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Dynamical Systems

English-language version

Dynamical Systems

A dynamical system is defined simply by:

$$\dot{x} = f(x),\ x, f \text{ vectors in } \mathcal{R}^N \tag{1}$$

N is called the number of degrees of freedom of the system.

Here are some examples of dynamical systems.

-Normal form of a saddle-node bifurcation ($N = 1$)

$$\dot{x} = \mu - x^2 \tag{2}$$

–Lorenz model ($N = 3$)

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10(y - x) \\ -xz + rx - y \\ xy - 8z/3 \end{pmatrix} \tag{3}$$

–Navier-Stokes equations($N \gg 1$)

$$\frac{d}{dt}\mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \frac{1}{Re}\Delta\mathbf{u} \tag{4}$$

For the Navier-Stokes equations, $\mathbf{u}(\mathbf{x}) = (u(x, y, z), v(x, y, z), w(x, y, z))$ and $N = \infty$. $N = 3 \times 100^3 = 3 \times 10^6$. In a typical three-dimensional numerical simulation, one uses a spatial discretization of $N_x = N_y = N_z = 100$, leading to $N = 3 \times 100^3 = 3 \times 10^6$.

Other systems can be re-written as dynamical systems, by writing additional variables, in particular, a *non-autonomous* system:

$$\dot{x} = f(x, t) \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} f(x, \theta) \\ 1 \end{pmatrix} \text{ with } \theta \equiv t \tag{5}$$

or a system of higher temporal order:

$$\ddot{x} = f(x, \dot{x}) \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ f(x, y) \end{pmatrix} \text{ with } y \equiv \dot{x} \tag{6}$$

1 Analysis of one-dimensional systems

1.1 Fixed points and linear stability

We begin with a dynamical system:

$$\dot{x} = f(x) \tag{7}$$

A fixed point $\bar{x}$ is a solution to:

$$0 = f(\bar{x}) \tag{8}$$

Fixed points $\bar{x}$, also called steady states, are thus roots of the function f. The *linear stability* of $\bar{x}$ can be

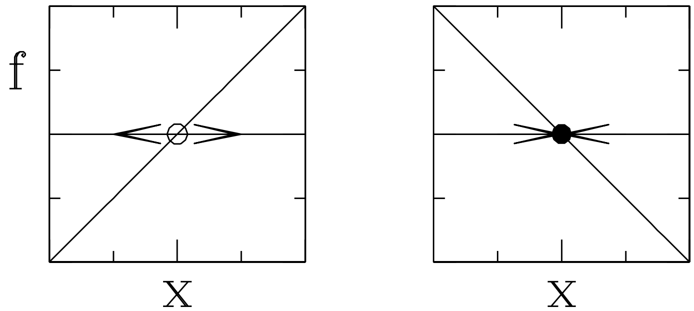


Figure 1: Unstable (left) and stable (right) fixed points. A fixed point of $\dot{x} = f(x)$ is a solution to $f(\bar{x}) = 0$. The dynamics causes x to increase (decrease) where f is positive (negative). If $f'(\bar{x}) > 0$, neighboring points evolve by leaving $\bar{x}$, which is thus unstable (left). If $f'(\bar{x}) < 0$, neighboring points evolve by approaching $\bar{x}$, which is thus stable (right).

studied by writing:

$$\begin{aligned} x(t) &= \bar{x} + \epsilon(t) && (9) \\ \frac{d}{dt}(\bar{x} + \epsilon) &= f(\bar{x} + \epsilon) \\ \cancel{\dot{\bar{x}}} + \dot{\epsilon} &= \cancel{f(\bar{x})} + f'(\bar{x})\epsilon + \frac{1}{2} f''(\bar{x})\epsilon^2 + \cdots \\ &\approx f'(\bar{x})\epsilon \\ \epsilon(t) &= e^{tf'(\bar{x})}\epsilon(0) && (10) \end{aligned}$$

A perturbation ϵ will grow exponentially in time $f'(\bar{x}) > 0$, i.e. if $\bar{x}$ is unstable. In contrast, if $f'(\bar{x}) < 0$, then ϵ decreases exponentially in time and $\bar{x}$ is stable, as shown in figure 1.

In what follows, we will assume that f depends on a parameter μ, for example a Reynolds or Rayleigh number measuring a velocity or temperature gradient imposed on a fluid. A *steady bifurcation* is defined as a change in the number of fixed points (roots of f). We will see that this is closely connected to stability.

1.2 Saddle-node bifurcations

A linear function cannot change its number of roots. The simplest function that can change the number of its roots as μ is varied is a quadratic polynomial, like that shown in figure 2.

$$f(x, \mu) = c_{00} + c_{10}x + c_{01}\mu + c_{20}x^2 \tag{11}$$

is assumed to represent the first terms of a Taylor expansion of a general function. We re-write (11) as:

$$f(x, \mu) = c_{01}\mu + c_{00} - \frac{c_{10}^2}{4c_{20}} + c_{20}\left(x + \frac{c_{10}}{2c_{20}}\right)^2 \tag{12}$$

If $c_{20} < 0$ and $c_{01} > 0$, we can define

$$\tilde{\mu} \equiv c_{01}\mu + c_{00} - \frac{c_{10}^2}{4c_{20}} \qquad \tilde{x} \equiv \sqrt{-c_{20}}\left(x + \frac{c_{10}}{2c_{20}}\right) \tag{13}$$

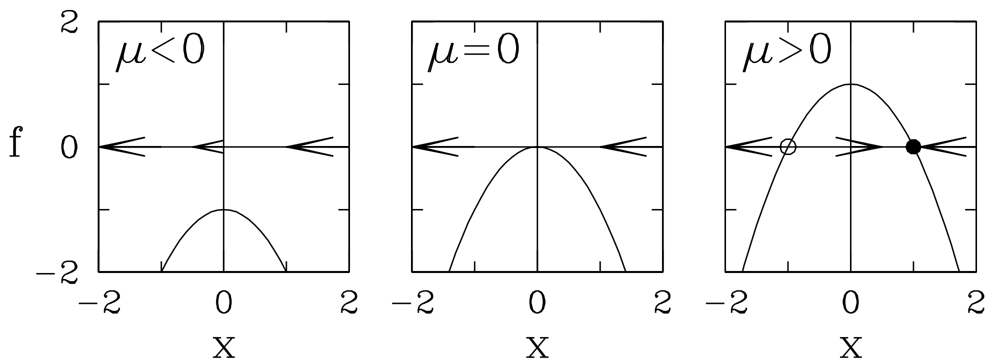


Figure 2: The function $f = \mu - x^2$ has 0, 1, or 2 roots, if $\mu < 0$, $\mu = 0$, or $\mu > 0$.

and write

$$f = \tilde{\mu} - \tilde{x}^2 \tag{14}$$

or, re-defining $\tilde{\mu} \to \mu$, $\tilde{x} \to x$,

$$f(x, \mu) = \mu - x^2 \tag{15}$$

We call (15) the *normal form* of the *saddle-node bifurcation*. Let us study the behavior of (15). The fixed points of (15) are

$$\bar{x}_{\pm} = \pm\sqrt{\mu} \tag{16}$$

which exist only for $\mu > 0$. Their stability is determined by

$$f'(\bar{x}_{\pm}) = -2\bar{x}_{\pm} = -2(\pm\sqrt{\mu}) = \mp 2\sqrt{\mu} \tag{17}$$

By looking at the sign of $f'(\bar{x})$, we see that $\bar{x}_+ = \sqrt{\mu}$ is stable, whereas $\bar{x}_- = -\sqrt{\mu}$ is unstable.

If $c_{20} > 0$ or $c_{01} < 0$, we deduce one of the following forms:

$$\begin{aligned} f(x, \mu) &= -\mu + x^2 && (18) \\ f(x, \mu) &= \mu - x^2 && (19) \\ f(x, \mu) &= -\mu - x^2 && (20) \end{aligned}$$

In each case, there is a transition at $\mu = 0$ between no fixed points and two fixed points, one stable and one unstable. Figure 3 summarizes this information for each case on what is called its *bifurcation diagram*.

