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Dynamical Systems

Dynamical Systems

$$\dot{x} = f(x), \quad x, f \text{ vectors in } \mathcal{R}^N$$

Examples:

-Normal form of saddle-node bifurcation ($N = 1$)

$$\dot{x} = \mu - x^2$$

-Lorenz model ($N = 3$)

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10(y - x) \\ -xz + rx - y \\ xy - 8z/3 \end{pmatrix}$$

-Navier-Stokes equations ($N \gg 1$)

$$\frac{d}{dt} \mathbf{u} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \frac{1}{Re} \Delta \mathbf{u}$$

Navier-Stokes:

$$\mathbf{u}(\mathbf{x}) = (u(x, y, z), v(x, y, z), w(x, y, z)) \quad N = \infty$$

3D Numerical Discretization:

$$N = 3 \times N_x \times N_y \times N_z \approx 3 \times 100^3 = 3 \times 10^6$$

Non-autonomous system:

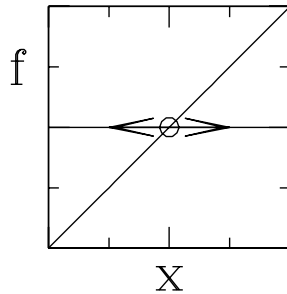
$$\dot{\mathbf{x}} = f(\mathbf{x}, t) \implies \frac{d}{dt} \begin{pmatrix} \mathbf{x} \\ \theta \end{pmatrix} = \begin{pmatrix} f(\mathbf{x}, \theta) \\ 1 \end{pmatrix} \quad \text{with } \theta \equiv t$$

Higher-order system:

$$\ddot{\mathbf{x}} = f(\mathbf{x}, \dot{\mathbf{x}}) \implies \frac{d}{dt} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{y} \\ f(\mathbf{x}, \mathbf{y}) \end{pmatrix} \quad \text{with } \mathbf{y} \equiv \dot{\mathbf{x}}$$

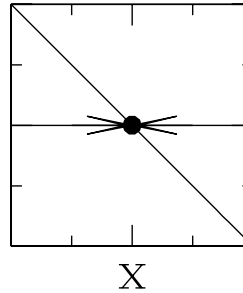
Fixed points and linear stability. $\dot{x} = f(x)$

unstable



$$0 = f(\bar{x})$$

stable



Fixed point \bar{x}

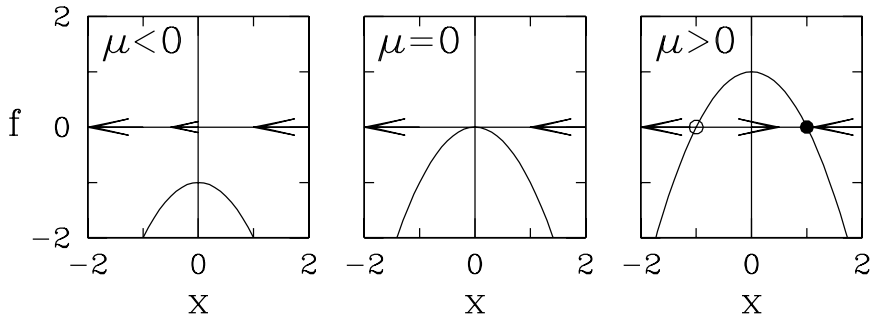
$$\frac{d}{dt}(\bar{x} + \epsilon(t)) = f(\bar{x} + \epsilon)$$

Linear stability of \bar{x}

$$\dot{\epsilon} = f(\bar{x}) + f'(\bar{x})\epsilon + \frac{1}{2}f''(\bar{x})\epsilon^2 + \dots \approx f'(\bar{x})\epsilon$$

$$\epsilon(t) = e^{tf'(\bar{x})}\epsilon(0) \begin{cases} \text{increases if } f'(\bar{x}) > 0 \\ \text{decreases if } f'(\bar{x}) < 0 \end{cases}$$

Saddle-node Bifurcations



Normal form:

$$\dot{x} = f(x) = \mu - x^2$$

Fixed points:

$$\bar{x}_{\pm} = \pm\sqrt{\mu} \quad \text{for } \mu > 0$$

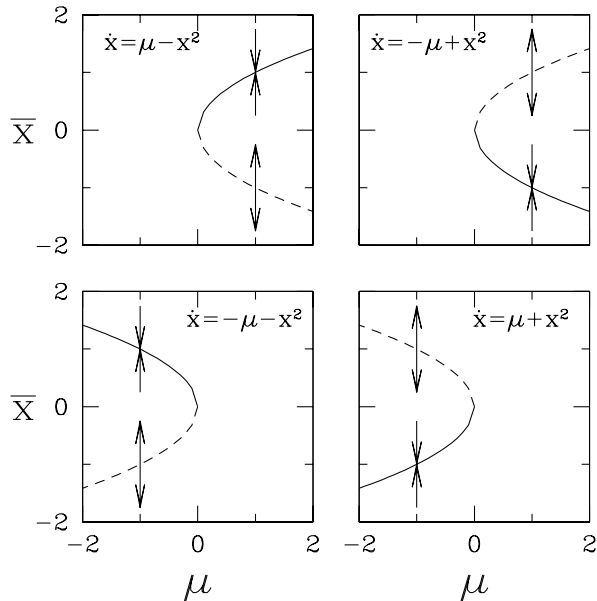
Stability:

$$f'(\bar{x}_{\pm}) = -2\bar{x}_{\pm} = -2(\pm\sqrt{\mu}) = \mp 2\sqrt{\mu}$$

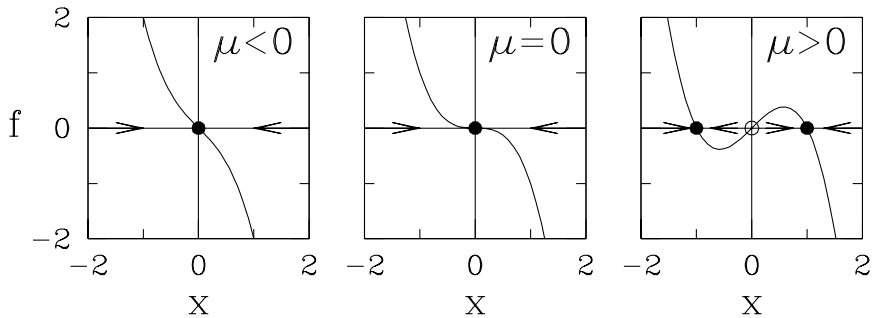
$$f'(\bar{x}_{+}) = f'(\sqrt{\mu}) = -2\sqrt{\mu} < 0 \implies \bar{x}_{+} \text{ stable}$$

$$f'(\bar{x}_{-}) = f'(-\sqrt{\mu}) = 2\sqrt{\mu} > 0 \implies \bar{x}_{-} \text{ unstable}$$

$$\begin{aligned}
 f(x, \mu) &= c_{00} + c_{10}x + c_{01}\mu + c_{20}x^2 + \dots && \text{general quadratic polynomial} \\
 &= \left(c_{01}\mu + c_{00} - \frac{c_{10}^2}{4c_{20}} \right) + c_{20} \left(x + \frac{c_{10}}{2c_{20}} \right)^2 \\
 &= \pm \tilde{\mu} \pm \tilde{x}^2 && \text{four cases, depending on signs of } c\text{'s}
 \end{aligned}$$



Pitchfork Bifurcations

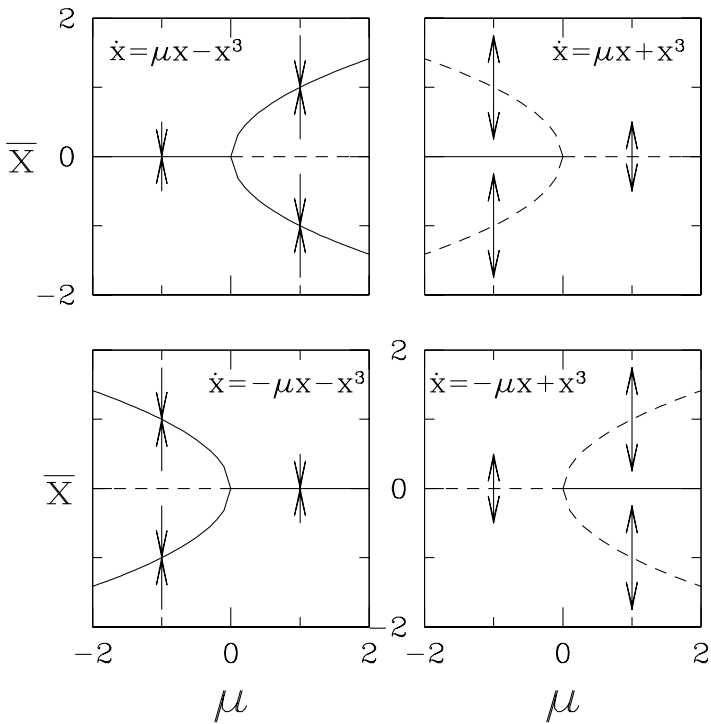


one fixed point

three fixed points

Normal form: $\dot{x} = f(x) = \mu x - x^3$

Symmetry $\implies f(x)$ odd in x .



