhoff theorem applies recursively to each one!





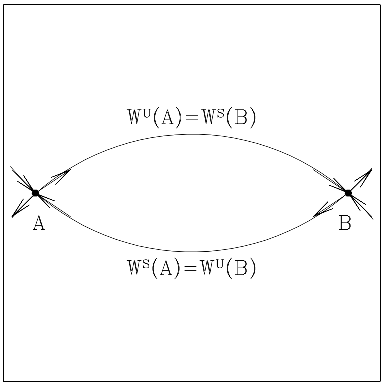
Fixed points of T_0^n $\Longrightarrow$ elliptic or hyperbolic points of T_ϵ^n

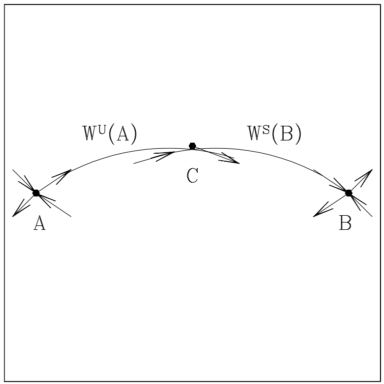
What happens to new hyperbolic points?

Unstable and stable manifold of hyperbolic A and B are points approaching them in iterating backwards or forwards:

$$W^U(A) \equiv \left\{x : \lim_{k\to\infty} T^{-k}(x) = A\right\}$$

$$W^S(B) \equiv \left\{x : \lim_{k\to\infty} T^{k}(x) = B\right\}$$





$W^U(A) = W^S(B)$ **intersect transfersely**

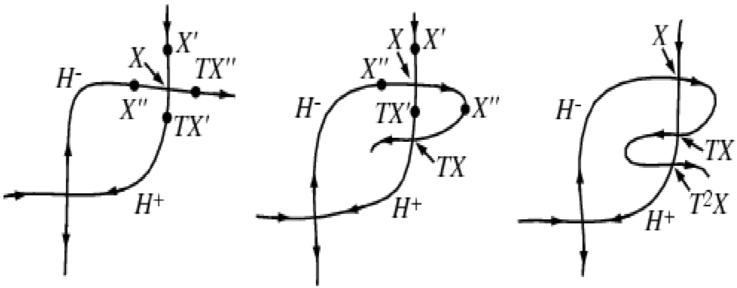
integrable $\mathcal{H}$ **non-integrable $\mathcal{H}$**

Non-integrable pert $\Longrightarrow$ $W^U(A)$ and $W^S(B)$ cross at C

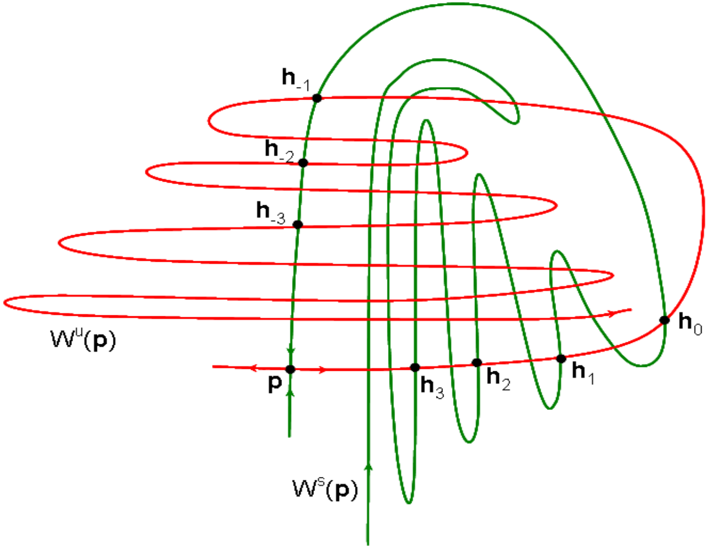
T^k and T^{-k} map C into other points, all in $W^U(A) \cap W^S(B)$

Infinite number of intersections accumulate at A and B

Intersections along decreasing distances $+$ area conservation $\Longrightarrow$ perpendicular directions increase $\Longrightarrow$ homoclinic tangles near A and B $\Longrightarrow$ chaos = separation of nearby points = sensitivity to initial conditions (SIC)



From E. Weisstein, *Homoclinic Tangle*, MathWorld: A Wolfram Web Resource
http://mathworld.wolfram.com/HomoclinicTangle.html