1.3 Pitchfork bifurcations

For reasons of symmetry, to which we shall return later, it may be that $f(x)$ is restricted to be an odd function of x. (x is to be considered as a deviation from some special state, called a base state, rather than the distance from zero.) There is then no constant or quadratic term in x, and a cubic term must be included, as in figure 4, for a bifurcation to take place. We can reduce a cubic polynomial to four cases:

$$\begin{aligned} f(x, \mu) &= \mu x - x^3 && (21) \\ f(x, \mu) &= \mu x + x^3 && (22) \\ f(x, \mu) &= -\mu x + x^3 && (23) \\ f(x, \mu) &= -\mu x - x^3 && (24) \end{aligned}$$

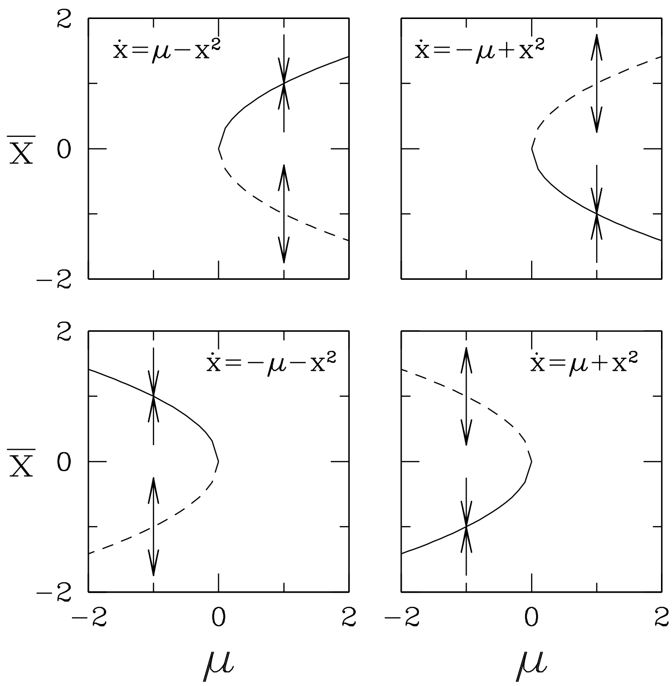


Figure 3: Saddle-node bifurcation diagrams. In each case, there exist two branches of fixed points, one stable and one unstable, on one side of $\mu = 0$, and no fixed points on the other side.

The four corresponding bifurcation diagrams are given in figure 1.3. Equation (21) is called the normal form of a *supercritical pitchfork bifurcation*. Its fixed points are calculated by:

$$0 = \bar{x}(\mu - \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \text{for all } \mu \\ \bar{x} = \pm\sqrt{\mu} & \text{for } \mu > 0 \end{cases} \tag{25}$$

We now determine the stability of these fixed points.

$$f'(\bar{x}) = \mu - 3\bar{x}^2 = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ \mu - 3\mu = -2\mu & \text{for} \bar{x} = \pm\sqrt{\mu} \end{cases} \tag{26}$$

The fixed point $\bar{x} = 0$ is therefore stable for $\mu < 0$ and becomes unstable at $\mu = 0$, where the new branches of fixed points $\bar{x} = \pm\sqrt{\mu}$ are created. These new fixed points are stable. We now repeat the calculation for (22), which is the normal form of a *subcritical pitchfork bifurcation*.

$$0 = \bar{x}(\mu + \bar{x}^2) \Longrightarrow \begin{cases} \bar{x} = 0 & \text{for all } \mu \\ \bar{x} = \pm\sqrt{-\mu} & \text{for } \mu < 0 \end{cases} \tag{27}$$

$$f'(\bar{x}) = \mu + 3\bar{x}^2 = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ \mu + 3(-\mu) = -2\mu & \text{for } \bar{x} = \pm\sqrt{\mu} \end{cases} \tag{28}$$

As in the supercritical case, the fixed point $\bar{x} = 0$ is stable for $\mu < 0$ and becomes unstable at $\mu = 0$. But, contrary to the supercritical case, the other fixed points $\pm\sqrt{\mu}$ exist in the région where $\bar{x} = 0$ is stable; the other fixed points are unstable.

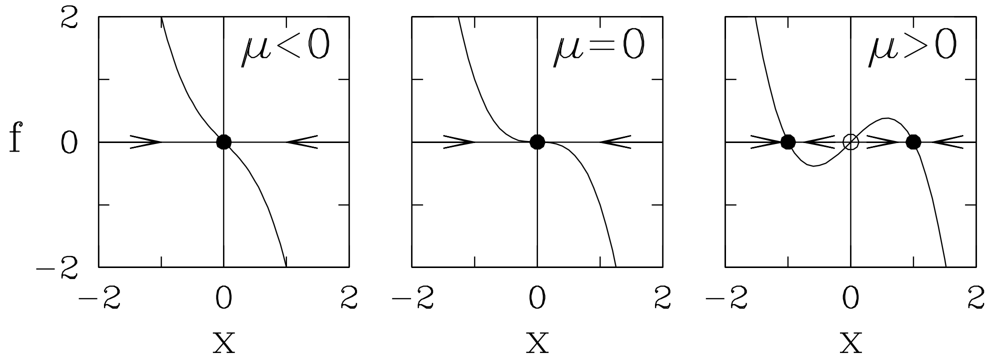


Figure 4: The function $f = \mu x - x^2$ has 1 or 3 roots, according to whether $\mu < 0$, $\mu = 0$, or $\mu > 0$.

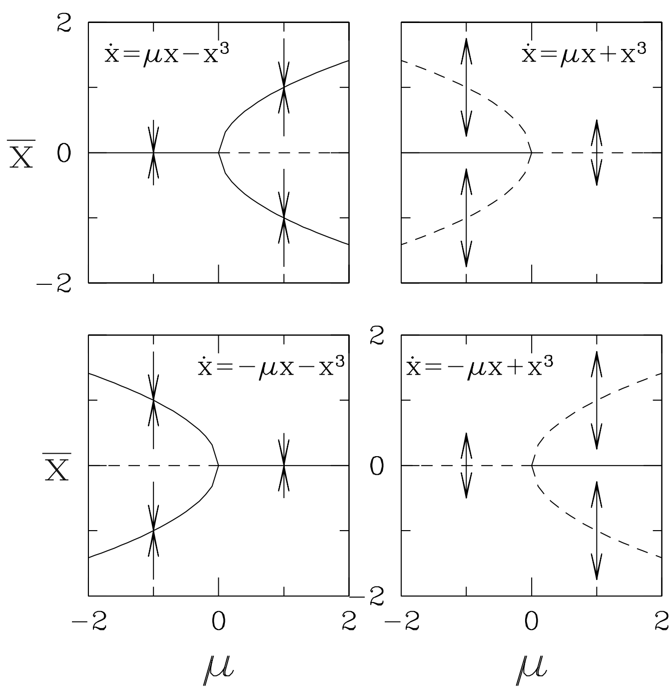


Figure 5: Pitchfork bifurcation diagrams. A branch of fixed points gives rise to two new branches when a critical value of μ is crossed. The bifurcation is called a supercritical (subcritical) pitchfork if the new branches are stable (unstable).

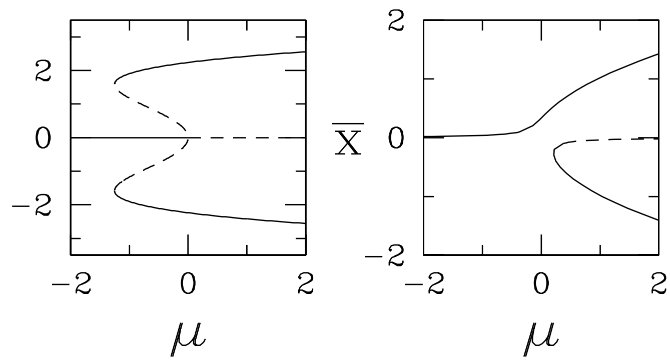


Figure 6: Left: bifurcation diagram for $\dot{x} = \mu x + x^3 - x^5/10$. The fifth-order term stabilizes the trajectories near a subcritical pitchfork bifurcation. This term causes two saddle-node bifurcations. As μ is increased, there is first one fixed point, then five, then finally three fixed points. Right: Diagram for an imperfect pitchfork bifurcation $\dot{x} = 1/27 + \mu x - x^3$. The constant term represents an imperfection, that causes the system to prefer one of the two branches. The pitchfork bifurcation has been transformed into a saddle-node bifurcation.

