Comment on "Bifurcation Structure and the Eckhaus Instability"

In their Letter, Tsiveriotis and Brown [1] present a bifurcation-theoretic analysis of the Eckhaus instability [2]. Such an analysis requires discrete spectra and well-separated solution branches, necessitating a bounded domain of length L. The scenario is as follows: As a control parameter R is increased, pure-mode branches, each characterized by a single wave number $k_n = n\pi/L$, bifurcate successively off the trivial branch. All but the first of the pure-mode branches are unstable when created and are then restabilized by a sequence of secondary bifurcations, which in turn create unstable mixed-mode states.

For the complex version of the Swift-Hohenberg equation[3] studied in Ref. [1], the primary bifurcations are at

$$R_n = (1-k_n^2)^2, \tag{1}$$

so that the onset of instability takes place at $(k_{\rm cr}, R_{\rm cr}) = (1,0)$. The secondary bifurcations from the k_n branch are indexed by l and occur at

$$R_l^n = \frac{2R_n(R_l + R_{2n-l}) - R_l R_{2n-l} - 3R_n^2}{R_l + R_{2n-1} - 2R_n}. \tag{2}$$

The following approximation to (2), valid for $k \approx k_{\rm cr}$, is then derived:

$$R_l^n \approx 12(1-k_n)^2 - 2(k_l - k_n)^2. \tag{3}$$

The maximum over l of (3) gives the final secondary bifurcation which renders the k_n branch stable: This is the Eckhaus instability. Tsiveriotis and Brown assert that this maximum is

$$R_{\rm Eck}(k_n) = 12(1-k_n)^2, \tag{4}$$

coinciding with the classic Eckhaus curve, independent of L (except for discretization of k).

It is this last claim with which we disagree. Equation (4) is obtained for an infinite domain by taking $k_l - k_n$ arbitrarily small in (3). Discretization forbids this in a finite domain, and the Eckhaus instability instead corresponds to setting $l = n \pm 1$ in (3), leading to

$$R_{\rm Eck}(k_n) = 12(1-k_n)^2 - 2(\pi/L)^2. \tag{5}$$

Our Fig. 1 is based on that of Ref. [1], where $L = 8\pi/k_{\rm cr}$, and illustrates the difference between (4) and (5). The important qualitative distinction between (4) and (5) is this: Equation (5) is not tangent to the marginal stability curve (1) and the minimum $(k_{\rm cr}, R_{\rm cr})$ of the marginal stability curve does not lie on (5). It might be argued that the displacement $-2(\pi/L)^2 \to 0$ as $L \to \infty$. However, the distance in R between successive primary or secondary bifurcation points is of the same order. For example, for $R_n = R_{\rm cr} = 0$, $R_{n\pm1} \approx 4(\pi/L)^2$, and $R_{\rm Eck}(k_{n\pm1}) \approx 10(\pi/L)^2$.

Indeed, the downward displacement is perhaps the most important feature distinguishing the finite-domain from the infinite-domain analysis and is its only experimental consequence. This stabilizing effect was first mentioned by Kramer and Zimmermann [4]; Ahlers *et al.* [5] noted that the displacement creates a "gap," where the Eckhaus curve falls below the neutral stability curve, and whose width (π/L) accommodates exactly one allowed wave number. The entire secondary bifurcation structure can be viewed geometrically in terms of such gaps. These and other aspects of the Eckhaus instability, as analyzed for the Ginzburg-Landau equation, are discussed in our recent article [6].

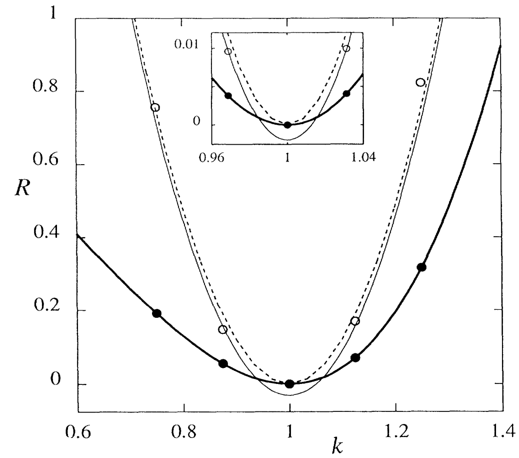

FIG. 1. The bold, dashed, and thin curves denote the neutral stability curve (1), the asymptotic Eckhaus formula (4) of Ref. [1], and our corrected version (5), respectively. [The dotted portion of (5) which falls below (1) has no physical significance.] Solid and open circles denote primary and secondary bifurcation points; the latter are computed from the exact formula (2) using $l = n \pm 1$ and correspond to the points in Fig. 1 of Ref. [1]. In the main figure $L = 8\pi$. Note the downwards displacement of the Eckhaus points relative to the dashed curve. The deviation between the Eckhaus points and the thin curve is due to the asymptotic nature ($k \approx 1$) of formula (5), and is reduced in the inset, for which $L = 32\pi$.

Laurette S. Tuckerman
Department of Mathematics
and Center for Nonlinear Dynamics
University of Texas
Austin, Texas 78712

Dwight Barkley
Program in Applied and Computational Mathematics
Princeton University
Princeton, New Jersey 08544

Received 26 July 1990
PACS numbers: 44.30.+v, 61.50.Cj, 68.45.−v

[1] K. Tsiveriotis and R. A. Brown, Phys. Rev. Lett. **63**, 2048 (1989).
[2] W. Eckhaus, *Studies in Nonlinear Stability Theory* (Springer, New York, 1965).
[3] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).
[4] L. Kramer and W. Zimmermann, Physica (Amsterdam) **16D**, 221 (1985).
[5] G. Ahlers, D. S. Cannell, M. A. Dominguez-Lerma, and R. Heinrichs, Physica (Amsterdam) **23D**, 202 (1986).
[6] L. S. Tuckerman and D. Barkley, Physica (Amsterdam) **46D**, 57 (1990)

