Magnetohydrodynamics in a finite cylinder

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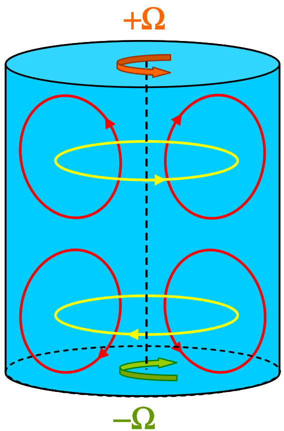


Axisymmetric flow in cylinder with counter-rotating disks

poloidal (ϕ) flow lines, toroidal (ψ) flow lines

Governing Equations

$$\left(\partial_t - \frac{1}{Re}\Delta\right)\mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} - (\mathrm{B}\cdot\nabla)\mathrm{B} = -\nabla\left(p + \frac{B^2}{2}\right)$$

$$\nabla\cdot\mathrm{u} = 0$$

$$\left(\partial_t - \frac{1}{Re_m}\Delta\right)\mathrm{B} - \nabla\times(\mathrm{u}\times\mathrm{B}) = 0$$

$$\nabla\cdot\mathrm{B} = 0$$

$(\mathrm{u}\cdot\nabla)\mathrm{u} - (\mathrm{B}\cdot\nabla)\mathrm{B} \equiv \mathrm{s}_u$ **<u>is not</u> divergence-free**

$p + \frac{B^2}{2}$ **such as to remove divergence**

$-\nabla\times(\mathrm{u}\times\mathrm{B}) \equiv \mathrm{s}_B$ **<u>is</u> divergence-free**

Poloidal-Toroidal Decomposition

divergence-free by construction

$$\mathbf{F} = \boldsymbol{\nabla} \times (\psi \hat{\mathrm{e}}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\phi \hat{\mathrm{e}})$$

($\hat{\mathrm{e}}$ = unit vector)

If $\hat{\mathrm{e}} = \hat{\mathrm{e}}_z$ or $\hat{\mathrm{e}} = \hat{\mathrm{e}}_\rho$ (ρ = spherical radius):

$$\begin{aligned} \hat{\mathrm{e}} \cdot \mathbf{F} &= -\boldsymbol{\Delta}_h \phi \\ \hat{\mathrm{e}} \cdot \boldsymbol{\nabla} \times \mathbf{F} &= -\boldsymbol{\Delta}_h \psi \\ \hat{\mathrm{e}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{F} &= \boldsymbol{\Delta} \boldsymbol{\Delta}_h \phi \end{aligned}$$

$\boldsymbol{\Delta}_h$ = horizontal Laplacian ($\perp$ to $\hat{\mathrm{e}}$)

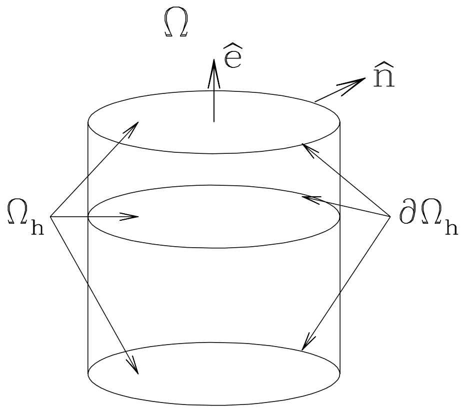
Equivalence ?

Marques (Phys. Fluids 1990; Comp. Meth. App. Mech. Eng. 1993)

$$\mathbf{g} = 0 \quad \text{in } \Omega \iff \begin{cases} \hat{\mathbf{e}} \cdot \mathbf{g} = 0 & \text{in } \Omega \\ \hat{\mathbf{e}} \cdot \nabla \times \mathbf{g} = 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{g} = 0 & \text{in } \Omega \\ \hat{\mathbf{n}} \cdot \mathbf{g} = 0 & \text{on } \partial\Omega_h \end{cases}$$

- $\hat{\mathbf{e}} \cdot \mathbf{g} = 0$ in Ω: $\hat{\mathrm{e}}$ component
- $\hat{\mathbf{e}} \cdot \nabla \times \mathbf{g} = 0$ in Ω: $\hat{\mathrm{e}}$ comp of curl
- $\nabla \cdot \mathbf{g} = 0$ in Ω: div-free
- $\hat{\mathbf{n}} \cdot \mathbf{g} = 0$ on $\partial\Omega_h$: compatibility

