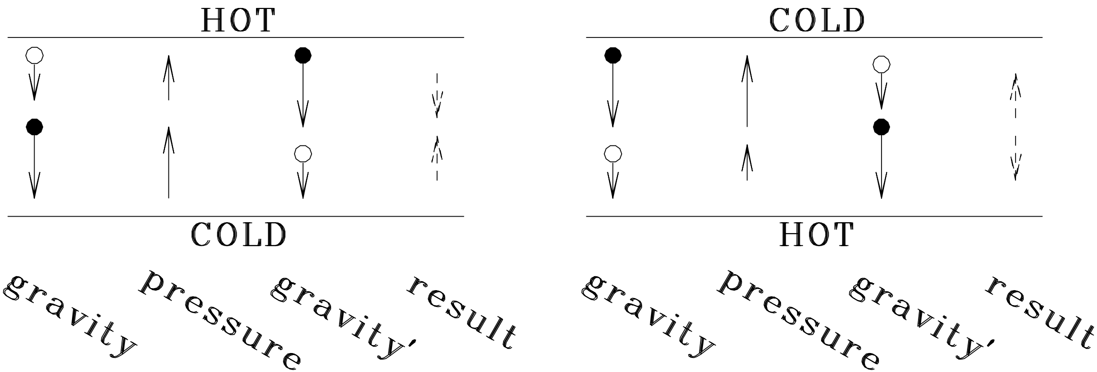
Laurette TUCKERMAN

laurette@pmmh.espci.fr

Rayleigh-Bénard Convection

and Lorenz Model

Rayleigh-Bénard Convection



Rayleigh-Bénard Convection

Boussinesq Approximation

Calculation and subtraction of the basic state

Non-dimensionalisation

Boundary Conditions

Linear stability analysis

Lorenz Model

Inclusion of nonlinear interactions

Seek bifurcations

Boussinesq Approximation

μ (viscosity $\sim$ diffusivity of momentum), κ (diffusivity of temperature),
ρ (density) constant except in buoyancy force.
Valid for $T_0 - T_1$ not too large.

$$\begin{aligned} \rho(T) &= \rho_0\left[1 - \alpha(T - T_0)\right] \\ \nabla \cdot \mathrm{U} &= 0 \end{aligned}$$

Governing equations:

$$\begin{aligned} \rho_0\left[\partial_t + (\mathrm{U}\cdot\nabla)\right]\mathrm{U} &= \mu\Delta\mathrm{U} - \nabla P - g\rho(T)\mathrm{e_z} \\ \left[\partial_t + (\mathrm{U}\cdot\nabla)\right]T &= \kappa\Delta T \end{aligned}$$

$\uparrow$ **advection** $\uparrow$ **diffusion** $\uparrow$ **buoyancy**

Boundary conditions:

$$\begin{aligned} \mathrm{U} &= 0 \qquad \textbf{at} \quad z = 0, d \\ T &= T_{0,1} \qquad \textbf{at} \quad z = 0, d \end{aligned}$$

Calculation and subtraction of base state

Conductive solution: (U^*, T^*, P^*)

Motionless: $U^* = 0$

uniform temperature gradient: $T^* = T_0 - (T_0 - T_1)\dfrac{z}{d}$

density: $\rho(T^*) = \rho_0 \left[1 + \alpha(T_0 - T_1)\dfrac{z}{d}\right]$

Hydrostatic pressure counterbalances buoyancy force:

$$\begin{aligned} P^* &= -g \int dz\ \rho(T^*) \\ &= P_0 - g\rho_0 \left[z + \alpha(T_0 - T_1)\frac{z^2}{2d}\right] \end{aligned}$$

Write:

$$T = T^* + \hat{T} \qquad P = P^* + \hat{P}$$

Buoyancy:

$$\begin{aligned}
\rho(T^* + \hat{T}) &= \rho_0(1 - \alpha(T^* + \hat{T} - T_0)) \\
&= \rho_0(1 - \alpha(T^* - T_0)) - \rho_0\alpha\hat{T} \\
&= \rho(T^*) - \rho_0\alpha\hat{T} \\
-\nabla P - g\rho(T)\mathrm{e_z} &= -\nabla P^* - g\rho(T^*) - \nabla\hat{P} + g\rho_0\alpha\hat{T}\mathrm{e_z} \\
&= -\nabla\hat{P} + g\rho_0\alpha\hat{T}\mathrm{e_z}
\end{aligned}$$

Advection of temperature:

$$\begin{aligned}
(U \cdot \nabla)T &= (U \cdot \nabla)T^* + (U \cdot \nabla)\hat{T} \\
&= (U \cdot \nabla)\left(T_0 - (T_0 - T_1)\frac{z}{d}\right) + (U \cdot \nabla)\hat{T} \\
&= -\frac{T_0 - T_1}{d} U \cdot \mathrm{e_z} + (U \cdot \nabla)\hat{T}
\end{aligned}$$

Governing equations:

$$
\begin{aligned}
\rho_0 \left[\partial_t + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\right]\boldsymbol{U} &= -\boldsymbol{\nabla}\hat{P} + g\rho_0\alpha\hat{T}\mathrm{e}_z + \mu\Delta\boldsymbol{U} \\
\boldsymbol{\nabla}\cdot\boldsymbol{U} &= 0 \\
\left[\partial_t + (\boldsymbol{U}\cdot\boldsymbol{\nabla})\right]\hat{T} &= \frac{T_0 - T_1}{d}\boldsymbol{U}\cdot\mathrm{e}_z + \kappa\Delta\hat{T}
\end{aligned}
$$

Homogeneous boundary conditions:

$$
\begin{aligned}
\boldsymbol{U} &= 0 \ \ \textbf{at} \ \ z = 0, d \\
\hat{T} &= 0 \ \ \textbf{at} \ \ z = 0, d
\end{aligned}
$$

