Eur. J. Mech., B/Fluids, **10**, n° 2 - Suppl., 205-210, 1991

Corrected according to Erratum,
Eur. J. Mech., B/Fluids, **10**, no. 2 – Suppl., pp. 205-210, (1991).

Periodic traveling waves with nonperiodic pressure

W.S. Edwards, R.P. Tagg, B.C. Dornblaser, L.S. Tuckerman and Harry L. Swinney

Center for Nonlinear Dynamics and Department of Physics
University of Texas at Austin
Austin, Texas 78712

Abstract. Nonlinear spiral vortices in fluid flow between counter-rotating cylinders are shown to induce an average axial pressure gradient when experiments are conducted between closed ends where the axial mean velocity must be zero. Previous calculations for spiral vortices have assumed periodic boundary conditions on the pressure, which in general is not compatible with zero mean axial velocity. Such calculations predicted wavespeeds which differed from experiment at leading nonlinear order, typically by 10% at 5% above criticality. In contrast, our numerical calculations with the constraint of zero mean axial flow agree well with the measured wavespeeds in the weakly nonlinear regime. Our results suggest a minor modification of previous analyses could incorporate the new constraint: the periodicity of the pressure could be replaced by periodicity of the pressure *gradient*, chosen to yield zero axial mean flow.

1 Introduction

Recently there has been much interest in the nonlinear traveling waves which arise in some incompressible fluid flows in simple geometries. Two experimental paradigms have received special attention; they are convection in binary fluid mixtures[Cross, 1986, 1988; Fineberg *et al*, 1988; Kolodner *et al*, 1986; Kolodner & Surko, 1988] and the flow between counter-rotating cylinders (the Couette-Taylor system). We have investigated the latter, which gives rise to spiral vortices for some parameters. These vortices arise as a primary instability of the basic laminar flow state known as circular Couette flow. For wide ranges of parameters they are observed to be spatially periodic with a nearly uniform wavelength in the axial direction.

The linear theory of spiral vortices has been well studied[Krueger *et al*, 1966; Langford *et al*, 1988; Tagg *et al*, 1990] and the results are in good agreement with experiment[Snyder, 1968; L *et al*, 1988].

0997-7546/91/ 205 6/$ 2.60/

Several authors[DiPrima & Grannick, 1971; Babenko, Afendikov & Yur'ev, 1982; Demay & Iooss, 1984; Chossat & Iooss, 1985; Golubitsky & Stewart, 1986; Friederich *et al*, 1987; Golubitsky & Langford, 1988] have investigated the nonlinear behavior of spiral vortices and their mode interactions. To date these results have not been reconciled quantitatively with experimental observations (cf. [Tagg *et al*, 1989]), although there have been some successes in predicting qualitatively the observed behaviors.

In this article we compare quantitatively the experimentally measured wavespeeds of nonlinear spiral vortices with the predictions of a cubic-order amplitude equation and with fully nonlinear numerical calculations. Both the amplitude equation and fully nonlinear models assume periodic boundary conditions (in the axial direction) on the velocity field. Fully nonlinear numerical calculations with periodic *pressure* boundary conditions yield amplitudes and wavespeeds which agree with the amplitude equation near onset. However, when we compare wavespeeds from experiments with closed ends (such that no axial mean flow is allowed), we find differences at leading nonlinear order. We show that by constraining the solution in the numerical model to have zero axial mean velocity (by imposing an overall pressure gradient), we recover quantitative agreement with experimental wavespeeds.

2 Problem Statement

An incompressible fluid of kinematic viscosity ν occupies the annular region between cylinders of radii a and b, with $b > a$. The inner and outer cylinders rotate with angular velocities Ω_1 and Ω_2 respectively, where by convention Ω_1 is positive, and thus for the counter-rotating cases we consider Ω_2 will be negative. We define three dimensionless parameters (R, μ, η) which are respectively the Reynolds number $R \equiv \Omega_1 a(b-a)/\nu$, the speed ratio $\mu \equiv \Omega_2/\Omega_1$ and the radius ratio $\eta \equiv a/b$. In this study the radius ratio is fixed at $\eta = 0.7992$. We focus on comparisons of wavespeeds because these are easily measured experimentally to high precision and thus they provide a sensitive test of theoretical models. The problem, then, is to measure experimentally and to compute theoretically the wavespeeds of spiral vortices both at criticality and for several Reynolds numbers in the nonlinear regime beyond criticality.