Supercritical

$f(x, \mu) = \mu x - x^3$
 $f(x, \mu) = -\mu x - x^3$

Subcritical

$f(x, \mu) = \mu x + x^3$
 $f(x, \mu) = -\mu x + x^3$

Supercritical pitchfork bifurcations

Fixed points:

$$0 = \bar{x}(\mu - \bar{x}^2) \implies \begin{cases} \bar{x} = 0 & \text{for all } \mu \\ \bar{x} = \pm\sqrt{\mu} & \text{for } \mu > 0 \end{cases}$$

Stability:

$$f'(\bar{x}) = \mu - 3\bar{x}^2 = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ \mu - 3\mu = -2\mu & \text{for } \bar{x} = \pm\sqrt{\mu} \end{cases} \quad \text{stable}$$

Subcritical pitchfork bifurcations

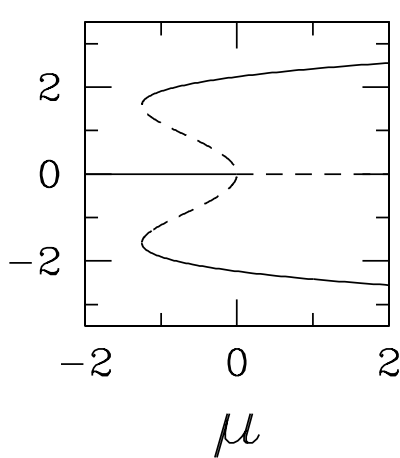
Fixed points:

$$0 = \bar{x}(\mu + \bar{x}^2) \implies \begin{cases} \bar{x} = 0 & \text{for all } \mu \\ \bar{x} = \pm\sqrt{-\mu} & \text{for } \mu < 0 \end{cases}$$

Stability:

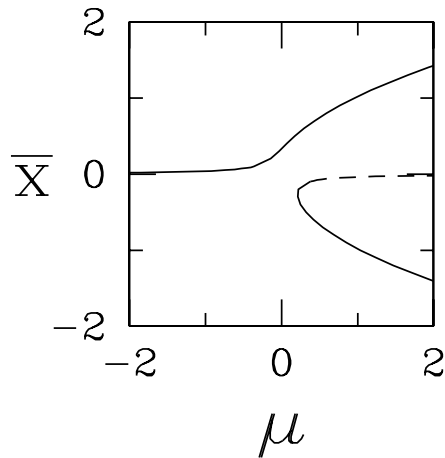
$$f'(\bar{x}) = \mu + 3\bar{x}^2 = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ \mu + 3(-\mu) = -2\mu & \text{for } \bar{x} = \pm\sqrt{-\mu} \end{cases} \quad \text{unstable}$$

Variations of pitchfork bifurcations



$$\dot{x} = \mu x + x^3 - \frac{x^5}{10}$$

stabilizing term
prevents trajectories
from evolving to ∞
pitchfork + 2 saddle-nodes