Note that most trajectories in the subcritical case evolve towards infinity. When we know that trajectories in a physical system do not behave in this way but remain bounded, we sometimes add to (22) a stabilizing term of higher order, illustrated in figure 6.

$$\dot{x} = \mu x + x^3 - \alpha x^5 \tag{29}$$

The fixed points of (29) are :

$$0 = \bar{x}(\mu + \bar{x}^2 - \alpha\bar{x}^4) \Longrightarrow \begin{cases} \bar{x} = 0 & \text{for all } \mu \\ \bar{x} = \pm\sqrt{(1 + \sqrt{1+4\alpha\mu})/(2\alpha)} & \text{for } -1 < 4\alpha\mu \\ \bar{x} = \pm\sqrt{(1 - \sqrt{1+4\alpha\mu})/(2\alpha)} & \text{for } -1 < 4\alpha\mu < 0 \end{cases} \tag{30}$$

In fact, (29) contains three bifurcations: a subcritical pitchfork bifurcation at $\mu = 0$ and two saddle-node bifurcations at $\mu = -1$. A dynamical system that obeys (29) displays *hysteresis*. Starting at $\bar{x} = 0$ and $\mu < -1$ and increasing μ, transition takes place at $\mu = 0$, at the subcritical pitchfork bifurcation at which $\bar{x} = 0$ becomes unstable. If we then decrease μ, transition takes place instead at $\mu = -1$, where the saddle-node bifurcations destroy the non-zero branches.

Another variant, also illustrated in figure 6, is the *imperfect pitchfork bifurcation*, whose equation is given by

$$\dot{x} = \alpha + \mu x - x^3 \tag{31}$$

The constant term "breaks" the pitchfork bifurcation. One of the two new branches becomes a continuation of the stable part of the original branch. The other branch joins with the unstable part of the original branch in a saddle-node bifurcation.

1.4 Simple mechanical examples

A loop of wire of length L will bend over in one direction or the other when L becomes sufficiently large. This bifurcation is *subcritical*: in decreasing the length, the loop will straighten only when a length L_0 is attained, which is smaller than the value L_c at which it bent during the increase in length, as shown in figure 7.

A load P is applied to a perfectly straight and symmetric beam, which will buckle either to the left or to the right, as shown in figure 8. This corresponds to a pitchfork bifurcation. If the beam has any defect, buckling will occur systematically on that side. Buckling in the other direction is possible, but only by forcing the system in some way to counteract the defect. This corresponds to an imperfect pitchfork bifurcation.

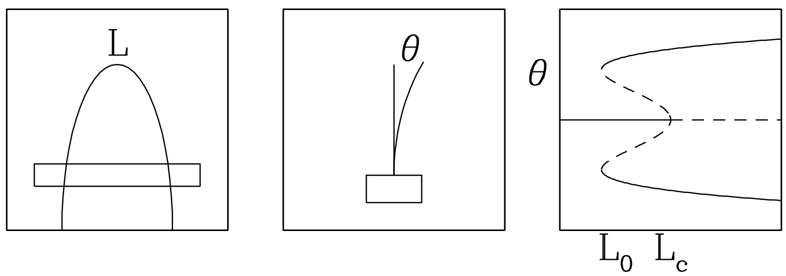


Figure 7: A loop of wire of length L will bend to the left or to the right when its length exceeds L_c. When decreasing the length, the loop will continue to bend until the length becomes less than L_0. Left: front view. Middle: side view. Right: bifurcation diagram for a subcritical pitchfork.

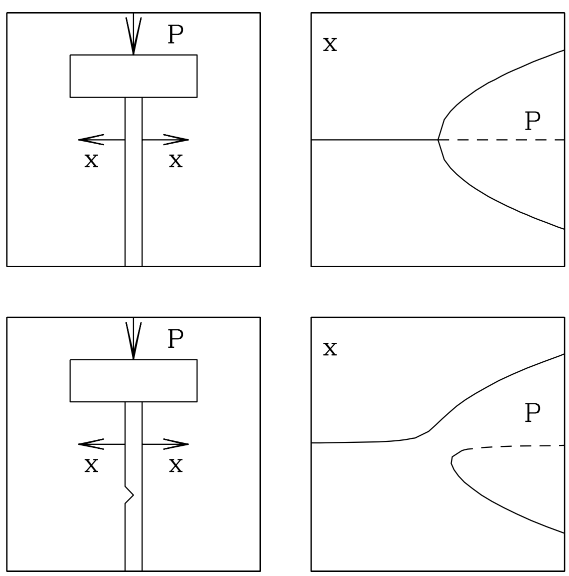


Figure 8: An ideal beam subjected to a load will buckle to the right or to the left with equal probability, meaning the buckling is described by a pitchfork bifurcation. A weakness on one side of the beam will favor buckling in that direction, meaning that the pitchfork is imperfect.

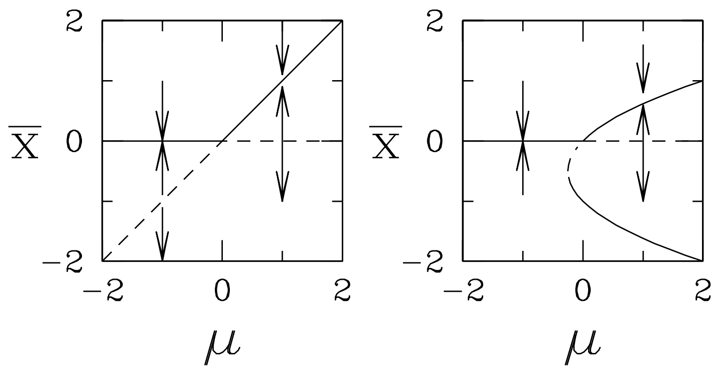


Figure 9: Bifurcation diagrams for a transcritical bifurcation (left) and for a transcritical bifurcation with an added cubic term, leading to an additional stabilizing saddle-node bifurcation (right).

1.5 Transcritical bifurcations

If the constraints on the problem are such as to forbid a constant term in the Taylor expansion of f (recall that x is to be considered as a deviation from some base state, rather than the distance from zero), then the truncated Taylor expansion leads to the normal form of a *transcritical bifurcation*, which is the final steady one-dimensional bifurcation:

$$\dot{x} = \mu x - x^2 \tag{32}$$

The usual analysis yields

$$0 = \bar{x}(\mu - \bar{x}) \Longrightarrow \begin{cases} \bar{x} = 0 \\ \bar{x} = \mu \end{cases} \tag{33}$$

$$f'(\bar{x}) = \mu - 2\bar{x} = \begin{cases} \mu & \text{for } \bar{x} = 0 \\ -\mu & \text{for } \bar{x} = \mu \end{cases} \tag{34}$$

Thus $\bar{x} = 0$ is stable for $\mu < 0$, unstable pour $\mu > 0$, whereas $\bar{x} = \mu$ does the opposite: these fixed points merely exchange their stability. Since trajectories of (32) go to infinity, a higher-order term is sometimes added to the normal form in order to stabilize the trajectories, as was done for the subcritical pitchfork bifurcation; see figure 9.

1.6 General conditions

If f is any function that can be expanded in a Taylor series, the table below shows the conditions which $f(x, \mu)$ must satisfy in order for $(\bar{x}, \bar{\mu})$ to be a point of the designated type.

	f	f_x	f_μ	f_{xx}	$f_{x\mu}$	f_{xxx}
steady state	0					
bifurcation	0	0	$\neq 0$			
saddle-node	0	0	$\neq 0$	$\neq 0$		
transcritical	0	0	0	$\neq 0$	$\neq 0$	
pitchfork	0	0	0	0	$\neq 0$	$\neq 0$

The normal forms (15), (21), (22), (32) are truncated expansions of f about $(\bar{x}, \bar{\mu})$ of a function satisfying the conditions above, after re-definition of x, μ as described in (11)-(15).