3 Experiments

Experiments were performed in an apparatus with annulus length $L = 36(b-a)$ which supports typically 18 axial wavelengths of the spiral pattern. The apparatus is oriented vertically and the end boundary conditions are established by teflon rings which rotate with the outer cylinder. There is a very narrow gap between each of these rings and the inner cylinder, through which fluid is free to pass; however, the bottom of the apparatus is sealed and thus no *mean* mass flux is possible. The spiral flow states were prepared by following a parameter path of fixed μ, beginning with circular Couette flow and increasing R slowly through the critical Reynolds number $R_c(\mu)$ at which spirals first appear. The working fluid was a mixture of 65% glycerol, 34% water and 1% Kalliroscope AQ-1000 visualization material. Wavespeeds were measured by a photodetector which monitored the intensity of light from a 10 milliwatt He-Ne laser that reflected from the visualization material in a small volume of the fluid. The signal from the reflectance probe was digitized and Fourier analyzed to determine the frequency ω of the wave. In all cases the dominant peak in the Fourier spectrum was sharp and was clearly that of the rotating helical pattern.

4 Weakly nonlinear analysis

Close to onset, the velocity field (minus Couette flow) is given by

$$\mathbf{U}(r,\theta,z,t) = A_L(t)\mathbf{U}_L(r)e^{i(m_c\theta+k_cz)} + A_R(t)\mathbf{U}_R(r)e^{i(m_c\theta-k_cz)} + \text{c.c.} + \mathbf{U}'(r,\theta,z,t;A_L,A_R,\varepsilon) \quad (1)$$

where c.c. denotes complex conjugate and where the residual field $\mathbf{U}'(r,\theta,z,t;A_L,A_R,\varepsilon)$ is "slaved" to the complex amplitudes A_L and A_R of the unstable left- and right-handed spiral eigenmodes $\mathbf{U}_L(r)e^{i(m_c\theta+k_cz)}$ and $\mathbf{U}_R(r)e^{i(m_c\theta-k_cz)}$; m_c and k_c are respectively the critical azimuthal and axial wavenumbers. Symmetry considerations lead to the coupled amplitude equations

$$\begin{aligned}\dot{A}_L &= -i\omega_c A_L + \tau_0^{-1}(1+ic_0)\varepsilon A_L - g(1+ic_2)|A_L|^2A_L - h(1+ic_3)|A_R|^2A_L + \cdots & (2)\\ \dot{A}_R &= -i\omega_c A_R + \tau_0^{-1}(1+ic_0)\varepsilon A_R - g(1+ic_2)|A_R|^2A_R - h(1+ic_3)|A_L|^2A_R + \cdots & (3)\end{aligned}$$

Here $\varepsilon = (R-R_c)/R_c$ is a small parameter measuring the distance from criticality.

Formal reductions of the Navier-Stokes equations have been carried out by various workers [D & I, 1984; G & L, 1988; F *et al*, 1987] to arrive at numerical values for the nonlinear coefficients. On this basis, the lowest-order nonlinear correction to amplitude and frequency can be quantitatively predicted. For this analysis we have used the code of [D & I, 1984] to compute the values of g, c_2, h and c_3 for our parameters. In the following we assume the left-handed spiral solution $A_L \neq 0$, $A_R = 0$. We further represent $A_L \equiv \rho e^{i\phi}$ and obtain (to leading nonlinear order):

$$\begin{aligned}\dot{\rho} &= \tau_0^{-1}\varepsilon\rho - g\rho^3 & (4)\\ \omega = -\dot{\phi} &= \omega_c - \tau_0^{-1}c_0\varepsilon + gc_2\rho^2 & (5)\end{aligned}$$

For steady-state modulus of the amplitude ($\dot{\rho}=0$),

$$\begin{aligned}\rho &= (\tau_0 g)^{-1/2}\varepsilon^{1/2} & (6)\\ \omega &= \omega_c - \tau_0^{-1}(c_0-c_2)\varepsilon & (7)\end{aligned}$$