Non-dimensionalization

Scales:

$$z = d\bar{z}, \quad t = \frac{d^2}{\kappa}\bar{t}, \quad U = \frac{\kappa}{d}\bar{U}, \quad \hat{T} = \frac{\mu\kappa}{d^3 g \rho_0 \alpha}\bar{T}, \quad \hat{P} = \frac{\mu\kappa}{d^2}\bar{P}$$

Equations :

$$\begin{aligned}
\frac{\kappa^2\rho_0}{d^3}\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\nabla})\right]\bar{U} &= -\frac{\mu\kappa}{d^3}\bar{\nabla}\bar{P} + \frac{\mu\kappa}{d^3}\bar{T}\mathrm{e_z} + \frac{\mu\kappa}{d^3}\bar{\Delta}\bar{U} \\
\frac{\kappa}{d^2}\bar{\nabla}\cdot\bar{U} &= 0 \\
\frac{\mu\kappa^2}{d^5 g\rho_0\alpha}\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\nabla})\right]\bar{T} &= \frac{\kappa}{d}\frac{T_0 - T_1}{d}\bar{U}\cdot\mathrm{e_z} + \frac{\mu\kappa^2}{d^5 g\rho_0\alpha}\bar{\Delta}\bar{T}
\end{aligned}$$

Dividing through, we obtain:

$$\begin{aligned}
\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\nabla})\right]\bar{U} &= \frac{\mu}{\rho_0\kappa}\left[-\bar{\nabla}\bar{P} + \bar{T}\mathrm{e_z} + \bar{\Delta}\bar{U}\right] \\
\left[\partial_{\bar{t}} + (\bar{U}\cdot\bar{\nabla})\right]\bar{T} &= \frac{(T_0 - T_1)d^3 g\rho_0\alpha}{\kappa\mu}\bar{U}\cdot\mathrm{e_z} + \bar{\Delta}\bar{T}
\end{aligned}$$

Non-dimensional parameters:

the Prandtl number: $$Pr \equiv \frac{\mu}{\rho_0 \kappa}$$

momentum diffusivity / thermal diffusivity

the Rayleigh number: $$Ra \equiv \frac{(T_0 - T_1) d^3 g \rho_0 \alpha}{\kappa \mu}$$

non-dimensional measure of thermal gradient

Boundary conditions

Horizontal direction: periodicity $2\pi/q$

Vertical direction: at $z = 0, 1$

$$T = 0|_{z=0,1} \quad \text{perfectly conducting plates}$$
$$w = 0|_{z=0,1} \quad \text{impenetrable plates}$$

Rigid boundaries at $z = 0, 1$:

$$u|_{z=0,1} = v|_{z=0,1} = 0 \quad \text{zero tangential velocity}$$

Incompressibility

$$\partial_x u + \partial_y v + \partial_z w = 0$$
$$\Longrightarrow \partial_z w = -(\partial_x u + \partial_y v)$$

$$u|_{z=0,1} = v|_{z=0,1} = 0 \Longrightarrow \quad \partial_x u|_{z=0,1} = \partial_y v|_{z=0,1} = 0$$
$$\Longrightarrow \quad \partial_z w|_{z=0,1} = 0$$

Free surfaces at $z = 0, 1$ to simplify calculations:

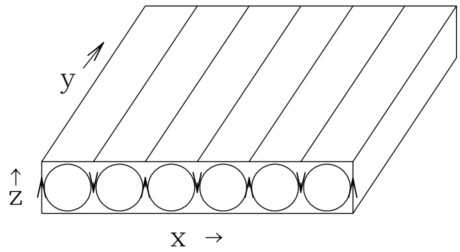
$$[\partial_z u + \partial_x w]_{z=0,1} \quad = [\partial_z v + \partial_y w]_{z=0,1} = 0$$

zero tangential stress

$$\begin{aligned} w|_{z=0,1} = 0 \quad & \Longrightarrow \partial_x w|_{z=0,1} = \partial_y w|_{z=0,1} = 0 \\ & \Longrightarrow \partial_z u|_{z=0,1} = \partial_z v|_{z=0,1} = 0 \\ & \Longrightarrow \partial_x \partial_z u|_{z=0,1} = \partial_y \partial_z v|_{z=0,1} = 0 \\ & \Longrightarrow \partial_{zz} w|_{z=0,1} = -\,\partial_z(\partial_x u + \partial_y v)|_{z=0,1} = 0 \end{aligned}$$

Not realistic, but allows trigonometric functions $\sin(k\pi z)$

Two-dimensional case



$$U = \nabla \times \psi \, \mathrm{e_y} \Longrightarrow \left\{ \begin{array}{c} u = -\partial_z \psi \\ w = \partial_x \psi \end{array} \right\} \Longrightarrow \nabla \cdot U = 0$$

No-penetration boundary condition:

$$0 = w = \partial_x \psi \Longrightarrow \begin{cases} \psi = \psi_1 & \text{at } z = 1 \\ \psi = \psi_0 & \text{at } z = 0 \end{cases}$$

Horizontal flux:

$$\int_{z=0}^{1} dz \; u(x,z) = -\int_{z=0}^{1} dz \; \partial_z \psi(x,z) = -\left.\psi(x,z)\right]_{z=0}^{1} = \psi_0 - \psi_1$$

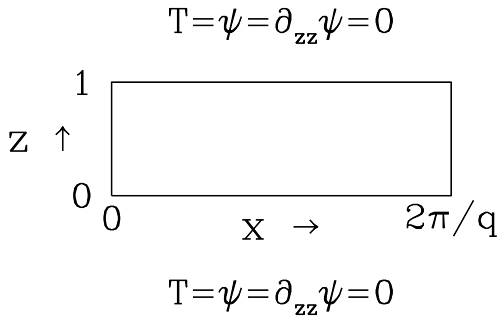
Arbitrary constant $\Longrightarrow \psi_0 = 0$