From P. So, *Unstable periodic orbits*, **Scholarpedia 2(2): 1353**

The combined complexity of the chains of elliptic and hyperbolic points and the homoclinic tangles was said by Poincaré to be too complicated to describe. Arnold tried:

Solid ellipses: surviving tori, whose winding numbers are sufficiently far from any rational. Others break into alternating elliptic and hyperbolic points. Around each elliptic point is a set of elliptical trajectories. Each hyperbolic point is surrounded by a chaotic region. From V.I. Arnol'd, *Small denominators and problems of stability of motion in classical and celestial mechanics*, **Russian Mathematical Surveys 18:6, 85–191 (1963). Reprinted in** *Hamiltonian Dynamical Systems: a reprint collection*, **ed. R.S. MacKay & J.D. Meiss**

KAM Theorem

Kolmogorov (1954), Arnold (1961-3), Moser (1962)

Poincaré-Birkhoff Theorem: tori with rational winding numbers w are destroyed by non-integrable perturbation

What about tori with irrational w?
If the perturbation is sufficiently small, some survive.

A torus whose w is close to a rational with small denominator, ("not very irrational") is destroyed by a small perturbation.

A torus whose w is sufficiently far from all rationals (i.e. is "very irrational") requires a large perturbation to be destroyed.
"Most irrational" number is golden mean $(1+\sqrt{5})/2$, whose torus is last one destroyed, i.e. perturbation required is largest.

$\exists K(\epsilon)$ such that $$\lim_{\epsilon \to 0} K(\epsilon) = 0$$

and such that if w satisfies

$$\forall m, n \quad \left| w - \frac{m}{n} \right| > \frac{K(\epsilon)}{n^{5/2}}$$

then torus with winding number w survives pert of size ϵ

Estimate measure of interval of w of surviving tori:

Each denominator n corresponds to $\sim n$ rationals $\frac{1}{n}, \frac{2}{n}, \ldots \frac{n-1}{n}$

Surrounding each rational is w-interval of destroyed tori

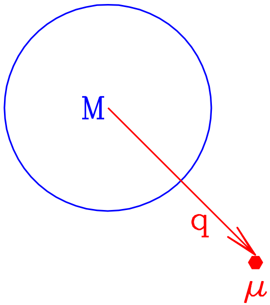
$\frac{m}{n} - \frac{K}{n^{5/2}} < w < \frac{m}{n} + \frac{K}{n^{5/2}}$

$$\sum_{n=1}^{\infty} \frac{2K(\epsilon)}{n^{5/2}} n = 2K(\epsilon) \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \leq 2K \int_{n=1}^{\infty} dx \frac{1}{x^{3/2}}$$

$$= 2K(\epsilon) \left[-\frac{2}{3} \frac{1}{x^{1/2}} \right]_1^{\infty} = \frac{4K(\epsilon)}{3} < 1 \text{ for small } \epsilon$$

For small ϵ, set of surviving w has finite (non-zero) measure!

Celestial Mechanics: I. Two-body problem



Mass μ **has**
position $\mathrm{q} = (r, \phi)$
momentum $\mathrm{p} = (p_r, p_\phi) = (\mu\dot{r}, \mu r^2\dot{\phi})$

$$\mathcal{H}_0(\mathrm{q},\mathrm{p}) = \frac{|\mathrm{p}|^2}{2\mu} - \frac{GM\mu}{r} = \frac{p_r^2 + p_\phi^2/r^2}{2\mu} - \frac{GM\mu}{r}$$

$$\frac{dr}{dt} = \frac{\partial \mathcal{H}_0}{\partial p_r} = \frac{p_r}{\mu} \qquad \frac{dp_r}{dt} = -\frac{\partial \mathcal{H}_0}{\partial r} = \frac{p_\phi^2}{\mu r^3} - \frac{GM\mu}{r^2}$$

$$\frac{d\phi}{dt} = \frac{\partial \mathcal{H}_0}{\partial p_\phi} = \frac{p_\phi}{\mu r^2} \qquad \frac{dp_\phi}{dt} = -\frac{\partial \mathcal{H}_0}{\partial \phi} = 0$$

$N = 2$ **degrees of freedom** (r, p_r, ϕ, p_ϕ)