By scaling the frequency ω by the inner cylinder angular velocity Ω_1, experimental uncertainty in viscosity is removed. We therefore write

$$\begin{aligned}\omega/\Omega_1 &= \omega_c/\Omega_1 - \tau_0^{-1}(c_0-c_2)\varepsilon/\Omega_1\\ &= \omega_c/\left[\Omega_{1c}(1+\varepsilon)\right] - \tau_0^{-1}(c_0-c_2)\varepsilon/\left[\Omega_{1c}(1+\varepsilon)\right]\\ &= \omega_c/\Omega_{1c} - \left[\tau_0^{-1}(c_0-c_2)/\Omega_{1c} + \omega_c/\Omega_{1c}\right]\varepsilon + O(\varepsilon^2) & (8)\end{aligned}$$

Equations (6) and (8) respectively were used to produce the dotted curves in figure 1(a) and (b).

5 Fully nonlinear numerical calculations

Solutions of the fully nonlinear incompressible Navier-Stokes equations were obtained by pseudo-spectral calculations performed with periodic boundary conditions on the velocity field (in the axial direction). The computational domain was of fixed axial length $\lambda = 2\pi/k_c$ where k_c is the critical wavenumber from linear theory. Thus the calculations did *not* model any nonlinear wavelength change. The velocity field $\mathbf{U} = \mathbf{U}(r, \theta, z, t)$ with components (U_r, U_θ, U_z) was assumed to have a helical symmetry, $\partial_\theta \mathbf{U} = (m_c/k_c)\partial_z \mathbf{U}$. In the cases we studied m_c was always 2 in both experiment and numerics. The imposition of the helical symmetry allowed the numerical problem to be posed on a two-dimensional (r, z) domain with $a \leq r \leq b$ and $0 \leq z \leq \lambda$. We used an accurate spectral discretization with 25 Chebyshev modes in r and 27 Fourier modes in z. The time-asymptotic traveling-wave solutions were found by time-evolving the discretized Navier-Stokes equations using a general nonlinear stiff-equation solver[Friesner *et al*, 1989] and by quasi-Newton iteration. Details of our numerical method will be published elsewhere[Edwards *et al*, to appear]; here we note only that the accurate solution of this quasi-two-dimensional Navier-Stokes problem in a simple geometry and at low Reynolds numbers (less than 200) poses no serious technical challenge and can be accomplished by a variety of methods. For the case of periodic pressure boundary conditions, wavespeeds obtained with our code were compared with those calculated with the fully three-dimensional code of [Marcus, 1984], which also assumes periodic pressure. The two codes produced wavespeeds that agreed to four significant figures.

For the zero-mean-flow calculations, we added to the equations of motion a mean pressure gradient term $-\bar{p}\hat{e}_z$, where $\bar{p}$ is a constant independent of position. The value of $\bar{p}$ was adjusted until the mean axial flow $< U_z >$ of the traveling-wave solution was zero.

The amplitudes in figure 1(a) were obtained by projecting the numerically computed flow $\mathbf{U}(r, \theta, z, t)$ onto the eigenfunction $\mathbf{U}_L(r)e^{i(m_c\theta + k_c z)}$. To do this the adjoint eigenfunction $\tilde{\mathbf{U}}_L(r)e^{-i(m_c\theta+k_c z)}$ was found and the amplitude was computed according to

$$A_L(t) = \frac{\left\langle \tilde{\mathbf{U}}_L(r)e^{-i(m_c\theta+k_c z)} \middle| \mathbf{U}(r, \theta, z, t) \right\rangle}{\left\langle \tilde{\mathbf{U}}_L(r)e^{-i(m_c\theta+k_c z)} \middle| \mathbf{U}_L(r)e^{i(m_c\theta+k_c z)} \right\rangle} \tag{9}$$

where $\langle \cdot | \cdot \rangle$ is a suitable inner product. Here the normalization of $\mathbf{U}_L(r)e^{i(m_c\theta+k_c z)}$ was chosen to be compatible with the code of [D & I, 1984] as used in the previous section, so that $|A_L|$ as computed via (9) may be directly compared to ρ as computed via (6).