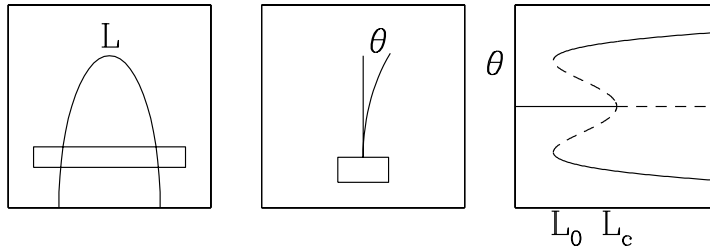


$$\dot{x} = \frac{1}{27} + \mu x - x^3$$

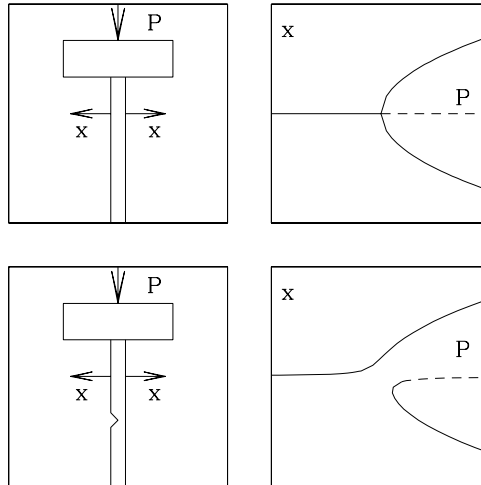
imperfect
symmetry broken
1 saddle-node

Simple mechanical examples

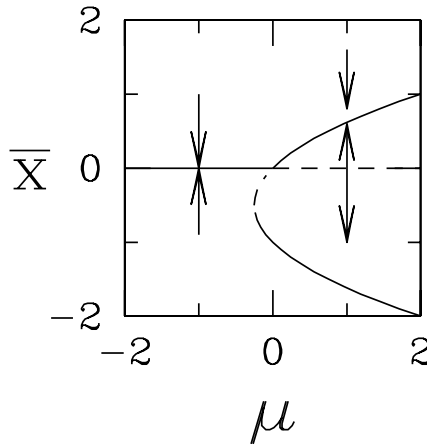
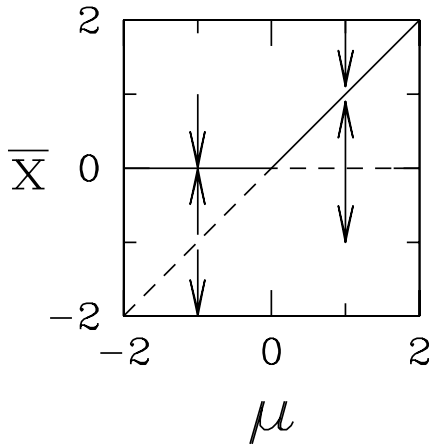
Wire loop: straight or leaning?



Weighted beam: weakness \rightarrow imperfect pitchfork



Transcritical Bifurcations



with stabilizing term

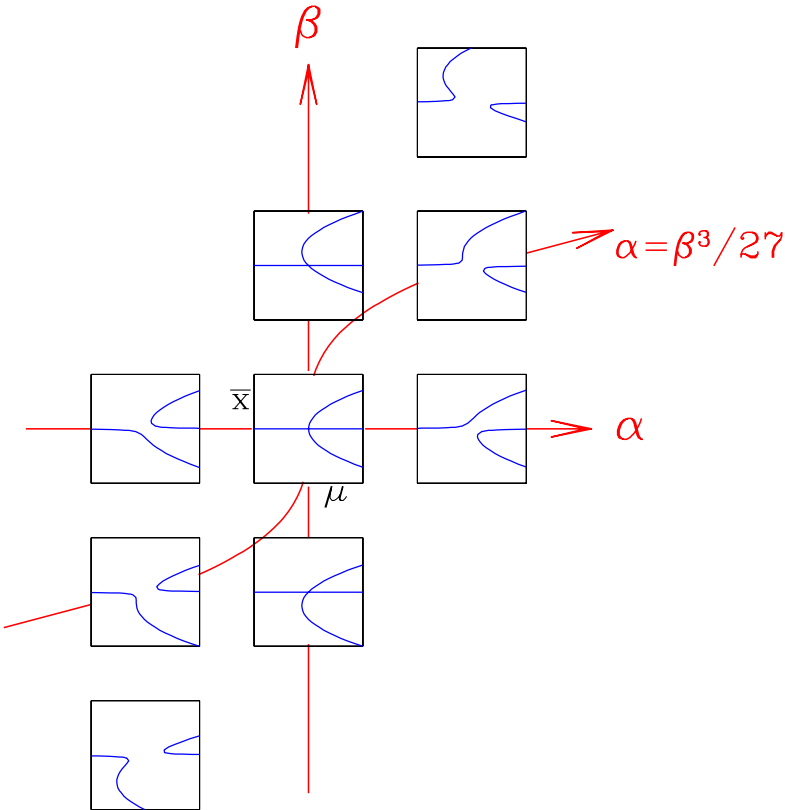
Normal form:

$$\dot{x} = \mu x - x^2$$

Fixed points: $0 = \bar{x}(\mu - \bar{x}) \implies \begin{cases} \bar{x} = 0 \\ \bar{x} = \mu \end{cases}$

Stability: $f'(\bar{x}) = \mu - 2\bar{x} = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ -\mu & \text{for } \bar{x} = \mu \end{cases}$

Unfolding of the pitchfork: $\dot{x} = \alpha + \mu x + \beta x^2 - x^3$



General conditions

$$\dot{x} = f(x, \mu) \text{ with } f(\bar{x}) = 0$$

	f	f_x	f_μ	f_{xx}	$f_{x\mu}$	f_{xxx}
steady state	0					
bifurcation	0	0	$\neq 0$			
saddle-node	0	0	$\neq 0$	$\neq 0$		
transcritical	0	0	0	$\neq 0$	$\neq 0$	
pitchfork	0	0	0	0	$\neq 0$	$\neq 0$

Systems with two or more dimensions

$$\dot{x} = f(x), \quad x, f \in \mathcal{R}^N$$

$$\text{Fixed points:} \quad 0 = f(\bar{x})$$

Stability:

$$\frac{d}{dt}(\bar{x} + \epsilon) = f(\bar{x} + \epsilon)$$

$$\dot{\epsilon} = f(\bar{x}) + Df(\bar{x}) \epsilon + \epsilon D^2 f(\bar{x}) \epsilon + \dots$$

$$\dot{\epsilon}_i = f_i(\bar{x}) + Df(\bar{x})_{ij} \epsilon_j + \epsilon_j [D^2 f(\bar{x})]_{ijk} \epsilon_k + \dots$$

$$= f_i(\bar{x}) + \frac{\partial f_i}{\partial x_j}(\bar{x}) \epsilon_j + \epsilon_j \frac{\partial^2 f_i}{\partial x_j \partial x_k}(\bar{x}) \epsilon_k + \dots$$