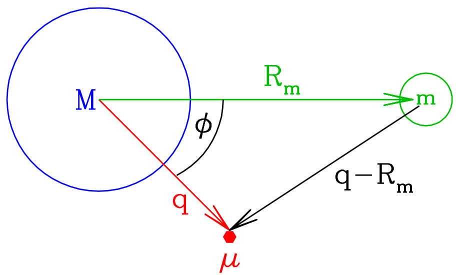
angular momentum p_ϕ **conserved** $\Longrightarrow$ **integrable**

Three-body problem

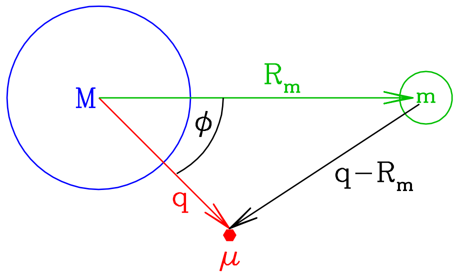
Historically, main motivation for studying celestial mechanics has been to determine whether the solar system is stable. Even three-body problem is known to be non-integrable.

Euler, Lagrange, Jacobi, Hill, Poincaré, Levi-Civita and Birkhoff studied circular restricted three-body problem (CRTBP):

$$M \gg m \gg \mu$$



m has circular orbit around M. Neither is affected by μ.



M **(sun)** $\gg$ m **(planet)** $\gg$ μ **(asteroid)**

Integrable two-body problem: μ **moves under influence of** M

Non-integrable perturbation: m **perturbs motion of** μ

μ **small:**

–neglect effect of μ **on** m **and** M

–identify center of mass of $M - m$ **system with center of** M

In addition, assume:

–m **rotates about** M **in circular orbit**

–μ **remains in plane containing** M **and** m

Integrable Hamiltonian system $\mathcal{H}_0$ without mass m

$$\mathcal{H}_0(\mathrm{q}, \mathrm{p}, t) = \frac{p_r^2 + p_\phi^2/r^2}{2\mu} - \frac{GM\mu}{r}$$

where p_ϕ is the angular momentum $\mu r^2 \dot{\phi}$

$$\frac{dr}{dt} = \frac{\partial \mathcal{H}_0}{\partial p_r} = \frac{p_r}{\mu}$$

$$\frac{d\phi}{dt} = \frac{\partial \mathcal{H}_0}{\partial p_\phi} = \frac{p_\phi}{\mu r^2} = \dot{\phi}$$

$$\frac{dp_r}{dt} = -\frac{\partial \mathcal{H}_0}{\partial r} = \frac{p_\phi^2}{\mu r^3} - \frac{GM\mu}{r^2}$$

$$= \frac{\mu^2 r^4 \dot{\phi}^2}{\mu r^3} - \frac{GM\mu}{r^2} = \mu r \dot{\phi}^2 - \frac{GM\mu}{r^2}$$

$$\frac{dp_\phi}{dt} = \frac{\partial \mathcal{H}_0}{\partial \phi} = 0 \Longrightarrow \text{conserved quantity}$$

Introduce intermediate mass m, at distance $|\mathrm{q} - \mathrm{R}_m|$ from μ:

$$\mathcal{H}(\mathrm{q}, \mathrm{p}, t) = \frac{|\mathrm{p}|^2}{2\mu} - \frac{GM\mu}{r} - \frac{Gm\mu}{|\mathrm{q} - \mathrm{R}_m(t)|}$$

Mass m follows circular orbit $\Longrightarrow \mathrm{R}_m(t)$ changes orientation $\Longrightarrow \mathcal{H}$ is non-autonomous (depends explicitly on time)

Autonomous $\mathcal{H}$ via rotating frame $\phi \to \phi'$ but retain p_ϕ:

$$\phi' = \phi - \Omega t$$

$$\frac{\partial \mathcal{H}'}{\partial p_\phi} = \frac{d\phi'}{dt} = \frac{d\phi}{dt} - \Omega = \frac{p_\phi}{\mu r^2} - \Omega$$

$$\mathcal{H}_0'(\mathrm{q}', \mathrm{p}) = \frac{p_r^2 + p_\phi^2/r^2}{2\mu} - \Omega p_\phi - \frac{GM\mu}{r}$$

$$\mathcal{H}'(\mathrm{q}', \mathrm{p}) = \frac{p_r^2 + p_\phi^2/r^2}{2\mu} - \Omega p_\phi - \frac{GM\mu}{r} - \frac{Gm\mu}{|\mathrm{q}' - \mathrm{R}_m|}$$

where R_m is now constant. Ωp_ϕ is Coriolis term.