6 Results

Figure 1 compares results from experiment, linear theory, the cubic-order amplitude equation and fully nonlinear numerical calculations. The experimental wave frequencies are seen to agree much better with the nonlinear calculations when the mean axial flow is constrained to be zero. Fully nonlinear calculations performed with periodic pressure boundary conditions yield frequencies which agree near criticality with the nonlinear correction predicted by the amplitude equation (which also assumes periodic pressure). However, the discrepancy between the periodic-pressure results and experiment appears to be of order $\varepsilon = (R - R_c)/R_c$; that is, the experimental nonlinear frequency falloff has a different slope $\partial_R(\omega/\Omega_1)$ near criticality than the slope predicted by either of the periodic-pressure methods.

7 Conclusions

Couette-Taylor spiral vortices are an attractive paradigm for the study of weakly nonlinear behavior of traveling waves. Theoretical predictions have been made for the nonlinear interaction of left- and right-

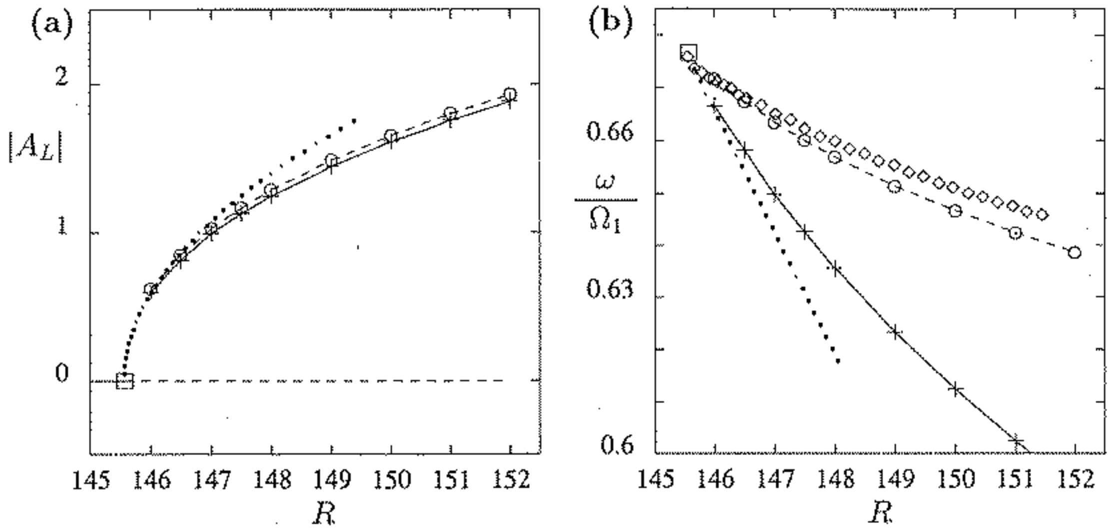

Figure 1: Amplitude (a) and frequency (b) of spiral vortices at criticality and in the weakly nonlinear regime. The figure compares results from linear stability analysis, experiment, a cubic-order amplitude equation, and fully nonlinear numerical calculations. The parameters are $\eta = 0.7992$, $\mu = -0.74$, for which linear stability analysis yields $R_c = \Omega_{1c}a(b-a)/\nu = 145.57$, $m_c = 2$, $k_c = 3.6997/(b-a)$, $\omega_c/\Omega_{1c} = 0.67664$, $\tau_0^{-1}/\Omega_{1c} = 1.0328$ and $c_0 = -0.7536$; linear stability results at criticality are shown by a square □ in each part. Diamonds ◇ (part (b) only) are experimental results for frequencies. Dotted lines are predictions of a center-manifold reduction to a cubic-order amplitude equation, computed using the code of [Demay & Iooss, 1984], which assumes periodic pressure; the resulting nonlinear coupling coefficients used to produce the dotted curves are $g = 0.00855\tau_0^{-1}$ and $c_2 = -3.38$. Plus signs + connected by solid lines are fully nonlinear numerical results assuming periodic pressure boundary conditions. Circles ○ connected by dashed lines are the same except that the axial mean flow is constrained to be zero by the imposition of a mean axial pressure gradient.