The behaviors described above are illustrated by the *unfolding* of the pitchfork, which is the classification of the behavior of all cubic polynomials

$$f(x) = \alpha + \mu x + \beta x^2 - x^3 \tag{35}$$

as parameters α and β are varied. Figure 10 displays all possible diagrams resulting from 35) and contains a pitchfork bifurcation (for $\alpha = \beta = 0$), transcritical bifurcations (for $\alpha = 0$), and saddle-node bifurcations for other parameter values.

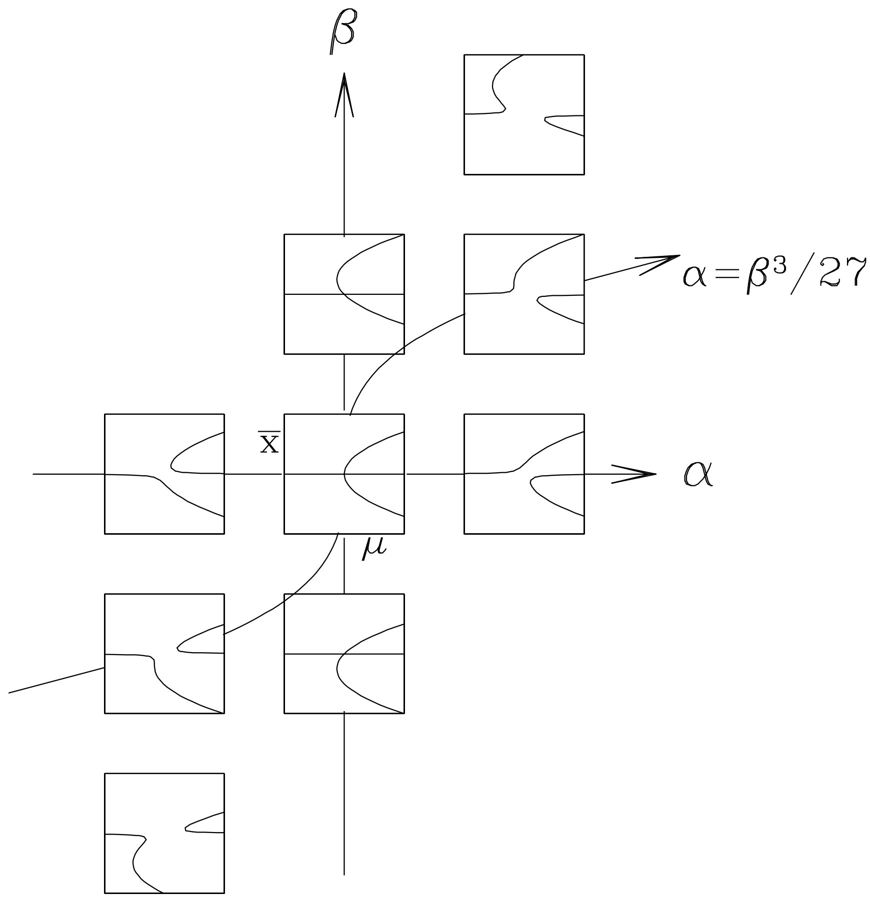


Figure 10: Unfolding of the pitchfork. Bifurcation diagrams $(\mu, \bar{x})$ showing the roots of $0 = \alpha + \mu x + \beta x^2 - x^3$ for nine values of (α, β). The pitchfork bifurcation occurs for $\alpha = \beta = 0$; transcritical bifurcations for $\alpha = 0$. For $0 < \alpha < \beta^3/27$ and $\beta^3/27 < \alpha < 0$, there are three saddle-node bifurcations; for other values there is a single one.

2 Systems with two or more dimensions

2.1 From one to many dimensions

As before, we consider a dynamical system:

$$\dot{x} = f(x), \quad x, f \in \mathcal{R}^N \tag{36}$$

whose fixed points are solutions of:

$$0 = f(\bar{x}) \tag{37}$$

To study the stabilitiy of $\bar{x}$, we perturb it by $\epsilon(t) \in \mathcal{R}^N$.

$$\begin{aligned} \frac{d}{dt}(\bar{x}+\epsilon) &= f(\bar{x}+\epsilon) \\ \cancel{\dot{\bar{x}}} + \dot{\epsilon} &= \cancel{f(\bar{x})} + Df(\bar{x})\,\epsilon + \epsilon\, D^2 f(\bar{x})\,\epsilon + \ldots \end{aligned} \tag{38}$$

Since quadratic terms in ϵ are infinitesimally smaller than linear ones, (38) is reduced to the linear differential system:

$$\dot{\epsilon} = Df(\bar{x})\,\epsilon \tag{39}$$

In (38)-(39), $Df(\bar{x})$ is the *Jacobian* of f, i.e. the matrix of partial derivatives, evaluated at the fixed point $\bar{x}$. (When x is infinite dimensional rather than a vector in $\mathcal{R}^N$, the operator analogous to the Jacobian matrix is called the *Fréchet derivative*.) To clarify the meaning of (38)-(39), we rewrite these equations explicitly for each component:

$$\begin{aligned} \cancel{\dot{\bar{x}}_i} + \dot{\epsilon}_i &= \cancel{f_i(\bar{x})} + Df(\bar{x})_{ij}\,\epsilon_j + \epsilon_j\,[D^2 f(\bar{x})]_{ijk}\,\epsilon_k + \ldots \\ &= \cancel{f_i(\bar{x})} + \frac{\partial f_i}{\partial x_j}(\bar{x})\,\epsilon_j + \epsilon_j\,\frac{\partial^2 f_i}{\partial x_j \partial x_k}(\bar{x})\,\epsilon_k + \ldots \\ \dot{\epsilon}_i &= \frac{\partial f_i}{\partial x_j}(\bar{x})\,\epsilon_j \end{aligned} \tag{40}$$

The solution to (39) is

$$\epsilon(t) = e^{Df(\bar{x})t}\epsilon(0) \tag{41}$$

We define the exponential of a matrix via its Taylor series, as we can do for any analytic function $f(A)$:

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \ldots \tag{42}$$

The behavior of (42) depends on the spectrum of the matrix, i.e. its set of eigenvalues. Let A have the eigenvector-eigenvalue decomposition $A = V\Lambda V^{-1}$. According to (42),

$$\begin{aligned} e^{tA} &= VV^{-1} + tV\Lambda V^{-1} + \frac{t^2}{2}V\Lambda V^{-1}\,V\Lambda V^{-1} + \frac{t^3}{6}V\Lambda V^{-1}\,V\Lambda V^{-1}\,V\Lambda V^{-1} + \ldots \\ &= V\left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \frac{t^3}{6}\Lambda^3 + \ldots\right]V^{-1} \\ &= Ve^{\Lambda t}V^{-1} \end{aligned} \tag{43}$$

Thus, we only need to know how to take exponentials of the matrix of eigenvalues.

For a matrix with real eigenvalues, we have, for the 2×2 case:

$$\Lambda^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \tag{44}$$

$$\Lambda^3 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix} \tag{45}$$

leading to

$$e^{t\Lambda} = \begin{pmatrix} 1 + t\lambda_1 + \frac{1}{2}(t\lambda_1)^2 + \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{1}{2}(t\lambda_2)^2 + \ldots \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \tag{46}$$

The question "stable or unstable?" becomes "stable or unstable *in which directions*?" The fixed point $\bar{x}$ is considered to be linearly stable if the real parts of *all* of the eigenvalues of $Df(\bar{x})$ are negative, and unstable if even *one* of the eigenvalues has a postive real part. The reasoning behind this is that initial random perturbations will contain components in all directions. If there is instability in one direction, then this component will grow and we will diverge away from $\bar{x}$, initially in the unstable direction.

In the simplest situation, we have $0 > \lambda_2 > \lambda_3 \ldots$, and λ_1 changes sign at a bifurcation. By projecting onto the eigenvector v_1 corresponding to λ_1, i.e. by taking the scalar product with the adjoint eigenvector v_1^T satisfying:

$$v_1^T Df(\bar{x}) = v_1^T \lambda_1 \tag{47}$$

we obtain a one-dimensional equation. (In the other directions, the behavior is uninteresting: there is only contraction along these directions towards the fixed point.) The first terms in the Taylor series of this equation correspond to a saddle-node, pitchfork, or transcritical bifurcation. It is in this way that we obtain bifurcations in physical systems with a large number of degrees of freedom, such as thermal convection. We emphasize the correspondence between realistic physical systems and the simple polynomial equations that we have just written down:

• Complicated equations in $N \gg 1$ variables.
Calculate fixed points $\bar{x}$, their Jacobians $Df(\bar{x})$ and their spectra $\{\lambda_1, \lambda_2, \ldots\}$.
Bifurcation if the real part of one of them changes sign.