Action variable $I_\phi = \frac{1}{2\pi}\oint_0^{2\pi} p_\phi \, d\phi = p_\phi$

Action variable I_r

$$
\begin{aligned}
p_r^2 &= 2\mu(H_0 + \Omega I_\phi + GM\mu/r) - I_\phi^2/r^2 \\
I_r &= \frac{1}{2\pi}\oint p_r dr = -I_\phi + \frac{GM\mu^2}{\sqrt{-2\mu(H_0 + \Omega I_\phi)}} \\
\frac{1}{I_r + I_\phi} &= \frac{\sqrt{-2\mu(H_0 + \Omega I_\phi)}}{GM\mu^2} \\
H_0 &= -\Omega I_\phi - \frac{1}{2\mu}\left(\frac{GM\mu^2}{I_r + I_\phi}\right)^2 \equiv \mathcal{H}_0(I_r, I_\phi)
\end{aligned}
$$

Frequencies

$$
\begin{aligned}
\omega_{0r} &= \frac{\partial \mathcal{H}_0}{\partial I_r} = \frac{(GM)^2\mu^3}{(I_r + I_\phi)^3} \equiv \omega_\mu \\
\omega_{0\phi} &= \frac{\partial \mathcal{H}_0}{\partial I_\phi} = -\Omega + \frac{(GM)^2\mu^3}{(I_r + I_\phi)^3} = -\Omega + \omega_\mu
\end{aligned}
$$

$$\mathcal{H}'(\mathrm{q}',\mathrm{p}) = \frac{p_r^2 + p_\phi^2/r^2}{2\mu} - \Omega p_\phi - \frac{GM\mu}{r} - \frac{Gm\mu}{|\mathrm{q}' - \mathrm{R}_m|}$$

$\mathcal{H}'$ depends on ϕ' through q' so

$$\frac{dp_\phi}{dt} = -\frac{\partial \mathcal{H}'}{\partial \phi'} = -\frac{\partial \mathcal{H}'}{\partial |q' - R_m|}\frac{\partial |q' - R_m|}{\partial \phi} \neq 0$$

$\mathcal{H}'$ non-integrable $\Longrightarrow$ m destroys tori with rational w of $\mathcal{H}_0'$

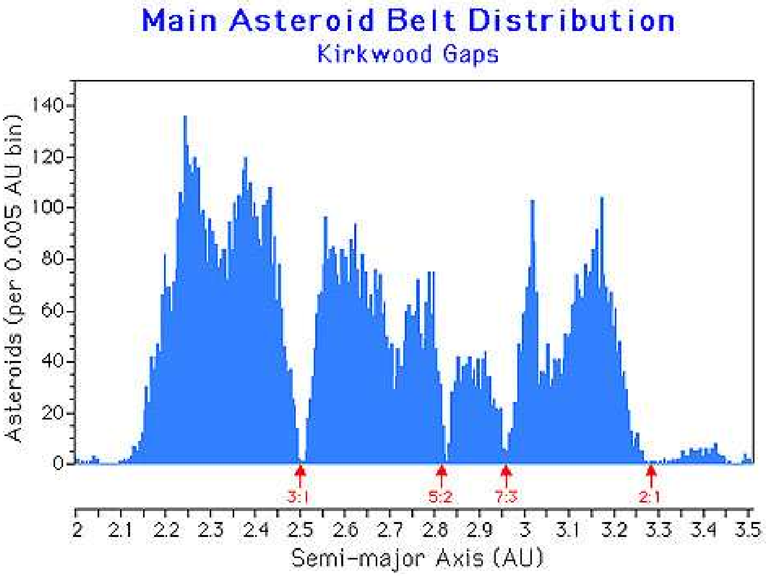
Winding number w is ratio

$$\frac{\omega_{0\phi}}{\omega_{0r}} = \frac{-\Omega + \omega_\mu}{\omega_\mu} = 1 - \frac{\Omega}{\omega_\mu}$$

For M=sun and m=Jupiter, Poincaré-Birkhoff theorem $\Longrightarrow$ Kirkwood gaps in frequencies (orbital paths) of asteroids μ

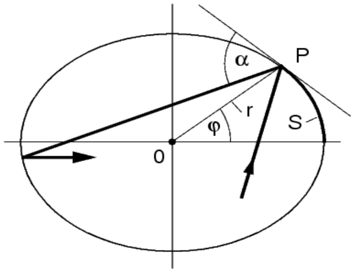
For M=Saturn and m=moon of Saturn, P-B theorem $\Longrightarrow$ gaps in the particles μ in the rings of Saturn