Zero flux $\Longrightarrow \psi_1 = 0$

Stress-free: $0 = \partial_z u = -\partial_{zz}^2 \psi$

Rigid: $0 = u = \partial_z \psi \quad$ **at** $z = 0, 1$

Two-dimensional case



Temperature equation:

$$\partial_t T + \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{T} = \boldsymbol{Ra}\boldsymbol{U} \cdot \mathrm{e_z} + \boldsymbol{\Delta} \boldsymbol{T}$$

$$\begin{aligned} \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{T} &= u\, \partial_x T + w\, \partial_z T \\ &= -\partial_z \psi\, \partial_x T + \partial_x \psi\, \partial_z T \equiv \boldsymbol{J}[\psi, \boldsymbol{T}] \end{aligned}$$

$$\partial_t T + \boldsymbol{J}[\psi, \boldsymbol{T}] = \boldsymbol{Ra}\, \partial_x \psi + \boldsymbol{\Delta} \boldsymbol{T}$$

Velocity equation

$$\partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \boldsymbol{\nabla})\boldsymbol{U} = Pr\left[-\boldsymbol{\nabla} P + T\mathbf{e_z} + \boldsymbol{\Delta U}\right]$$

Take $\mathbf{e_y} \cdot \boldsymbol{\nabla} \times$**:**

$$\begin{aligned}
\mathbf{e_y} \cdot \boldsymbol{\nabla} \times \partial_t \boldsymbol{U} &= \mathbf{e_y} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \partial_t \psi \mathbf{e_y} = -\partial_t \boldsymbol{\Delta} \psi \\
\mathbf{e_y} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} P &= 0 \\
\mathbf{e_y} \cdot \boldsymbol{\nabla} \times T\mathbf{e_z} &= -\partial_x T \\
\mathbf{e_y} \cdot \boldsymbol{\nabla} \times \boldsymbol{\Delta U} &= \mathbf{e_y} \cdot \boldsymbol{\nabla} \times \boldsymbol{\Delta \nabla} \times \psi \mathbf{e_y} = -\boldsymbol{\Delta}^2 \psi
\end{aligned}$$

$$\partial_t \boldsymbol{\Delta} \psi - \mathbf{e_y} \cdot \boldsymbol{\nabla} \times (\boldsymbol{U} \cdot \boldsymbol{\nabla})\boldsymbol{U} = Pr[\partial_x T + \boldsymbol{\Delta}^2 \psi]$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{\nabla \nabla} \cdot \boldsymbol{f} - \boldsymbol{\Delta f}$$

$$\mathbf{e_y} \cdot \boldsymbol{\nabla} \times (\boldsymbol{U} \cdot \boldsymbol{\nabla})\boldsymbol{U} \;=\; \partial_z(\boldsymbol{U} \cdot \boldsymbol{\nabla})u - \partial_x(\boldsymbol{U} \cdot \boldsymbol{\nabla})w$$

$$\begin{aligned}
&= \partial_z(u\partial_x u + w\partial_z u) - \partial_x(u\partial_x w + w\partial_z w) \\
&= \partial_z u\ \partial_x u + \partial_z w\ \partial_z u - \partial_x u\ \partial_x w - \partial_x w\ \partial_z w \\
&+\ u\ \partial_{xz} u + w\ \partial_{zz} u - u\ \partial_{xx} w - w\ \partial_{xz} w \\
&= \partial_z u\ (\partial_x u + \partial_z w) - \partial_x w\ (\partial_x u + \partial_z w) \\
&+\ u\ \partial_x(\partial_z u - \partial_x w) + w\partial_z(\ \partial_z u - \partial_x w) \\
&= (-\partial_z \psi)\partial_x(-\partial_{zz}\psi - \partial_{xx}\psi) \\
&+\ (\partial_x \psi)\partial_z(-\partial_{zz}\psi - \partial_{xx}\psi) \\
&= (\partial_z \psi)\partial_x(\Delta\psi) - (\partial_x \psi)\partial_z(\Delta\psi)
\end{aligned}$$

$$= -J[\psi, \boldsymbol{\Delta}\psi]$$

$$\partial_t \boldsymbol{\Delta}\psi + J[\psi, \boldsymbol{\Delta}\psi] = Pr[\partial_x T + \boldsymbol{\Delta}^2\psi]$$

Linear stability analysis

Linearized equations:

$$\begin{aligned}\partial_t \Delta\psi &= Pr[\partial_x T + \Delta^2\psi] \\ \partial_t T &= Ra\, \partial_x \psi + \Delta T\end{aligned}$$

Solutions:

$$\begin{aligned}\psi(x,z,t) &= \hat{\psi}\ \sin qx\ \sin k\pi z\ e^{\lambda t} \qquad q \in \mathcal{R},\ k \in \mathcal{Z}^+,\ \lambda \in \mathcal{C} \\ T(x,z,t) &= \hat{T}\ \cos qx\ \sin k\pi z\ e^{\lambda t}\end{aligned}$$

$\uparrow$ functions $\qquad$ $\uparrow$ scalars

$$\gamma^2 \equiv q^2 + (k\pi)^2$$

$$\begin{aligned}-\lambda\gamma^2\hat{\psi} &= Pr[-q\hat{T} + \gamma^4\hat{\psi}] \\ \lambda\hat{T} &= Ra\ q\ \hat{\psi} - \gamma^2\hat{T}\end{aligned}$$

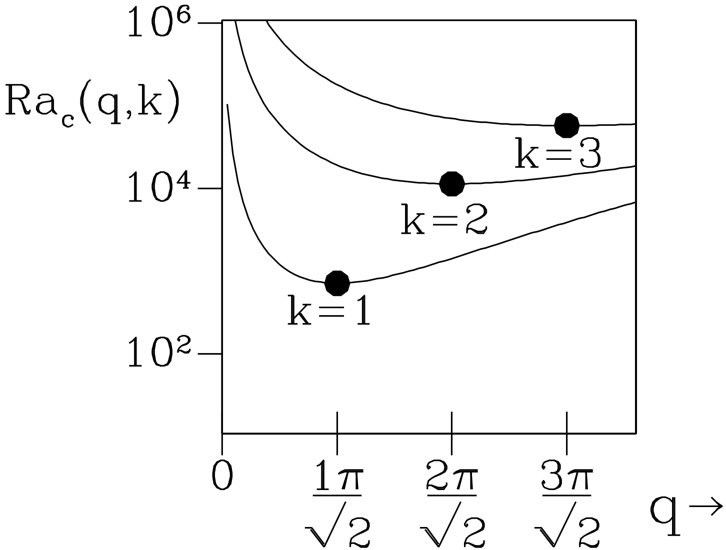
$$\lambda \begin{bmatrix} \hat{\psi} \\ \hat{T} \end{bmatrix} = \begin{bmatrix} -Pr\,\gamma^2 & Pr\,q/\gamma^2 \\ Ra\,q & -\gamma^2 \end{bmatrix} \begin{bmatrix} \hat{\psi} \\ \hat{T} \end{bmatrix}$$