• Project onto the corresponding adjoint eigenvector $\Longrightarrow$ Function of one variable.

• Taylor expand about the fixed point.
Minimal truncation giving observed behavior $\Longrightarrow$ Normal form of the bifurcation.

2.2 Linear systems with complex eigenvalues

We have become familiar with the situations which are one-dimensional or reducible to one dimension. We now discuss the more complicated situations which can occur in two dimensions, in particular when λ_1 and λ_2 are part of a complex conjugate pair. We consider the 2×2 matrix corresponding to an imaginary pair of eigenvalues $\pm i\omega$, which is skew-symmetric anti-diagonal, i.e.

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \tag{48}$$

$$A^2 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} \tag{49}$$

$$A^3 = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix} \tag{50}$$

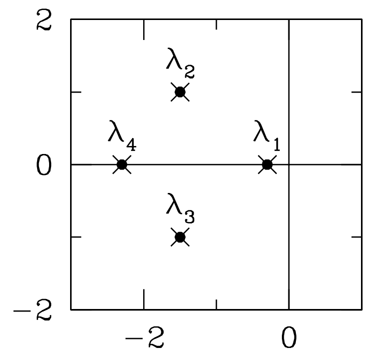


Figure 11: Eigenvalues of a Jacobian having two real eigenvalues and a pair of complex conjugate eigenvalues, with $0 > Re(\lambda_1) > Re(\lambda_2) = Re(\lambda_3) > Re(\lambda_4)$.

Using

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \dots \tag{51}$$

we have

$$e^{tA} = \begin{pmatrix} 1 - \frac{1}{2}(t\omega)^2 + \dots & -t\omega + \frac{1}{6}(t\omega)^3 + \dots \\ t\omega - \frac{1}{6}(t\omega)^3 + \dots & 1 - \frac{1}{6}(t\omega)^2 + \dots \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{52}$$

We can combine (46) with (52) to obtain

$$\exp\left[t \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\right] = \begin{pmatrix} e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) \\ e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) \end{pmatrix} \tag{53}$$

If a perturbation (ϵ_x, ϵ_y) of a fixed point with complex eigenvalues $\mu \pm i\omega$ is governed by:

$$\frac{d}{dt}\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} \epsilon_x \\ \epsilon_y \end{pmatrix} \tag{54}$$

then it evolves according to:

$$\epsilon_x(t) = e^{\mu t}\left[\cos(\omega t)\epsilon_x(0) - \sin(\omega t)\epsilon_y(0)\right] \tag{55}$$
$$\epsilon_y(t) = e^{\mu t}\left[\sin(\omega t)\epsilon_x(0) + \cos(\omega t)\epsilon_y(0)\right] \tag{56}$$

More generally, for a mixture of real and complex eigenvalues, as in figure 11, we have:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \mu & -\omega & 0 \\ 0 & \omega & \mu & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \Longrightarrow \exp(t\Lambda) = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\mu t}\cos(\omega t) & -e^{\mu t}\sin(\omega t) & 0 \\ 0 & e^{\mu t}\sin(\omega t) & e^{\mu t}\cos(\omega t) & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} \tag{57}$$

2.3 Jordan blocks and transient growth

We now consider two 2×2 matrices which have only λ as an eigenvalue, but which behave quite differently. The matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{58}$$

is a multiple of the identity. We calculate its eigenvectors:

$$\lambda x_1 + 0x_2 = \lambda x_1 \quad \Longrightarrow \quad x_1 \text{ arbitrary} \tag{59}$$
$$0x_1 + \lambda x_2 = \lambda x_2 \quad \Longrightarrow \quad x_2 \text{ arbitrary} \tag{60}$$

All vectors $(x_1, x_2)^T$ are eigenvectors and the eigenspace corresponding to the double eigenvalue λ is two-dimensional. In contrast, the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{61}$$

is called a *Jordan block*. We calculate its eigenvectors:

$$\lambda x_1 + x_2 = \lambda x_1 \quad \Longrightarrow \quad x_2 = 0 \tag{62}$$
$$0x_1 + \lambda x_2 = \lambda x_2 \quad \Longrightarrow \quad x_1 \text{ arbitrary} \tag{63}$$

The eigenspace is thus one-dimensional and consists of all multiples of $(1, 0)^T$, as shown on the left portion of figure 12. For most matrices A, whether or not the eigenvalues are multiple, there exist N linear independent eigenvectors, which are solutions to

$$(A - \lambda I)x = 0 \tag{64}$$

Any vector in $\mathcal{R}^N$ can thus be written as a sum of eigenvectors. This is not the case for a Jordan block. Where is the missing dimension? The Jordan block (61) also has a *generalized eigenvector*, which is a solution to

$$(A - \lambda I)v = x, \tag{65}$$

where x is an eigenvector. We calculate the generalized eigenvector of (61) as follows:

$$\lambda v_1 + 1v_2 - \lambda v_1 = c \quad \Longrightarrow \quad v_2 = c \neq 0 \tag{66}$$
$$0v_1 + \lambda v_2 - \lambda v_2 = 0 \quad \Longrightarrow \quad v_1 \text{ arbitrary} \tag{67}$$

Thus, any vector v satisfying $v_2 \neq 0$ is a generalized eigenvector of (61), as shown in the right portion of figure 12. A generalized eigenvector is determined, like an ordinary eigenvector, up to an arbitrary multiplicative constant, as is shown by (66), but also up to an arbitrary additive constant, since we can add any multiple of the eigenvector, as is shown by (67). This non-uniqueness can be eliminated using a scalar product, by requiring that the eigenvector be normalized, and that the scalar product of the generalized eigenvector with the eigenvector be zero. The eigenvector and generalized eigenvector of (61) selected by these criteria are then:

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{68}$$

The behavior of a system near a degenerate node is

$$x_1(t) = e^{\lambda t}(x_1(0) + x_2(0)t) \tag{69}$$
$$x_2(t) = e^{\lambda t} x_2(0) \tag{70}$$

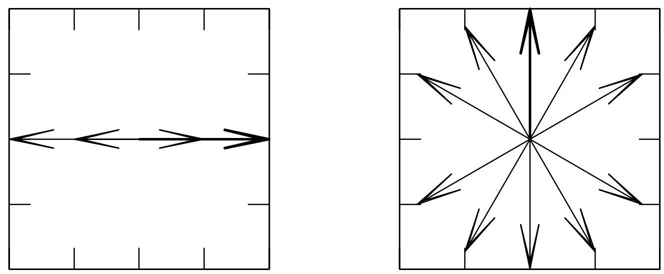


Figure 12: Left: the single eigenvector x of the 2×2 Jordan block (61) is directed along the x_1 axis and determined up to a multiplicative factor. Right: any vector v containing a non-zero x_2 component is a generalized eigenvector of (61). Wider arrows show x, v satisfying $||x|| = 1$, $||v|| = 1$ and $\langle x, v\rangle = 0$.

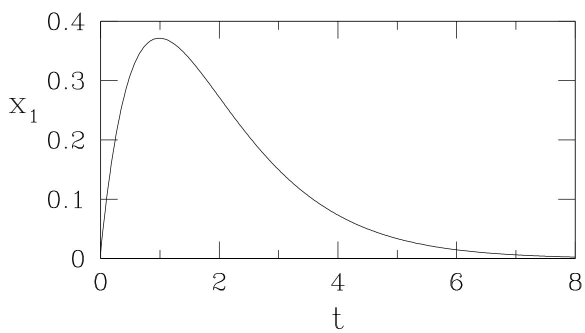
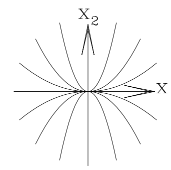
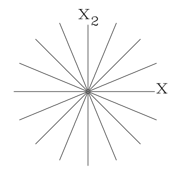
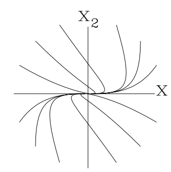
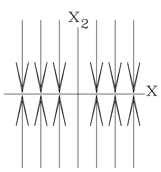
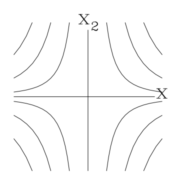


Figure 13: Transient growth. When (x_1, x_2) evolves according to a linear system which is a Jordan block, then $x_1(t)$ can start to grow, even if the negative eigenvalue of the matrix eventually leads to exponential decay. Here $\lambda = -1$, $x_1(0) = 0.01$ and $x_2(0) = 1$.