$$\dot{\epsilon} = Df(\bar{x}) \epsilon \quad Df(\bar{x}): \text{Jacobian matrix of } f \text{ at } \bar{x}$$

$$\epsilon(t) = \exp(Df(\bar{x})t)\epsilon(0)$$

Exponential of a matrix: real eigenvalues

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \dots$$

$$= VV^{-1} + tV\Lambda V^{-1} + \frac{t^2}{2}V\Lambda V^{-1} V\Lambda V^{-1} +$$

$$= V \left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \dots \right] V^{-1} = V e^{\Lambda t} V^{-1}$$

$$\Lambda^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$e^{t\Lambda} = \begin{pmatrix} 1 + t\lambda_1 + \frac{(t\lambda_1)^2}{2} + \dots & 0 \\ 0 & 1 + t\lambda_2 + \frac{(t\lambda_2)^2}{2} + \dots \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}$$

Stability in several dimensions

\bar{x} stable $\Leftrightarrow \text{Re}(\lambda_i) < 0$ for ALL λ_i

\bar{x} unstable $\Leftrightarrow \text{Re}(\lambda_1) > 0$ even for ONE value of λ_1
since perturbation in v_1 direction will increase

Scalar product with adjoint eigenvector v_1^T :

$$v_1^T Df(\bar{x}) = v_1^T \lambda_1$$

\Rightarrow 1D (or 2D) equation

Directions ($\text{Re}(\lambda_i) < 0$) are uninteresting: contraction only

Physical system \implies Polynomial equation:

- Complicated equations in $N \gg 1$ variables.

Calculate fixed points \bar{x} , Jacobians $Df(\bar{x})$ and spectra $\{\lambda_1, \lambda_2, \dots\}$.

\implies Bifurcation if $Re(\lambda_i)$ changes sign.

- Project onto corresponding adjoint eigenvector

\implies Function of a single variable.

- Taylor expansion about fixed point.

Minimal truncation giving observed behavior

\implies Normal form of the bifurcation.

Imaginary Eigenvalues

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad \lambda_{\pm} = \pm i\omega$$

$$A^2 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = \begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}$$

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} + \frac{t^3}{6} \begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}$$

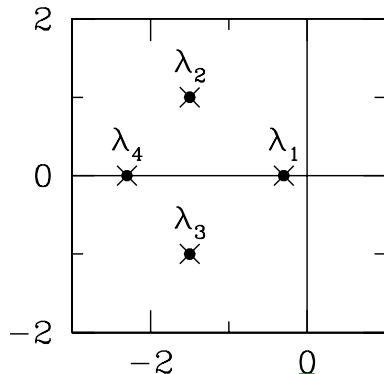
$$= \begin{pmatrix} 1 - \frac{1}{2}(t\omega)^2 + \dots & -t\omega + \frac{1}{6}(t\omega)^3 + \dots \\ t\omega - \frac{1}{6}(t\omega)^3 + \dots & 1 - \frac{1}{2}(t\omega)^2 + \dots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Complex Eigenvalues:

$$\exp \left[t \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \right] = e^{\mu t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

Mixed spectrum:



$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \mu & -\omega & 0 \\ 0 & \omega & \mu & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

$$\text{Re}(\lambda_1) \geq \text{Re}(\lambda_2) = \text{Re}(\lambda_3) \geq \text{Re}(\lambda_4)$$

$$\exp(\Lambda t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\mu t} \cos(\omega t) & -e^{\mu t} \sin(\omega t) & 0 \\ 0 & e^{\mu t} \sin(\omega t) & e^{\mu t} \cos(\omega t) & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix}$$

Jordan blocks and transient growth

Star node: multiple of the identity $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ with eigenvalues $\lambda = \mu$

Eigenvectors: $\mu x_1 + 0x_2 = \mu x_1 \implies x_1$ arbitrary
 $0x_1 + \mu x_2 = \mu x_2 \implies x_2$ arbitrary

Two-dimensional eigenspace: $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Degenerate node: Jordan block $\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$ with eigenvalues $\lambda = \mu$

Eigenvectors: $\mu x_1 + x_2 = \mu x_1 \implies x_2 = 0$
 $0x_1 + \mu x_2 = \mu x_2 \implies x_1$ arbitrary

One-dimensional eigenspace: $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Jordan blocks and transient growth

Eigenvector: $(A - \lambda I)x = 0$

Generalized eigenvector for Jordan block:

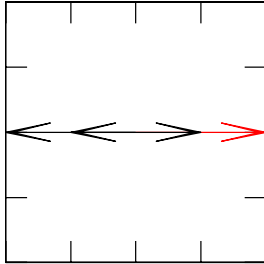
$$(A - \lambda I)v = x, \quad (\lambda, x) \text{ eigen (value,vector)}, \quad v \nparallel x$$

$$\left[\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} \quad \begin{matrix} c \neq 0 \\ v_2 \neq 0 \end{matrix}$$