Steady Bifurcation: $\lambda = 0$

$$Pr\,\gamma^4 - Pr\,Ra\,\frac{q^2}{\gamma^2} = 0$$

$$Ra = \frac{\gamma^6}{q^2} = \frac{(q^2 + (k\pi)^2)^3}{q^2} \equiv Ra_c(q, k)$$

Convection Threshold



Conductive state unstable at (q, k) for $Ra > Ra_c(q, k)$

Conductive state stable if

$$Ra < \min_{\substack{q \in \mathcal{R} \\ k \in \mathcal{Z}^+}} Ra_c(q,k)$$

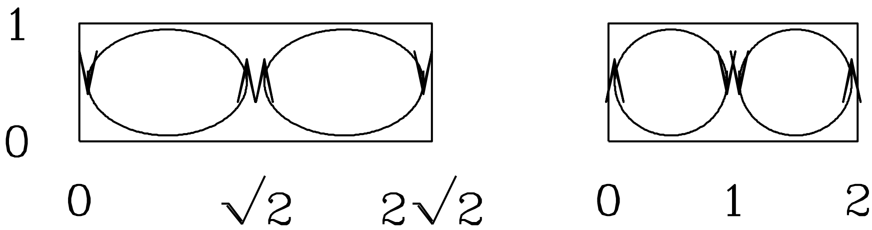
$$\begin{aligned}
0 &= \frac{\partial Ra_c(q,k)}{\partial q} = \frac{q^2 3(q^2+(k\pi)^2)^2 2q - 2q(q^2+(k\pi)^2)^3}{q^4} \\
&= \frac{2(q^2+(k\pi)^2)^2}{q^3}(3q^2-(q^2+(k\pi)^2)
\end{aligned}$$

$$\Longrightarrow \ q^2 = \frac{(k\pi)^2}{2}$$

$$Ra_c\left(q=\frac{k\pi}{\sqrt{2}}, k\right) = \frac{(k\pi)^2/2+(k\pi)^2)^3}{(k\pi)^2/2} = \frac{27}{4}(k\pi)^4$$

$$Ra_c \equiv Ra_c\left(q=\frac{\pi}{\sqrt{2}}, k=1\right) = \frac{27}{4}(\pi)^4 = 657.5$$

Rigid Boundaries



Calculation follows the same principle, but more complicated.

Boundaries damp perturbations $\Longrightarrow$ **higher threshold**

$q_c \downarrow \Longrightarrow \ell_c = \pi / q_c \uparrow \Longrightarrow$ **rolls** $\approx$ **circular**

	Ra_c	q_c	ℓ_c
stress-free boundaries	$\frac{27}{4}\pi^4 = 657.5$	$\frac{\pi}{\sqrt{2}}$	1.4
rigid boundaries	≈ 1700	$\approx \pi$	≈ 1

Lorenz Model:
including nonlinear interactions

$$
\begin{aligned}
J\,[\psi, \Delta\psi] &= J[\psi, -\gamma^2\psi] \\
&= \partial_x\psi\;\partial_z(-\gamma^2\psi) - \partial_x(-\gamma^2\psi)\partial_z\psi = 0
\end{aligned}
$$

$$
\begin{aligned}
J[\psi, T] &= \hat{\psi}\hat{T}\;[\partial_x(\sin qx \sin \pi z)\partial_z(\cos qx \sin \pi z) \\
&\quad -\partial_x(\cos qx\;\; \sin \pi z)\partial_z(\sin qx\;\; \sin \pi z)] \\
&= \hat{\psi}\hat{T}\;q\pi\;[\cos qx\;\; \sin \pi z\;\; \cos qx\;\; \cos \pi z \\
&\quad + \sin qx\;\; \sin \pi z\;\; \sin qx\;\; \cos \pi z] \\
&\quad +\hat{\psi}\hat{T}\;q\pi\;\;(\cos^2 qx + \sin^2 qx)\;\; \sin \pi z \cos \pi z \\
&= \hat{\psi}\hat{T}\;\frac{q\pi}{2}\;\sin 2\pi z
\end{aligned}
$$

$\uparrow\quad\uparrow\qquad\qquad\uparrow\uparrow$

functions **scalars**

$$
\begin{aligned}
\psi(x,z,t) &= \hat{\psi}(t)\sin qx\ \sin\pi z \\
T(x,z,t) &= \hat{T}_1(t)\cos qx\ \sin\pi z + \hat{T}_2(t)\sin 2\pi z
\end{aligned}
$$

$$
\begin{aligned}
J[\psi, T_2] &= \hat{\psi}\hat{T}_2\ [\partial_x(\sin qx \sin\pi z)\partial_z(\sin 2\pi z) \\
&\quad -\partial_x(\sin 2\pi z)\partial_z(\sin qx\ \sin\pi z)] \\
&= \hat{\psi}\hat{T}_2\ q\ 2\pi\ \cos qx\ \sin\pi z\ \cos 2\pi z \\
&= \hat{\psi}\hat{T}_2\ q\ \pi\cos qx\ (\sin\pi z + \sin 3\pi z)
\end{aligned}
$$

Including $\hat{T}_3(t)\cos qx \sin 3\pi z \Longrightarrow$ new terms $\Longrightarrow$
Closure problem for nonlinear equations

Lorenz (1963) proposed stopping at T_2.