The linear term in (69) can cause the system to display *transient growth*, even when the eigenvalue λ is negative, as shown in figure 13.

We classify all the possible linear behaviors of a fixed point of a two-dimensional system below:

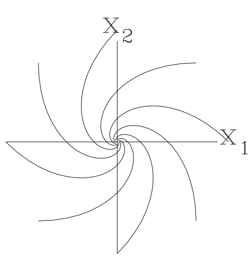
Name and classification	Matrix	Behavior	
Node : stable ($\lambda_2 < \lambda_1 < 0$) unstable ($\lambda_2 > \lambda_1 > 0$)	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$x_1 = e^{\lambda_1 t} x_1(0)$ $x_2 = e^{\lambda_2 t} x_2(0)$	
Star node: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$x_1 = e^{\lambda t} x_1(0)$ $x_2 = e^{\lambda t} x_2(0)$	
Degenerate node: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$x_1 = e^{\lambda t}(x_1(0) + t x_2(0))$ $x_2 = e^{\lambda t} x_2(0)$	
Non-isolated fixed points: stable ($\lambda < 0$) unstable ($\lambda > 0$)	$\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$	$x_1 = x_1(0)$ $x_2 = e^{\lambda t} x_2(0)$	
Saddle: $\lambda_2 < 0 < \lambda_1$	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$x_1 = e^{\lambda_1 t} x_1(0)$ $x_2 = e^{\lambda_2 t} x_2(0)$	

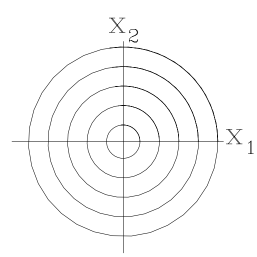
Spiral:
stable ($\mu < 0$)
unstable ($\mu > 0$)

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\mu t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

Center:

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$





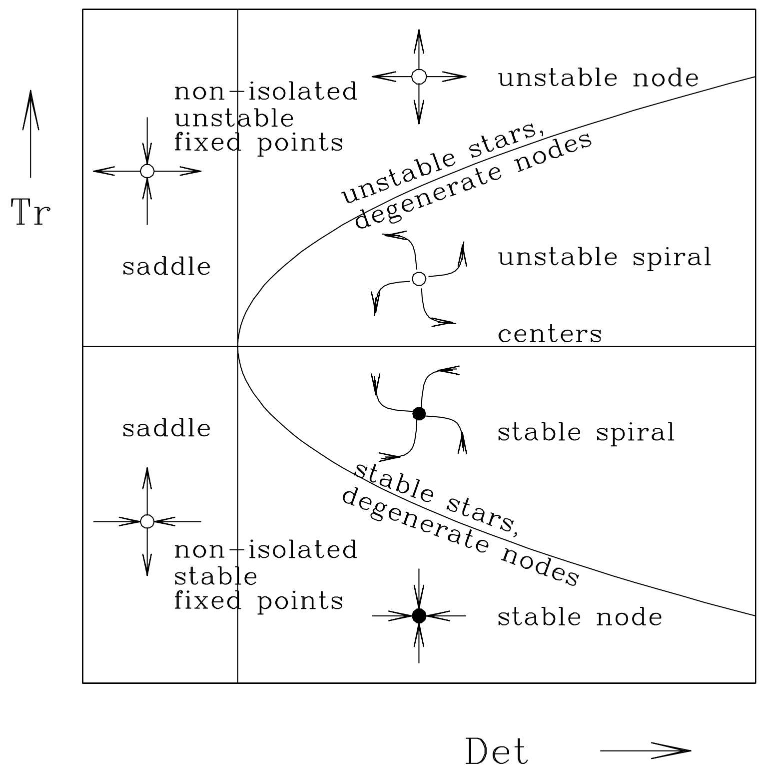


Figure 14: Behavior of two-dimensional linear systems as a function of the trace Tr and the determinant Det of the matrix. For $Det < 0$, the two eigenvalues are real and of opposite sign, leading to a saddle. For $Det > Tr^2/4$, the eigenvalues are complex conjugates and the behavior is thus oscillatory. For $Tr > 0$ at least one of the eigenvalues has a positive real part and the fixed point is thus unstable. Limiting cases are non-isolated fixed points ($Det = 0$), stars and degenerate nodes ($Tr^2 = 4Det$), and centers ($Tr = 0$, $Det > 0$).

All of these behaviors can be classified according to the *trace* Tr and the *determinant* Det of the matrix, as shown in figure 14. Recall that for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\lambda_{1,2} = \frac{1}{2}\left(\text{Tr} \pm \sqrt{\text{Tr}^2 - 4\text{Det}}\right) \quad (71)$$

$$\text{Tr} \equiv a + d = \lambda_1 + \lambda_2 \quad (72)$$

$$\text{Det} \equiv ad - bc = \lambda_1 \lambda_2 \quad (73)$$

Note that the (Det, Tr) diagram does not distinguish between the stars and the degenerate nodes, because the eigenvalues in both cases are double: $\lambda_1 = \lambda_2 = \lambda$.

2.4 Hopf Bifurcation

If λ_1, λ_2 are a complex conjugate pair whose real part μ changes sign, then a *Hopf bifurcation* takes place. Its normal form, i.e. the simplest nonlinear equation displaying this behavior, can be written:

$$\dot{z} = (\mu + i\omega)z - \alpha|z|^2 z \quad (74)$$

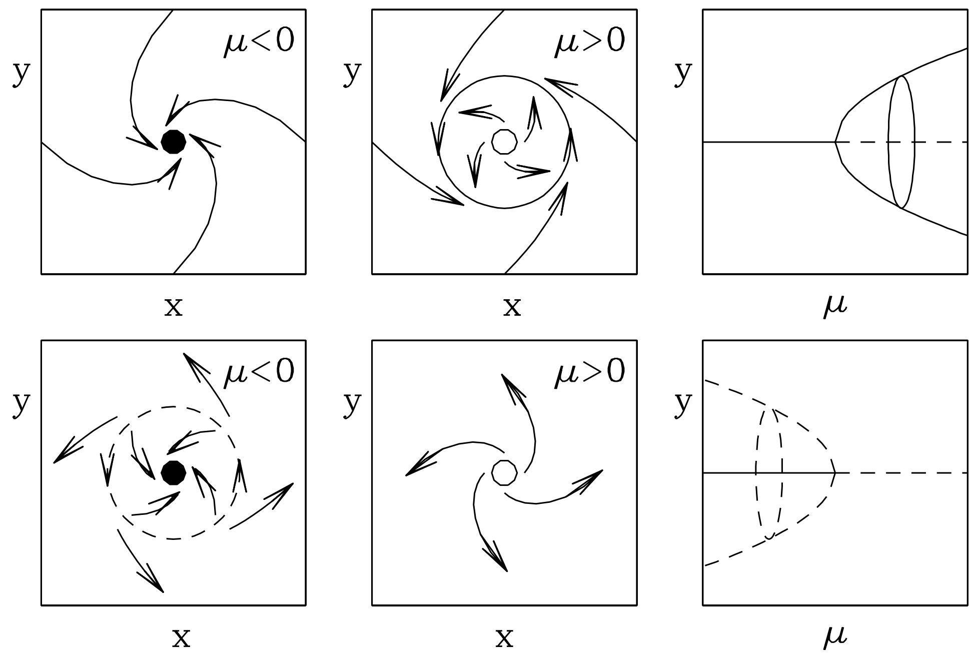


Figure 15: Hopf bifurcations. Top: supercritical case, $\dot{z} = \mu z - |z|^2 z$. Bottom: subcritical case, $\dot{z} = \mu z + |z|^2 z$. Left and middle: behavior of trajectories for $\mu < 0$ and for $\mu > 0$. The solid points and circles correspond to stable fixed points and limit cycles. The hollow points and dashed circles correspond to unstable fixed points and limit cycles. Right: bifurcation diagrams showing the solutions as a function of μ. The solid curves correspond to stable solution branches, the dashed curves to unstable solution branches. The ellipses represent one member of the branch of limit cycles.