Kirkwood gaps



Gaps in the distribution of main belt asteroids as function of semi-major axis (equivalent to orbital period). They correspond to the location of orbital resonances with Jupiter.
From Wikipedia, *Kirkwood gap*

Billiards



Generalized billiard table bounded by closed curve $r(\phi)$

From H.J. Korsch & F. Zimmer, *Chaotic Billiards*,
`http://kluedo.ub.uni-kl.de/frontdoor.php?source_opus=1202`
M.V. Berry, *Regularity and chaos in classical mechanics, illustrated by three deformations of a circular billiard*, **Eur. J. Phys 2, 91-102 (1981)**

Trajectory = sequence of bounces = (ϕ_n, α_n)
Bounce = (angular position ϕ, angle α with tangent to table)

$$(\phi_n, \alpha_n) \Longrightarrow (\phi_{n+1}, \alpha_{n+1})$$

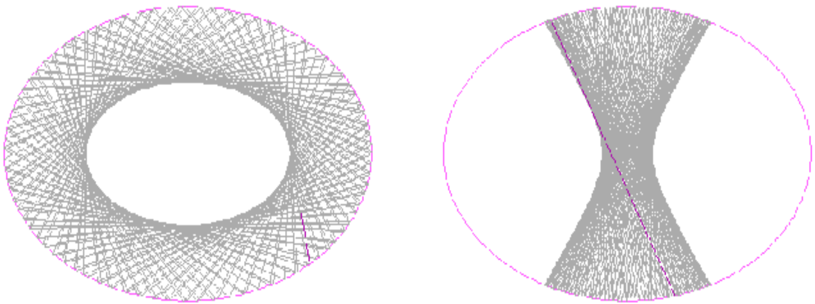
Alternately, use arclength S along table and/or $p = \cos(\alpha)$

$$(S_n, p_n) \Longrightarrow (S_{n+1}, p_{n+1})$$

Circle $r = c \Longrightarrow$ integrable dynamics since α is conserved

$$\text{Orbits are} \begin{cases} \text{periodic} & \text{if } \alpha/2\pi \text{ is rational} \\ \text{quasiperiodic} & \text{if } \alpha/2\pi \text{ is irrational} \end{cases}$$

Elliptical billiards are also integrable.



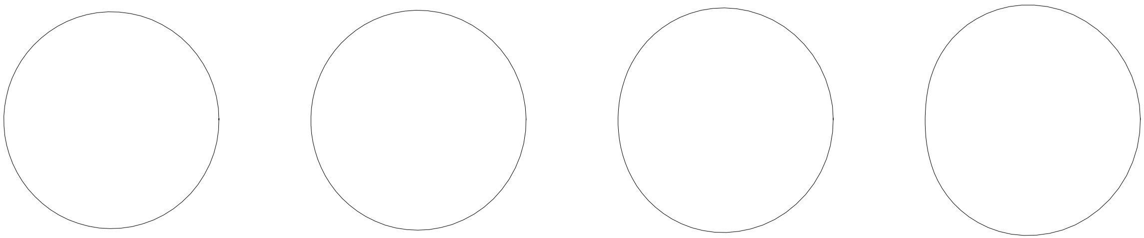
Trajectories in elliptical billiards. From H.J. Korsch & F. Zimmer, *Chaotic Billiards*

`http://kluedo.ub.uni-kl.de/frontdoor.php?source_opus=1202`.

Stadium (Bunimovich) = two line segments connected by two semi-circles. Dynamics are ergodic: all points are visited.