Lorenz Model

$$\begin{aligned}
\partial_t \hat{\psi} &= Pr(q\hat{T}_1/\gamma^2 - \gamma^2\hat{\psi}) && \sin qx \sin \pi z \\
\partial_t \hat{T}_1 + q\pi\hat{\psi}\hat{T}_2 &= Ra\ q\ \hat{\psi} - \gamma^2\hat{T}_1 && \cos qx \sin \pi z \\
\partial_t \hat{T}_2 + \frac{q\pi}{2}\hat{\psi}\hat{T}_1 &= -(2\pi)^2\hat{T}_2 && \sin 2\pi z
\end{aligned}$$

Define:

$$X \equiv \frac{\pi q}{\sqrt{2}\gamma^2}\hat{\psi}, \quad Y \equiv \frac{\pi q^2}{\sqrt{2}\gamma^6}\hat{T}_1, \quad Z \equiv \frac{\pi q^2}{\sqrt{2}\gamma^6}\hat{T}_2,$$

$$\tau \equiv \gamma^2 t, \quad r \equiv \frac{q^2}{\gamma^6}Ra, \quad b \equiv \frac{4\pi^2}{\gamma^2} = \frac{8}{3}, \quad \sigma \equiv Pr$$

Famous Lorenz Model:

$$\begin{aligned}
\dot{X} &= \sigma(Y - X) \\
\dot{Y} &= -XZ + rX - Y \\
\dot{Z} &= XY - bZ
\end{aligned}$$

$\sigma = Pr$ **(often set to 10, its value for water)**

$r = Ra/Ra_c$

Damping $\Longrightarrow -\sigma X, -Y, -bZ$

Advection $\Longrightarrow XZ, XY$

Symmetry between (X, Y, Z) **and** $(-X, -Y, Z)$

Lorenz Model

Pitchfork Bifurcation

Steady states:

$$0 = \sigma(Y - X) \implies X = Y$$

$$0 = -XZ + rX - Y \implies X = 0 \text{ or } Z = r - 1$$

$$0 = XY - bZ \implies Z = 0 \text{ or } X = Y = \pm\sqrt{b(r-1)}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{pmatrix}$$

Jacobian:

$$Df = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z & -1 & -X \\ Y & X & -b \end{pmatrix}$$

For $(X, Y, Z) = (0, 0, 0)$:

$$Df(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

Eigenvalues:

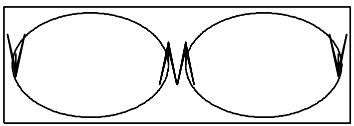
$$\begin{aligned} \lambda_1 + \lambda_2 &= Tr = -\sigma - 1 < 0 \\ \lambda_1 \lambda_2 &= Det = \sigma(1 - r) \\ \lambda_3 &= -b < 0 \end{aligned}$$

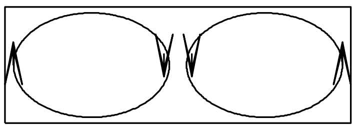
$$0 < r < 1 \Longrightarrow \boldsymbol{\lambda}_{1,2,3} < 0 \Longrightarrow \textbf{stable node}$$

$$r > 1 \Longrightarrow \boldsymbol{\lambda}_{1,3} < 0, \boldsymbol{\lambda}_2 > 0 \Longrightarrow \textbf{saddle}$$

Pitchfork bifurcation at $r = 1$ creates

$$X = Y = \pm\sqrt{b(r-1)}, Z = r - 1$$





Lorenz Model: Hopf Bifurcation

For $X = Y = \pm\sqrt{b(r-1)}, Z = r - 1,$

$$Df = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

Eigenvalues:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0$$

Hopf bifurcation $\lambda = i\omega$ **:**

$$-i\omega^3 - (\sigma + b + 1)\omega^2 + i(r + \sigma)b\omega + 2b\sigma(r-1) = 0$$

$$
\begin{aligned}
-(\sigma+b+1)\omega^2+2b\sigma(r-1) &= 0\\
-\omega^3+(r+\sigma)b\omega &= 0
\end{aligned}
$$

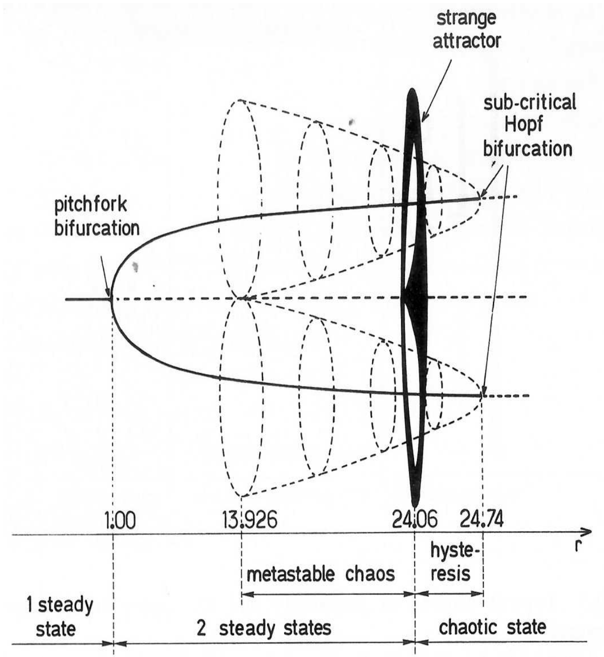
$$
\begin{aligned}
\frac{2b\sigma(r-1)}{\sigma+b+1} &= \omega^2=(r+\sigma)b\\
2b\sigma(r-1) &= (r+\sigma)b(\sigma+b+1)\\
2b\sigma r-2b\sigma &= rb(\sigma+b+1)+\sigma b(\sigma+b+1)
\end{aligned}
$$