Writing $z = x + iy$, (74) becomes

$$\begin{aligned}
\dot{x} + i\dot{y} &= (\mu + i\omega)(x + iy) - (\alpha_r + i\alpha_i)(x^2 + y^2)(x + iy) && (75)\\
\dot{x} &= \mu x - \omega y - (x^2 + y^2)(\alpha_r x - \alpha_i y) && (76)\\
\dot{y} &= \omega x + \mu y - (x^2 + y^2)(\alpha_i x + \alpha_r y) && (77)\\
& && (78)
\end{aligned}$$

We can also use a polar representation $z = re^{i\theta}$. The normal form (74) then becomes:

$$\begin{aligned}
(\dot{r} + ir\dot{\theta})e^{i\theta} &= (\mu + i\omega)re^{i\theta} - (\alpha_r + i\alpha_i)r^2 re^{i\theta} && (79)\\
\dot{r} &= \mu r - \alpha_r r^3 && (80)\\
\dot{\theta} &= \omega - \alpha_i r^2 && (81)
\end{aligned}$$

Equation (80) describes a pitchfork in the radial direction, and (81) describes rotation. The fixed points of (80) are $r = 0$ and $r = \sqrt{\mu}$ (where we retain only $r > 0$). For the normal form, we can calculate the *limit cycle*, that is, the periodic solution of (74) approached by all trajectories, regardless of initial condition.

$$z(t) = \sqrt{\mu} e^{i\omega(t - t_0)} \qquad (82)$$

As for the pitchfork, there also exists a *subcritical* version of the Hopf bifurcation, whose normal form is:

$$\dot{z} = (\mu + i\omega)z + \alpha|z|^2 z \qquad (\alpha_r > 0) \qquad (83)$$

The behavior of systems (74) and (83) in the neighborhood of a Hopf bifurcation is shown in figure 15.

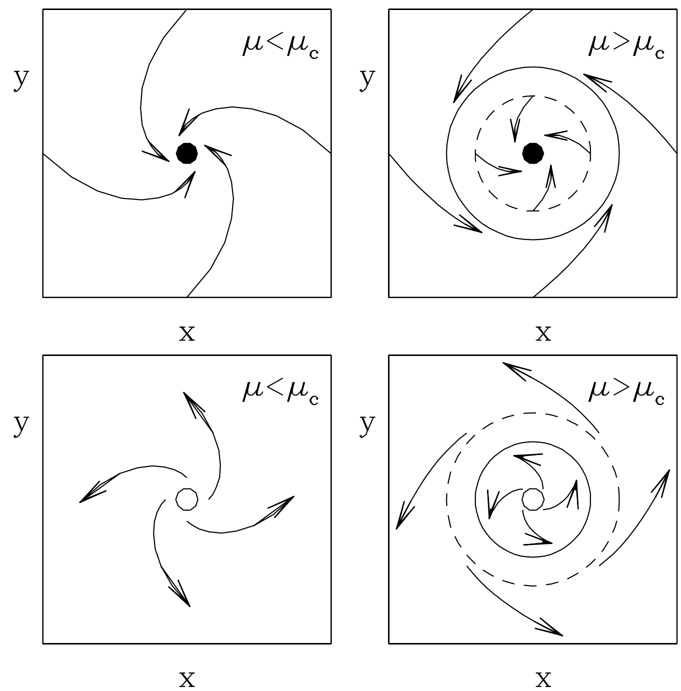


Figure 16: Saddle-node bifurcation *of* two periodic orbits.

2.5 Global bifurcations leading to limit cycles

In addition to Hopf bifurcations, other events, called *global bifurcations*, can create or destroy limit cycles in two dimensions.

Two limit cycles can undergo a saddle-node bifurcation, analogous to that undergone by two fixed points. Ths system

$$\dot{r} = \alpha r(\mu - \mu_c - (r^2 - r_c^2)^2) \quad (84)$$
$$\dot{\theta} = \omega \quad (85)$$

has two limit cycles with radius near $r = r_c$ when $\mu > \mu_c$, both encircling a fixed point at $r = 0$. Their stability depends on the sign of α. This transition, called a *saddle-node bifurcation of limit cycles*, is illustrated in figure 16.

A usual saddle-node bifurcation of two fixed points can also lead to a limit cycle if the fixed points are *on* a closed trajectory. For this to occur, it is necessary to break the angular symmetry. The following system of equations and figure 17 depict an example of such a transition, called a SNIPER, for Saddle-Node In a PERiodic orbit, or Saddle-Node Infinite PERiod; other names are a saddle-node homoclinic, a SNIC (Saddle-Node on Invariant Circle), or an Andronov bifurcation.

$$\dot{r} = r(1 - r^2) \quad (86)$$
$$\dot{\theta} = \mu + 1 + \cos(\theta) \quad (87)$$

The limit cycle produced in this way will not be traversed uniformly, as shown in figure 19. There will

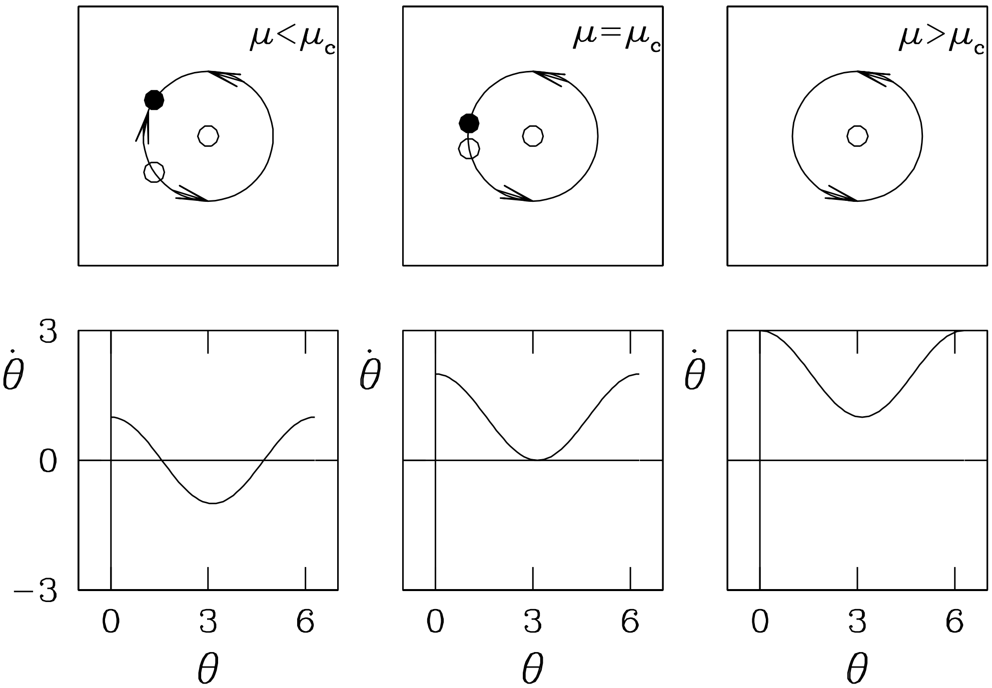


Figure 17: Saddle-node bifurcation *in* a periodic orbit.

be a slow phase while traversing the neighborhood of the former fixed points, sometimes called *ghosts of former fixed points*.