Cosine Billiards

$$r(\phi) = 1 + \epsilon \ \cos(\phi)$$



$\epsilon = 0$ $\epsilon = 0.1$ $\epsilon = 0.2$ $\epsilon = 0.3$

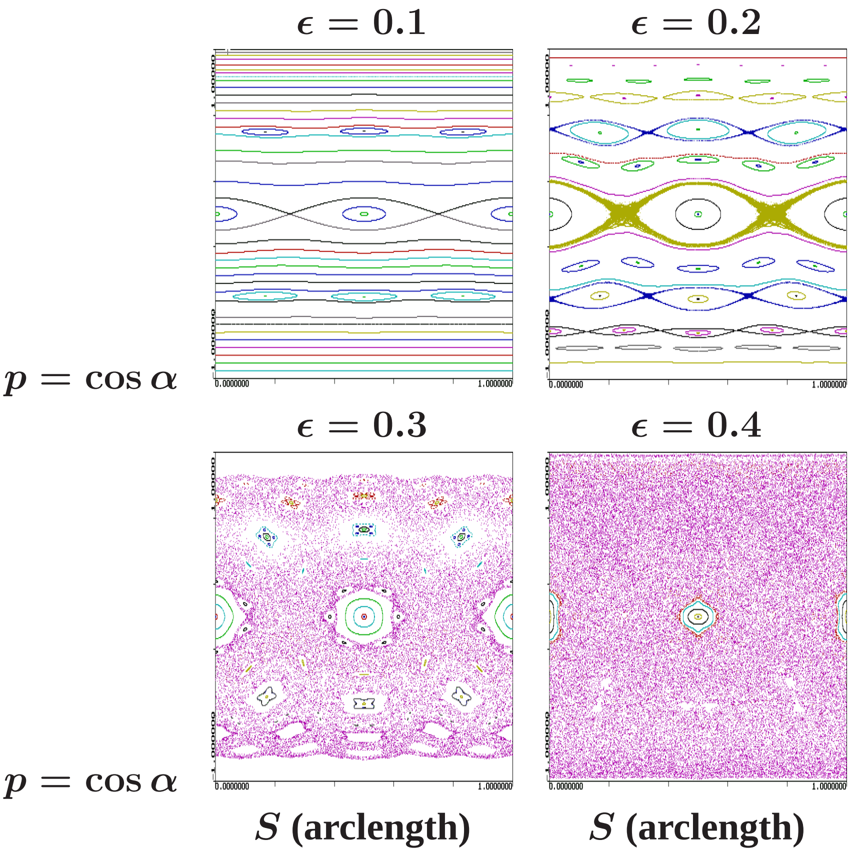
$\epsilon = 0$: integrable circle case

ϵ increases: rational tori break up,
destroying irrational tori close to small-denominator fractions

$\epsilon = \epsilon_c$: last torus, with $w = w^*$ golden mean, breaks up

Low-denominator fractions m/n with angles α

m/n	α	$p = \cos(\alpha)$	m/n	α	$p = \cos(\alpha)$
$1/2$	$90°$	0	$1/5$	$36°$	0.809
$1/3$	$60°$	$1/2$	$2/5$	$72°$	0.309
$2/3$	$120°$	$-1/2$	$3/5$	$108°$	-0.309
$1/4$	$45°$	$1/\sqrt{2}$	$4/5$	$144°$	-0.809
$3/4$	$135°$	$-1/\sqrt{2}$			
$\sqrt{2}$	$127.279°$	-0.6057	$w^* = 0.618$	$291.262°$	0.3626



From H.J. Korsch & F. Zimmer, *Chaotic Billiards*
`http://kluedo.ub.uni-kl.de/frontdoor.php?source_opus=1202`.

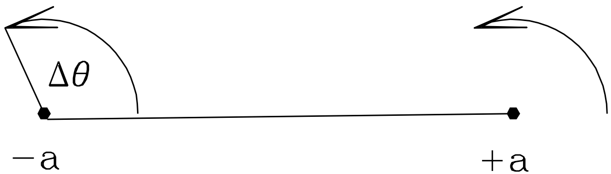
Fluid dynamics

Fluid-dynamical streamfunction can play role of Hamiltonian:

$$\frac{dx}{dt} = u = -\frac{\partial \psi}{\partial y} \qquad \frac{dy}{dt} = v = \frac{\partial \psi}{\partial x}$$

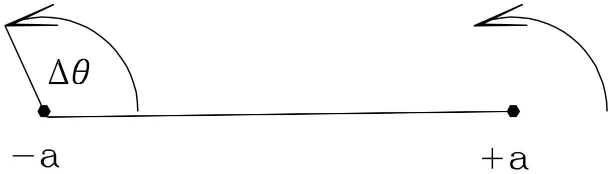
In fluid context, chaos can be desirable: promotes mixing

Non-integrable perturbation: blinking vortex



Vortex at $(+a, 0)$ is switched on for time T
then vortex at $(-a, 0)$ is switched on for time T

H. Aref: theory, 1984 Ottino: experiments, 1989



Rotate $\Delta\theta$ around $(+a, 0)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\Delta\theta & -\sin\Delta\theta \\ \sin\Delta\theta & \cos\Delta\theta \end{pmatrix} \begin{pmatrix} x-a \\ y \end{pmatrix}$$

where $\Delta\theta_+(x,y) = \kappa T/((x-a)^2 + y^2)$

Rotate $\Delta\theta$ around $(-a, 0)$:

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -a \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\Delta\theta & -\sin\Delta\theta \\ \sin\Delta\theta & \cos\Delta\theta \end{pmatrix} \begin{pmatrix} x'+a \\ y' \end{pmatrix}$$

where $\Delta\theta_-(x',y') = \kappa T/((x'+a)^2 + y'^2)$

Non-dimensional parameter controlling non-integrability:

$$\mu = \kappa T/a^2$$

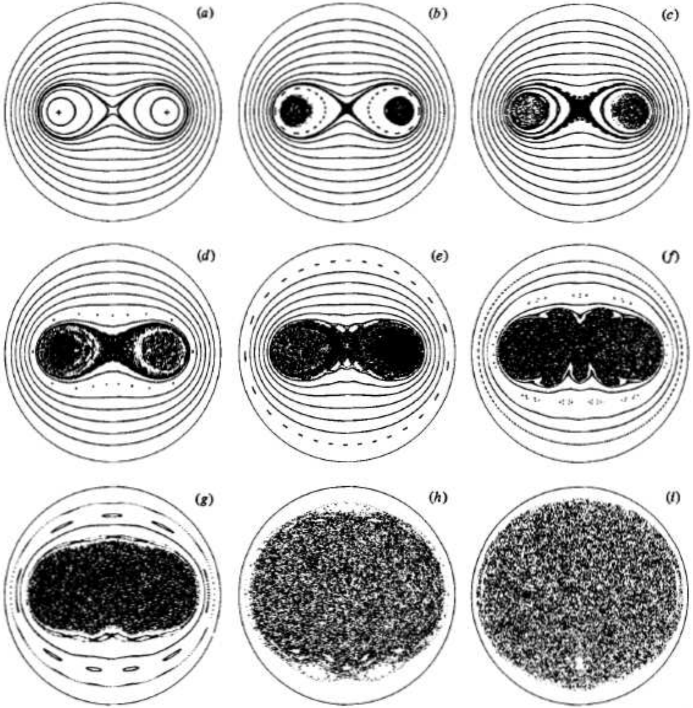


FIGURE 2. Iterated-map results described in §4. Parameters are $\beta = 0.5$ and (*a*) $\mu = 0.05$; (*b*) 0.10; (*c*) 0.125; (*d*) 0.15; (*e*) 0.20; (*f*) 0.35; (*g*) 0.50; (*h*) 1.0; (*i*) 1.5. Crosses indicate agitator positions.

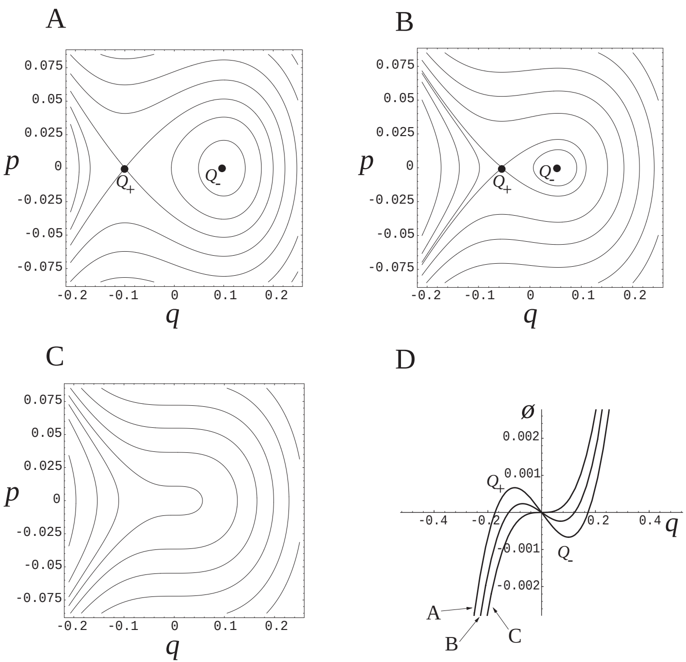
Simulation of blinking vortex flow shows increasing degree of chaos as μ increases.
From H. Aref, *Stirring by chaotic advection*, **J. Fluid Mech. 143, 1–21 (1984).**