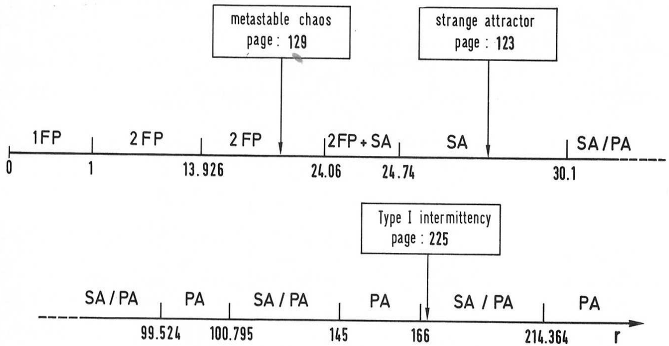
$$
r=\frac{\sigma(\sigma+b+3)}{\sigma-b-1}=24.74 \ \text{ for } \ \sigma=10,\ b=8/3
$$

At $r = 24.74$, the two steady states undergo a Hopf bifurcation (shown to be subcritical)

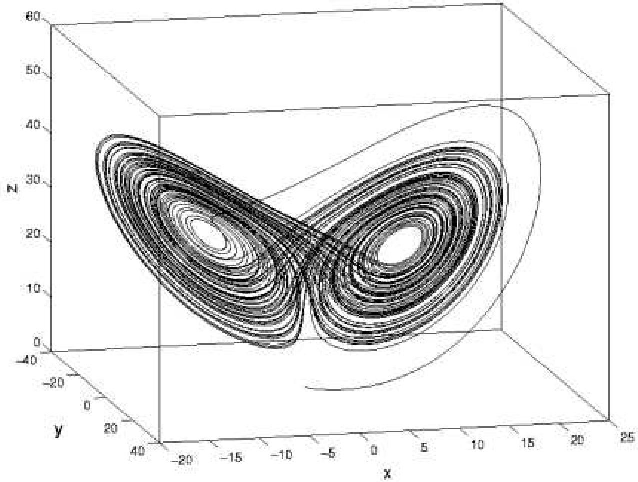
$\Longrightarrow$ **unstable limit cycles exist for $r < 24.74$**

Lorenz Model: Bifurcation Diagram

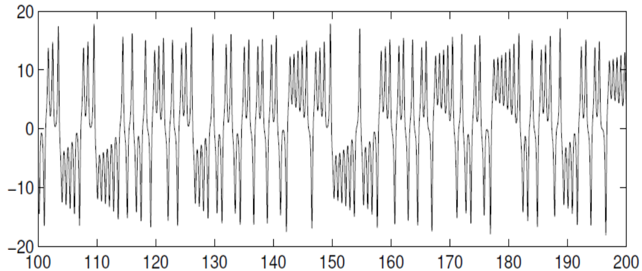




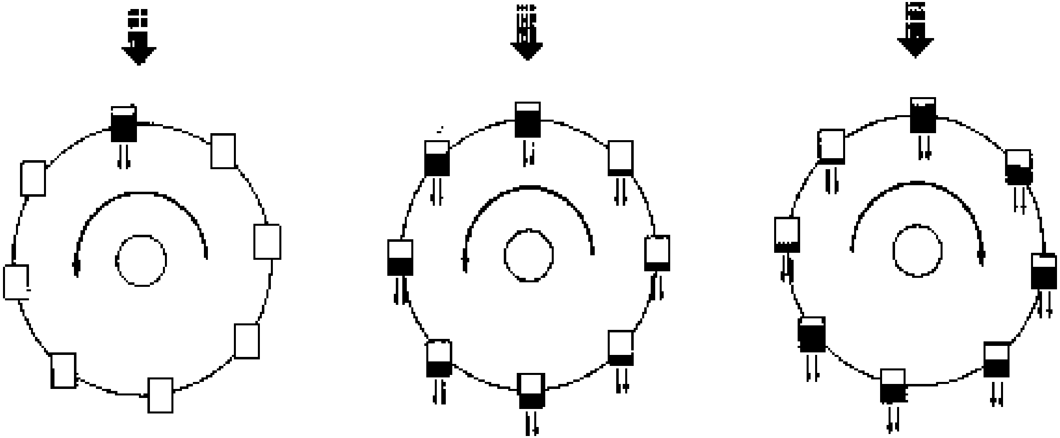
Lorenz Model: Strange Attractor for $r = 28$