A third way of creating or destroying a limit cycle relies on the proximity of two fixed points, one saddle and one spiral point. At the bifurcation, one of the trajectories leaving the saddle circles the spiral point and returns to the saddle. This trajectory is called a *homoclinic cycle* and takes an infinite time to complete. This *homoclinic bifurcation* is illustrated in figure 18. Here too, the temporal signal behaves in a non-uniform fashion: the limit cycle spends a long time near the saddle. The system of equations used to generate figure 18 is:

$$\dot{x} = y \tag{88}$$

$$\dot{y} = -\mu - x + x^2 - xy \tag{89}$$

In an experimental situation, we probably do not know the underlying equations, but only the behavior of one or several sampled quantities. In a numerical simulation, we know the equations, but probably not their projection onto the bifurcating directions. It is therefore necessary to determine the origin of a limit cycle from its properties as the bifurcation is traversed. The table below displays these properties for the bifurcations which give rise to a limit cycle. The length of the slow phases diverges as the bifurcation is approached. The notation $O(1)$ means that the quantity remains finite as the bifurcation is traversed, approaching neither zero nor infinity.

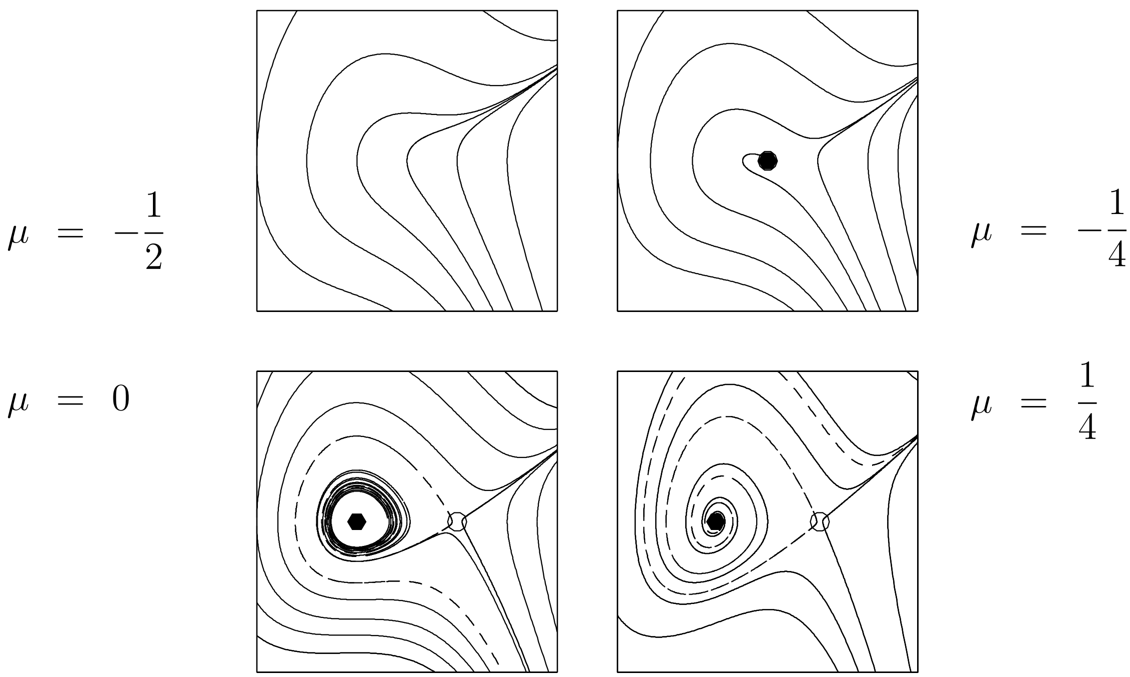


Figure 18: Homoclinic bifurcation. For $\mu < -1/4$, there are no fixed points. At $\mu = -1/4$, a saddle-node bifurcation gives rise to two steady states, a spiral node (solid dot) and a saddle (hollow dot). At $\mu = 0$, the spiral node undergoes a Hopf bifurcation, leading to the creation of a limit cycle, which is reached by all trajectories leaving the spiral node and some trajectories leaving the saddle. By $\mu = 1/4$, the limit cycle has been destroyed by colliding with the saddle.

	Amplitude	Period
Supercritical Hopf	$O(\mu^{\frac{1}{2}})$	$O(1)$
Saddle-node *of* periodic orbits	$O(1)$	$O(1)$
Saddle-node *in* periodic orbit	$O(1)$	$O(\mu^{-\frac{1}{2}})$
Homoclinic	$O(1)$	$O(\log(\mu))$

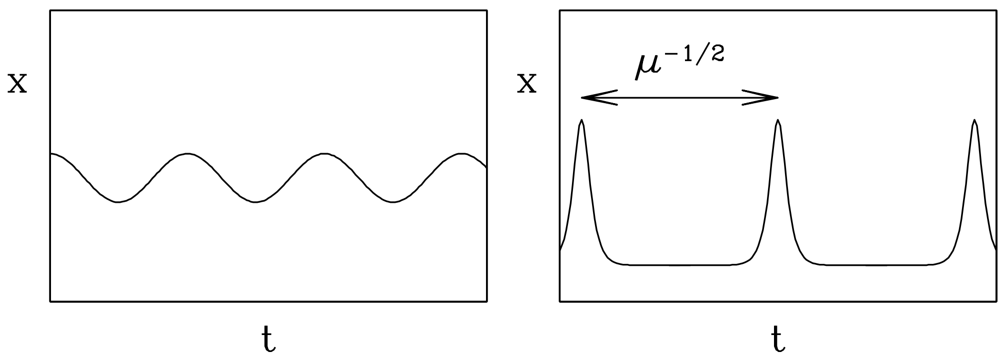


Figure 19: Timeseries during a limit cycle produced by a Hopf bifurcation (left) and a saddle-node bifurcation on a limit cycle (right), for parameter values near the bifurcation. The oscillation on the left is sinusoidal, whereas that on the right contains long phases during which the quantity plotted does not changed, separated by short phases of sudden change.

3 Web sites used in demonstrations

Lorenz model: http://to-campos.planetaclix.pt/fractal/lorenz_eng.html
Billiards: http://serendip.brynmawr.edu/chaos/javacode/stad/Stadium.html
Patterns: http://hopf.chem.brandeis.edu
Cylinder wake: http://personalpages.manchester.ac.uk/staff/david.d.apsley/images/cfd/one_cylinder.gif

4 Exercises

4.1 A one-dimensional dynamical system

Figure 20 is a bifurcation diagram showing the steady states $\bar{x}$ of a one-dimensional dynamical system:

$$\dot{x} = f(x) \tag{90}$$

The solution $\bar{x} = 0$ is stable where shown by the arrows. Complete the bifurcation diagram and the stability information, showing where the states are stable and unstable. Identify the bifurcations on the figure.

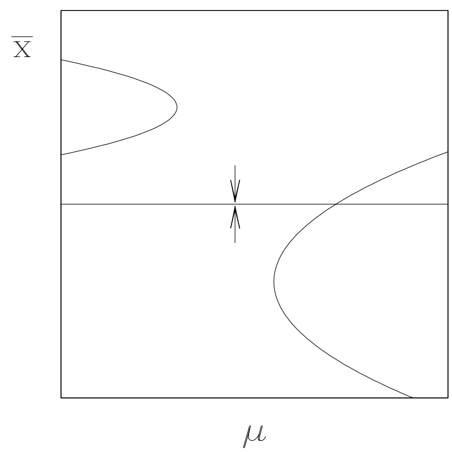


Figure 20: Bifurcation diagram for a 1D dynamical system $\dot{x} = f(x)$.

4.2 Lorenz Model

The Lorenz Model is given by the three equations:

$$\begin{aligned}\dot{x} &= 10(y - x) \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - \frac{8}{3}z\end{aligned}$$

Determine the fixed points and primary bifurcations. Then find a secondary Hopf bifurcation. (Bifurcations undergone by the trivial state are called primary bifurcations. These create new states which in turn undergo bifurcations called secondary bifurcations.) Steady bifurcations can be found by setting $\lambda = 0$, Hopf bifurcations by setting $\lambda = i\omega$.

4.3 Global bifurcations

Show that, near a saddle-node bifurcation, the time taken to traverse the neighborhood of the former fixed points ("ghosts") behaves like $\mu^{-1/2}$ by evaluating

$$\int \mathrm{dt} = \int_{-\infty}^{\infty} \frac{\mathrm{dx}}{\mu + x^2}$$