Hamiltonian saddle-node bifurcation

center and saddle meet and annihilate

$\delta = 0.2$

$\delta = 0.1$



$\delta = 0$

$\Phi(q)$

Normal form:

$$\ddot{q} = \delta - q^2 \Longleftrightarrow \begin{cases} \dot{q} = p = \frac{\partial \mathcal{H}}{\partial p} \\ \dot{p} = \delta - q^2 = -\frac{\partial \mathcal{H}}{\partial q} = -\Phi'(q) \end{cases}$$

where

$$\mathcal{H} = \frac{p^2}{2} + \Phi(q) = \frac{p^2}{2} + \frac{q^3}{3} - \delta q$$

Fixed points are extrema of Φ

$$\begin{aligned} p &= 0 \\ q &= \pm\sqrt{\delta} \text{ for } \delta > 0 \end{aligned}$$

Stability is determined by

$$\begin{bmatrix} 0 & 1 \\ -2q & 0 \end{bmatrix} \Longleftrightarrow \lambda(-\lambda) = 2q$$

$$q = +\sqrt{\delta} \Longleftrightarrow -\lambda^2 > 0 \Longleftrightarrow \lambda = \pm i\omega \Longleftrightarrow q \text{ a center}$$

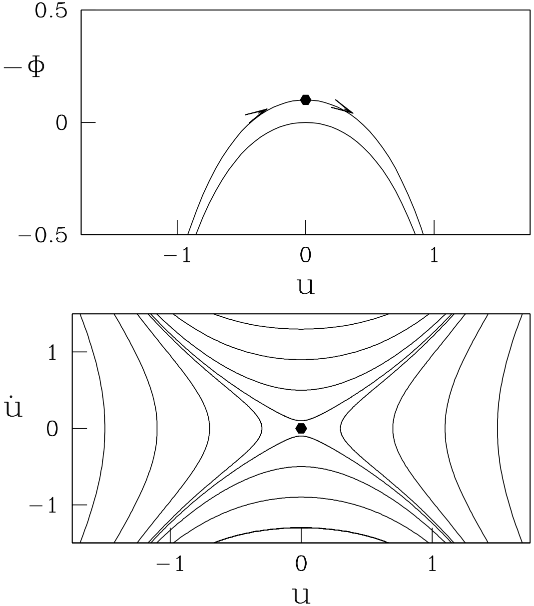
$$q = -\sqrt{\delta} \Longleftrightarrow -\lambda^2 < 0 \Longleftrightarrow \lambda = \pm\sigma \Longleftrightarrow q \text{ a saddle}$$

At $\delta = 0$, a saddle and center are created simultaneously

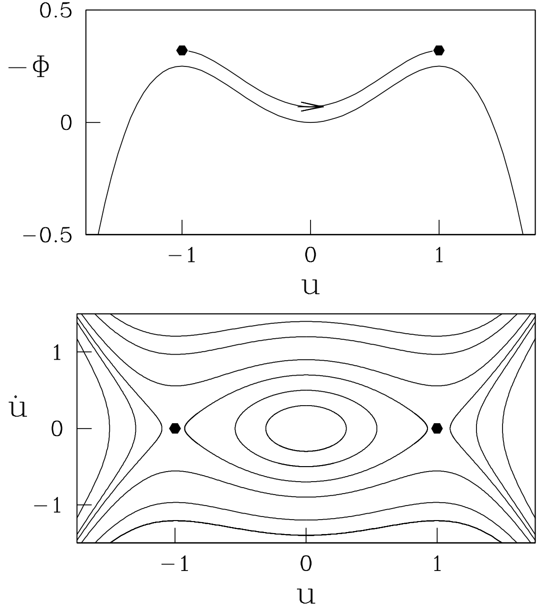
Hamiltonian pitchfork bifurcation

Saddle $\Longleftrightarrow$ Saddle-Center-Saddle

(or Center $\Longleftrightarrow$ Center-Saddle-Center)



$\mu = -1$



$\mu = +1$

$$\mathcal{H} = \frac{v^2}{2} - \Phi(u) = \frac{v^2}{2} - \frac{u^4}{4} + \mu\frac{u^2}{2}$$

$$\ddot{u} = u^3 - \mu u \iff \begin{cases} \dot{u} = v = \dfrac{\partial \mathcal{H}}{\partial v} \\ \dot{v} = u^3 - \mu u = -\dfrac{\partial \mathcal{H}}{\partial u} = \dfrac{\partial \Phi}{\partial u} \end{cases}$$