Lorenz Model: Time Series for $r = 28$

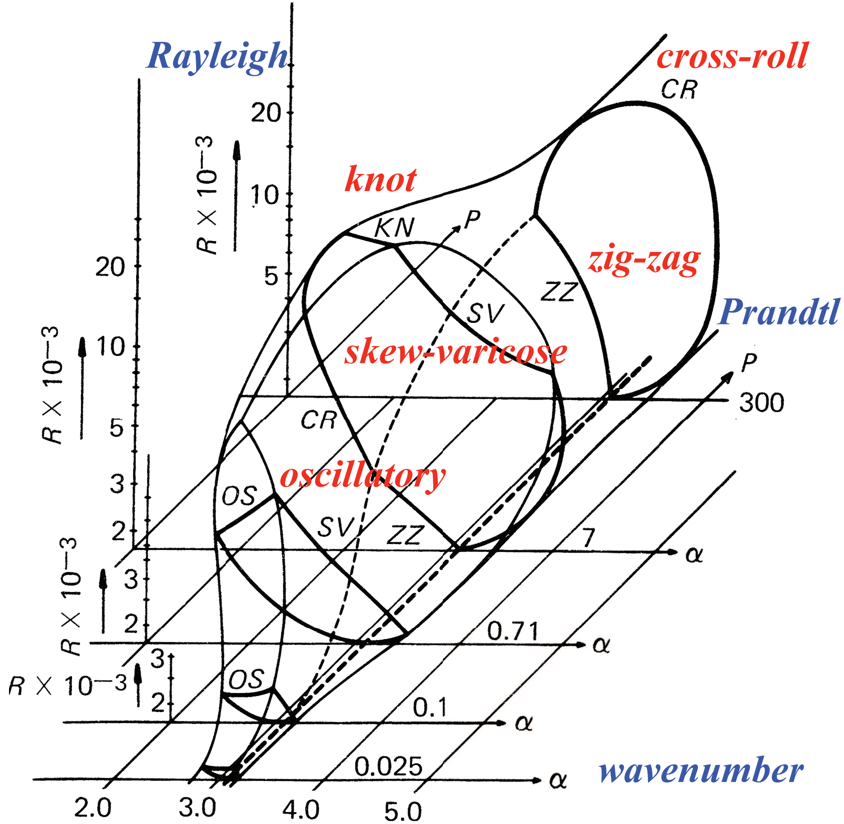


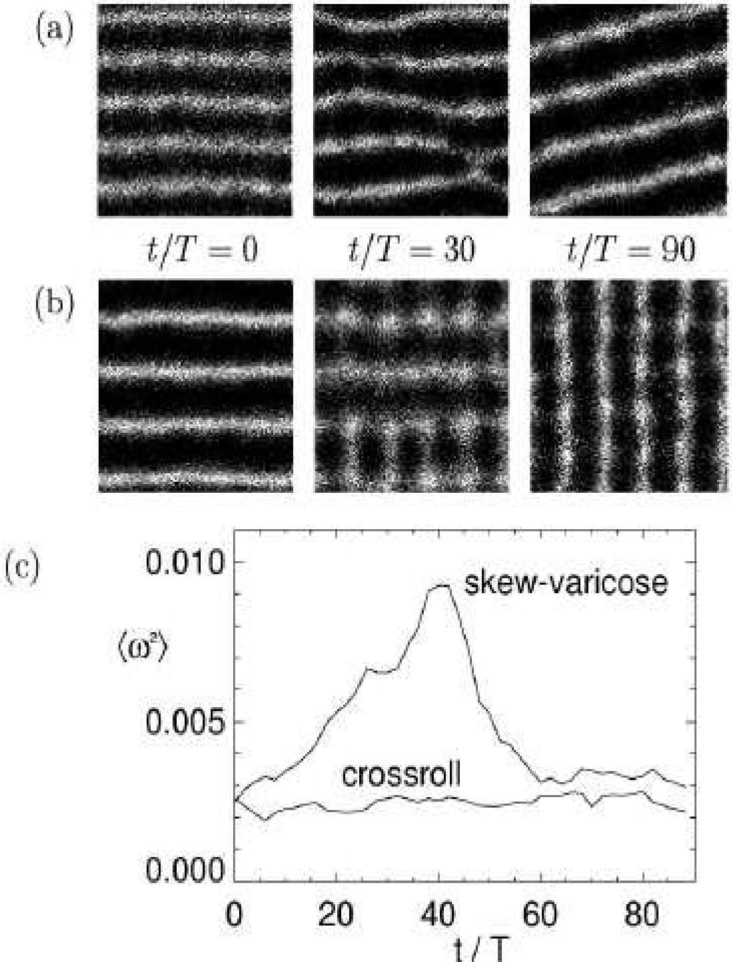
Water wheel (Malkus)



motion described by Lorenz model

Instabilities of straight rolls: “Busse balloon”





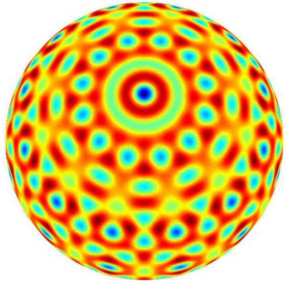
skew-varicose instability

cross-roll instability

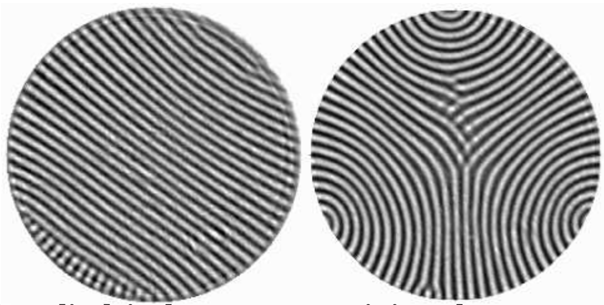
Continuum-type stability balloon in oscillated granulated layers, J. de Bruyn, C. Bizon, M.D. Shattuck, D. Goldman, J.B. Swift & H.L. Swinney, Phys. Rev. Lett. 1998.

Complex spatial patterns in convection

Experimental spiral defect chaos
Egolf, Melnikov, Pesche, Ecke
Nature 404 (2000)

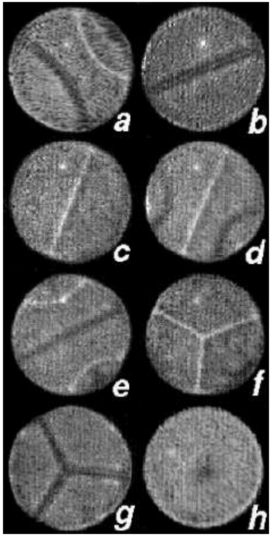


Spherical harmonic $\ell = 28$
P. Matthews
Phys. Rev. E. 67 (2003)

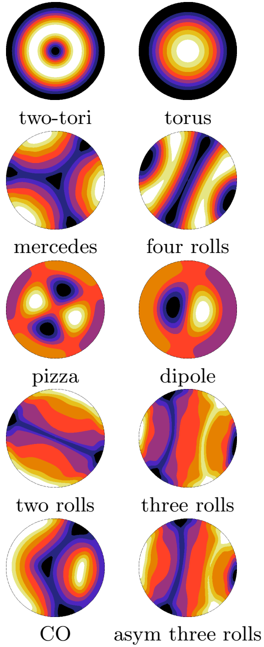


Convection in cylindrical geometry. Bajaj et al. J. Stat. Mech. (2006)