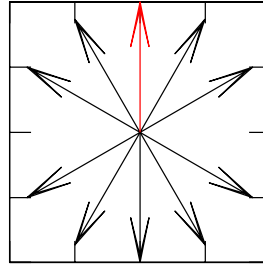
$$(\mu - \lambda)v_2 = 0 \implies \lambda = \mu$$

$$(\mu - \lambda)v_1 + 1v_2 = c \implies v_2 = c \neq 0, \quad v_1 \text{ arbitrary}$$

$$\|x\| = 1 \implies x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \left. \begin{matrix} \langle v, x \rangle = 0 \\ \|v\| = 1 \end{matrix} \right\} \implies v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



eigenvector x

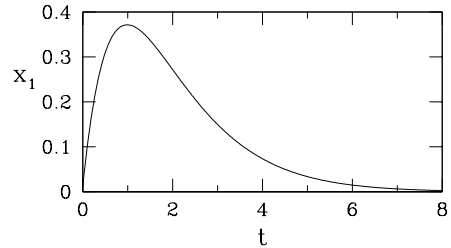


generalized eigenvector v

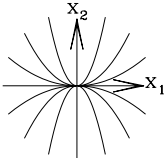
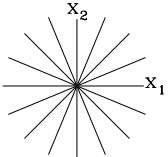
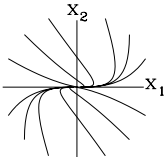
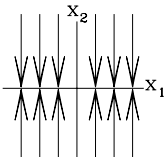
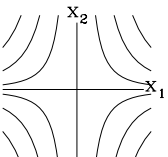
$$x_1 = e^{\lambda t}(x_1(0) + x_2(0)t)$$

$$x_2 = e^{\lambda t}x_2(0)$$

$$\lambda = -1 \implies$$



Linear behavior near a 2D fixed point

Name	Matrix	Behavior	
Node: stable ($\lambda_2 < \lambda_1 < 0$) unstable ($\lambda_2 > \lambda_1 > 0$)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$x_1 = e^{\lambda_1 t} x_1(0)$ $x_2 = e^{\lambda_2 t} x_2(0)$	
Star node: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$x_1 = e^{\lambda t} x_1(0)$ $x_2 = e^{\lambda t} x_2(0)$	
Degenerate node: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$x_1 = e^{\lambda t} (x_1(0) + t x_2(0))$ $x_2 = e^{\lambda t} x_2(0)$	
Non-isolated fixed points: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$	$x_1 = x_1(0)$ $x_2 = e^{\lambda t} x_2(0)$	
Saddle: $\lambda_2 < 0 < \lambda_1$	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$x_1 = e^{\lambda_1 t} x_1(0)$ $x_2 = e^{\lambda_2 t} x_2(0)$	

Linear behavior near a 2D fixed point

Name

Matrix

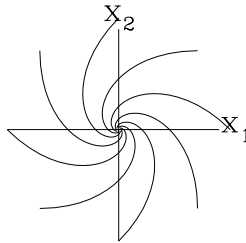
Behavior

Spiral:

stable ($\mu < 0$)
unstable ($\mu > 0$)

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

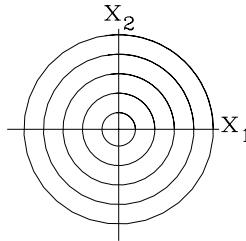
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$



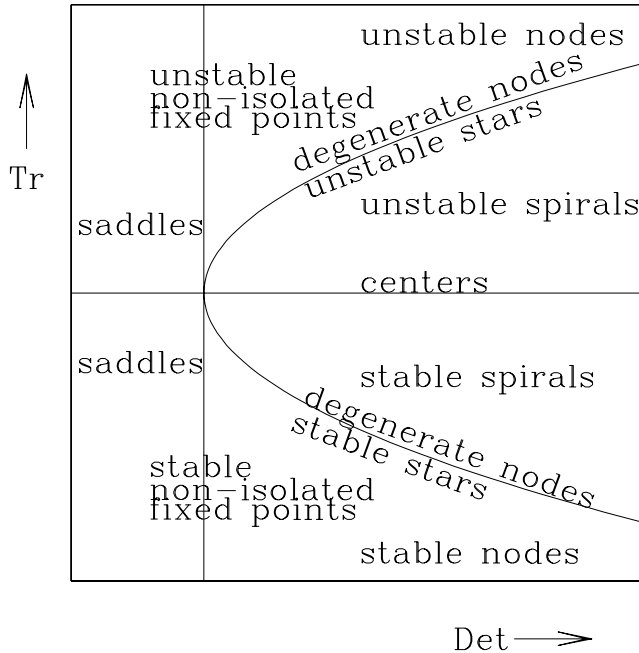
Center:

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$



Linear behavior near a 2D fixed point



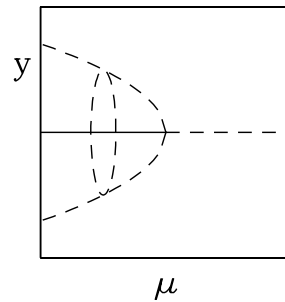
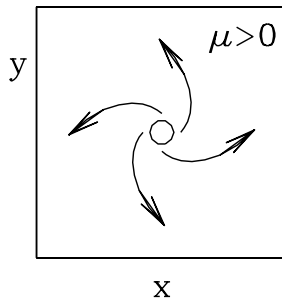
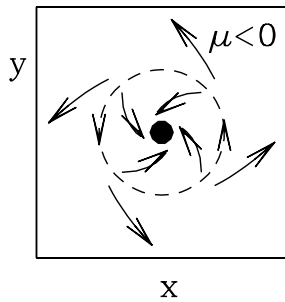
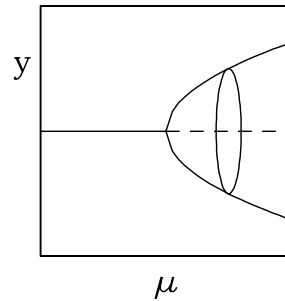
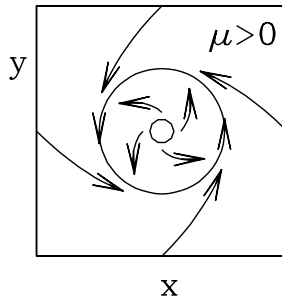
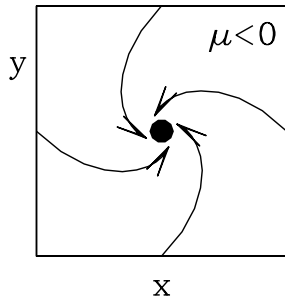
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\lambda_{1,2} = \frac{1}{2} \left(\text{Tr} \pm \sqrt{\text{Tr}^2 - 4\text{Det}} \right)$$

$$\text{Tr} \equiv a + d = \lambda_1 + \lambda_2$$

$$\text{Det} \equiv ad - bc = \lambda_1 \lambda_2$$

Hopf Bifurcation



Normal Form:

$$\dot{z} = (\mu + i\omega)z - \alpha|z|^2z$$

Cartesian Form:

$$\dot{x} + i\dot{y} = (\mu + i\omega)(x + iy) - (\alpha_r + i\alpha_i)(x^2 + y^2)(x + iy)$$

$$\dot{x} = \mu x - \omega y - (x^2 + y^2)(\alpha_r x - \alpha_i y)$$

$$\dot{y} = \omega x + \mu y - (x^2 + y^2)(\alpha_i x + \alpha_r y)$$

Polar Form:

$$(\dot{r} + ir\dot{\theta})e^{i\theta} = (\mu + i\omega)re^{i\theta} - (\alpha_r + i\alpha_i)r^2re^{i\theta}$$

$$\dot{r} = \mu r - \alpha_r r^3$$

$$\dot{\theta} = \omega - \alpha_i r^2$$

Trajectory:

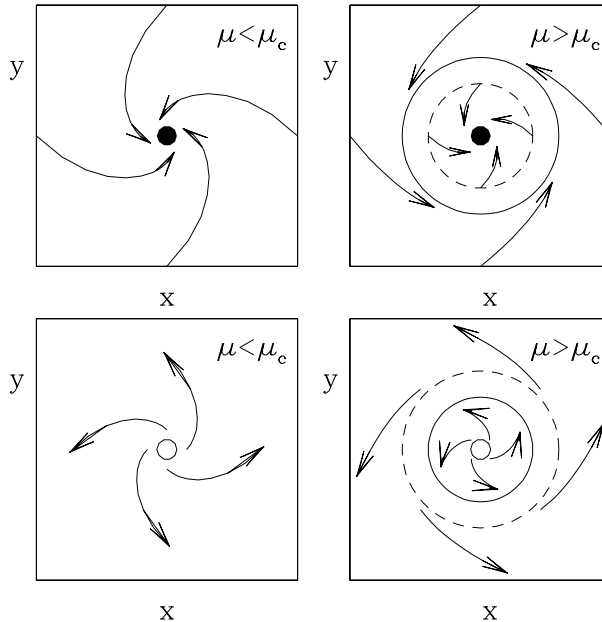
$$z(t) = \sqrt{\mu}e^{i\omega(t-t_0)}$$

Subcritical Form:

$$\dot{z} = (\mu + i\omega)z + \alpha|z|^2z \quad (\alpha_r > 0)$$

Global bifurcations \implies limit cycles

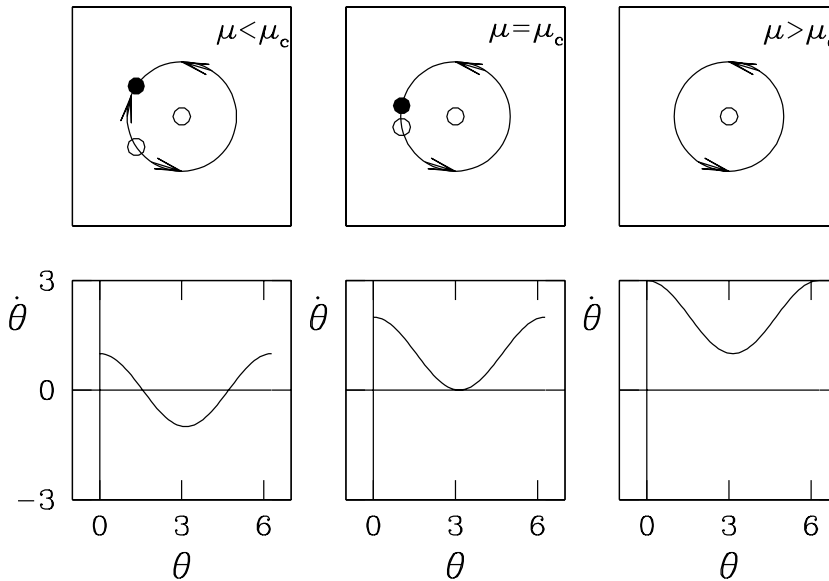
Saddle-node bifurcation OF two limit cycles



$$\dot{r} = \alpha r (\mu - \mu_c - (r^2 - r_c^2)^2)$$

$$\dot{\theta} = \omega$$

Saddle-node bifurcation ON a limit cycle

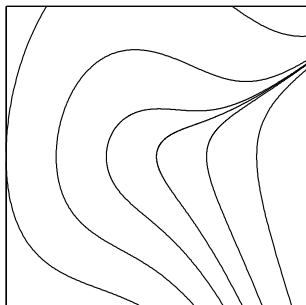


$$\dot{r} = r(1 - r^2)$$

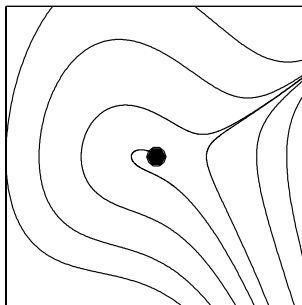
$$\dot{\theta} = \mu + 1 + \cos(\theta)$$

Homoclinic bifurcation

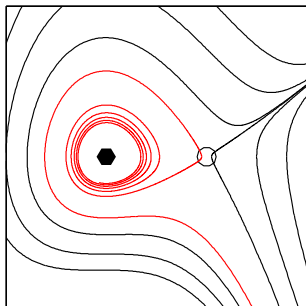
$$\mu = -\frac{1}{2}$$



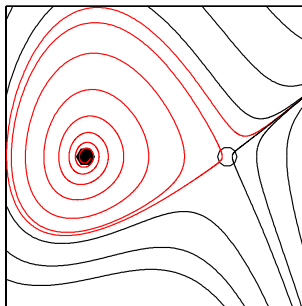
$$\mu = -\frac{1}{4}$$



$$\mu = 0$$



$$\mu = \frac{1}{4}$$



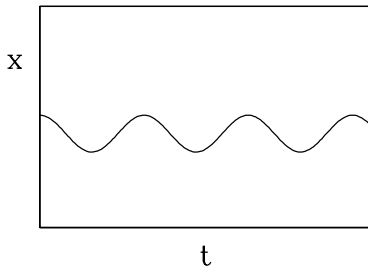
$$\dot{x} = y$$

$$\dot{y} = -\mu - x + x^2 - xy$$

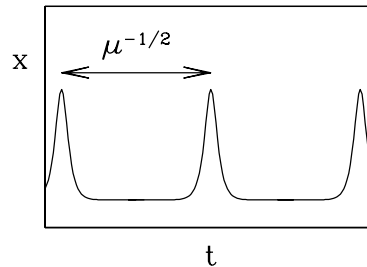
Global bifurcations

Non-uniform trajectory:

slow phases near saddles or near “ghosts” of fixed points



uniform trajectory



non-uniform trajectory

	Amplitude	Period
Supercritical Hopf	$O(\mu^{1/2})$	$O(1)$
Saddle-node <i>of</i> limit cycles	$O(1)$	$O(1)$
Saddle-node <i>on</i> limit cycle	$O(1)$	$O(\mu^{-1/2})$
Homoclinic	$O(1)$	$O(\log(\mu))$