handed spirals[D & G, 1971; D & I, 1984; C & I, 1985], the interaction of spirals with Taylor vortices[G & S, 1986; G & L, 1988], and the interaction of spirals of differing azimuthal wavenumbers[Chossat, Demay & Iooss, 1987]. In these analyses the Navier-Stokes equations are reduced to a simpler set of nonlinear amplitude equations which predict the dynamics of the system in the neighborhood of critical and bicritical points. Coefficients for these amplitude equations are computed in the reduction process. These theories all assume periodic boundary conditions on both the velocity and pressure fields. We have shown that some features of the nonlinear spirals occurring in experiments (eg. wavespeeds) can indeed be quantitatively modeled in a small periodic domain, but only if the velocity field is constrained to have zero axial mean flow, which in general is not compatible with periodic pressure. Our results also suggest, however, that the weakly nonlinear theories which have been proposed to date need only minor changes to incorporate the new constraint; rather than assume periodicity of the *pressure*, one would assume periodicity of the pressure *gradient*, with the mean pressure gradient chosen such that the axial mean flow is zero. The assumption of periodic pressure boundary conditions, on the other hand, leads to nonlinear predictions which are incorrect at order ε.

We conclude that relatively minor changes in existing weakly nonlinear theories can lead to improved quantitative agreement with experiment. Furthermore, predictions about the *qualitative* behavior of spiral vortices may be affected if the zero axial mean velocity constraint changes the signs or relative sizes of some nonlinear coefficients. It will be interesting to see if this in fact occurs.

More general theories, which allow space-dependent amplitudes, should also incorporate the zero mean flow constraint if they are to be compared to finite-length experiments with closed ends.

Alternatively, one may envision experiments in which fluid may enter and exit the ends of the annulus, in which an overall mean flow or pressure gradient is imposed externally. In principle it should be possible with such an experiment to zero the mean pressure gradient so that existing predictions based on periodic pressure boundary conditions may again be relevant.

Acknowledgements. This research was supported by the U.S. Office of Naval Research Nonlinear Dynamics Program. Computational resources for this work were provided by the University of Texas System Center for High Performance Computing. The authors wish to thank W.F. Langford for valuable discussions, and Y. Demay and P.S. Marcus for the use of their computer codes.

References

K.I. Babenko, A.L. Afendikov and S.P. Yur'ev, Sov. Phys. Dokl. **27**, 706 (1982)

P. Chossat, Y. Demay and G. Iooss, Arch. Rat. Mech. Anal. **99**, 213 (1987)

P. Chossat and G. Iooss, Japan J. Appl. Math. **2**, 37 (1985)

M.C. Cross, Phys. Rev. Lett. **57**, 2935 (1986)

M.C. Cross, Phys. Rev. A **38**, 3593 (1988)

Y. Demay and G. Iooss, J. Mécanique théorique et appliquée, **Numéro spécial**, 193 (1984)

R.C. DiPrima and R.N. Grannick, in *Instability of Continuous Systems*, edited by H. Leipholz (Springer, Berlin, 1971), p. 55

W.S. Edwards, L.S. Tuckerman and R.P. Tagg, to appear

W.S. Edwards, L.S. Tuckerman, R. Friesner, D. Sorensen, to appear

J. Fineberg, E. Moses and V. Steinberg, Phys. Rev. Lett. **61**, 838 (1988)

R. Friederich, T. Grauer and H. Haken, unpublished talk given at the Fifth Taylor-Vortex Flow Working Party, Phoenix, Arizona 1987

R.A. Friesner, L.S. Tuckerman, B.C. Dornblaser, T.V. Russo, J. Sci. Comput. **4**, 327 (1989)

M. Golubitsky and W.F. Langford, Physica D **32**, 362 (1988)

M. Golubitsky and I. Stewart, SIAM J. Math. Anal. **17** 2, 249 (1986)

P. Kolodner, A. Passner, C.M. Surko, and R.W. Walden, Phys. Rev. Lett. **56**, 2621 (1986)

P. Kolodner and C.M Surko, Phys. Rev. Lett. **61**, 842 (1988)

E.R. Krueger, A. Gross and R.C. DiPrima, J. Fluid Mech. **24**, 521 (1966)

W.F. Langford, R. Tagg, E.J. Kostelich, H.L Swinney and M. Golubitsky, Phys. Fluids **31**, 776 (1988)

P.S. Marcus, J. Fluid Mech. **146**, 45 (1984)

H.A. Snyder, Phys. Fluids **11**, 728 (1968)

R. Tagg, W.S. Edwards, H.L. Swinney and P.S. Marcus, Phys. Rev. A **39**, 3734 (1989)

R. Tagg, W.S. Edwards, H.L. Swinney, Phys. Rev. A **42**, 831 (1990)