Small containers: multiplicity of states
cylindrical container with R = 2H

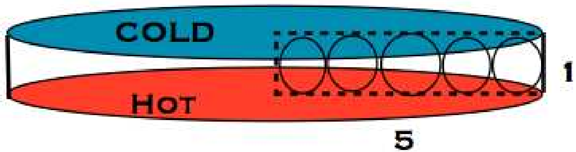


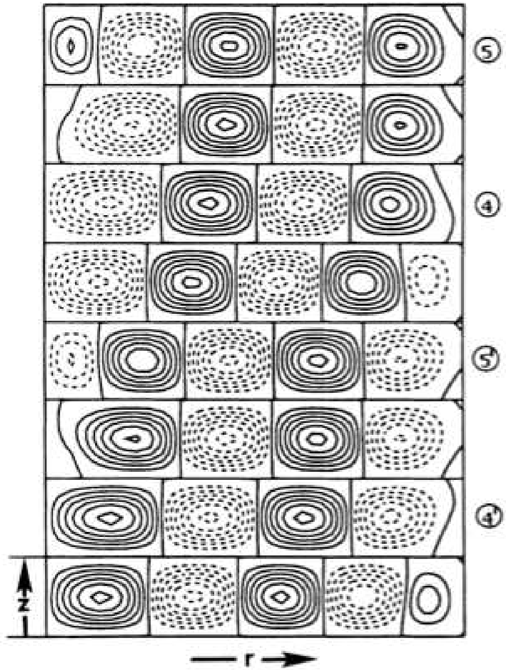
**experimental photographs by
Hof, Lucas, Mullin,
Phys. Fluids (1999)**



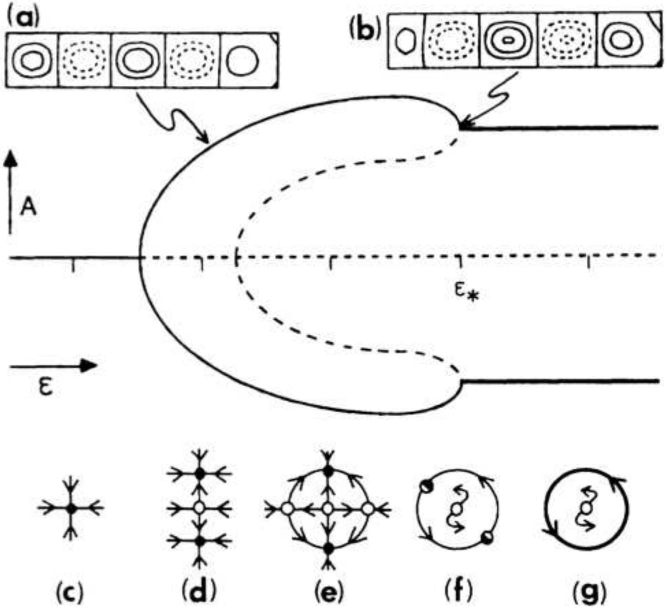
**numerical simulations by
Borońska, Tuckerman,
Phys. Rev. E (2010)**

Small containers: a SNIPER bifurcation in a cylindrical container with R = 5H



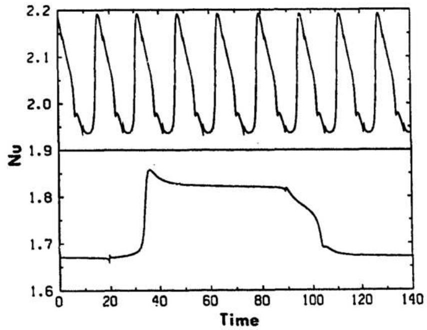


Pattern of five toroidal convection cells moves radially inwards in time. From Tuckerman, Barkley, Phys. Rev. Lett. (1988).



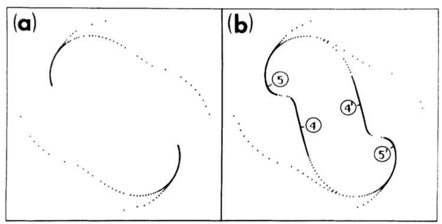
pitchfork pitchfork   saddle-nodes

Timeseries



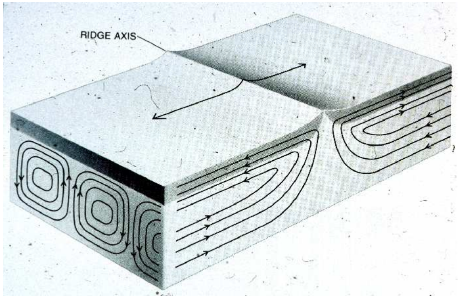
fast away from SNIPER
slow near SNIPER

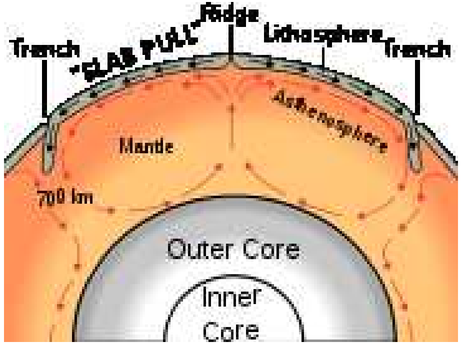
Phase portraits



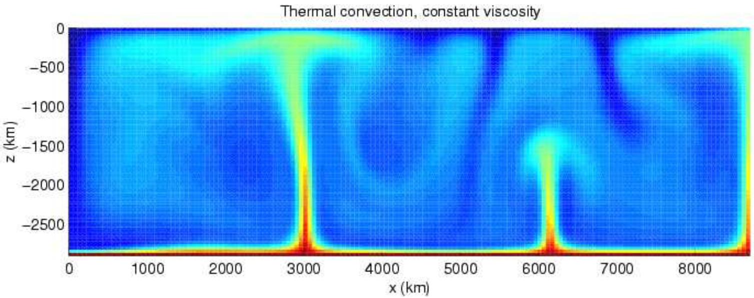
before   SNIPER   after

Geophysics





Convection and plate tectonics



Numerical simulation of convection in earth's mantle, showing plumes and thin boundary layers. By H. Schmeling, Wikimedia Commons.