(like constant of integration, not relevant for spheres or infinite planes)



Holds for simply-connected domain. For a spherical geometry, Ω_h is the sphere and has no boundary $\partial\Omega_h$. For a torus (cylinder of infinite/periodic length) or a two-torus (doubly infinite/periodic planar layer or annulus of infinite/periodic length), additional conditions apply; see Marques (1990).

$$
\begin{aligned}
\mathbf{f}_u &\equiv \left(\partial_t - \frac{1}{Re}\Delta\right)\mathbf{u} + \mathbf{s}_u = -\nabla\left(p + B^2/2\right) \\
\mathbf{g}_u &\equiv \nabla \times \mathbf{f}_u = 0 \\
\mathbf{g}_B &\equiv \left(\partial_t - \frac{1}{Re_m}\Delta\right)\mathbf{B} + \mathbf{s}_B = 0
\end{aligned}
$$

Velocity: $\mathbf{g}_u = 0$ **with** $\nabla \cdot \mathbf{g}_u = 0$

$$
\begin{aligned}
0 &= \hat{\mathbf{e}} \cdot \mathbf{g}_u = (\partial_t - \frac{1}{Re}\Delta)\Delta_h \psi_u - \hat{\mathbf{e}} \cdot \nabla \times \mathbf{s}_u \\
0 &= \hat{\mathbf{e}} \cdot \nabla \times \mathbf{g}_u = (\partial_t - \frac{1}{Re}\Delta)\Delta\Delta_h \phi_u + \hat{\mathbf{e}} \cdot \nabla \times \nabla \times \mathbf{s}_u
\end{aligned}
$$

Magnetic field: $\mathbf{g}_B = 0$ **with** $\nabla \cdot \mathbf{g}_B = 0$

$$
\begin{aligned}
0 &= \hat{\mathbf{e}} \cdot \mathbf{g}_B = (\partial_t - \frac{1}{Re_m}\Delta)\Delta_h \phi_B - \hat{\mathbf{e}} \cdot \mathbf{s}_B \\
0 &= \hat{\mathbf{e}} \cdot \nabla \times \mathbf{g}_B = (\partial_t - \frac{1}{Re_m}\Delta)\Delta_h \psi_B - \hat{\mathbf{e}} \cdot \nabla \times \mathbf{s}_B
\end{aligned}
$$

Velocity conditions

$$0 \;=\; \hat{e}\cdot g_u = (\partial_t - \frac{1}{Re}\Delta)\Delta_h\psi_u - \hat{e}\cdot\nabla\times s_u$$

$$0 \;=\; \hat{e}\cdot\nabla\times g_u = (\partial_t - \frac{1}{Re}\Delta)\Delta\Delta_h\phi_u + \hat{e}\cdot\nabla\times\nabla\times s_u$$

Vertical directions: order 2+4=6 $\Longrightarrow$ Need 6/2=3 BCs at $z=\pm h/2$

$\Longrightarrow$ 3 no-slip conditions on u_z, u_r, u_θ.

Horizontal directions: order 4+6=10 $\Longrightarrow$ Need 10/2=5 BCs at $r=1$

$\Longrightarrow$ 3 no-slip conditions on u_z, u_r, u_θ. Two more conditions needed.

Gauge at $r=1$: **(u unaffected by addition to ϕ of any solution to $\Delta_h\phi=0$)**

$$0 = \phi|_{r=1}$$

Compatibility at $r=1$:

$$0 \;=\; \hat{e}_r\cdot g|_{r=1} = \partial^2_{rz}\Delta_h\psi - \frac{1}{r}\partial_\theta\Delta\Delta_h\phi\Big]_{r=1} \qquad \text{(using gauge and BCs)}$$

Alternative approach:

Use $\hat{e} = e_r$

$$
\begin{aligned}
\hat{e} \cdot \mathbf{F} &= -r^2 \Delta_C \phi \\
\hat{e} \cdot \nabla \times \mathbf{F} &= -r^2 \Delta_C \psi + 2\phi_{\theta z}
\end{aligned}
$$

where $\Delta_C = \frac{1}{r^2}\partial_{\theta\theta} + \partial_{zz}$, i.e. Laplacian $\perp$ to $\hat{e} = \hat{e}_r$

Disadvantage: governing equations couple ϕ and $\psi \Longrightarrow$
feasible only if θ, z are both homogeneous (periodic) directions.

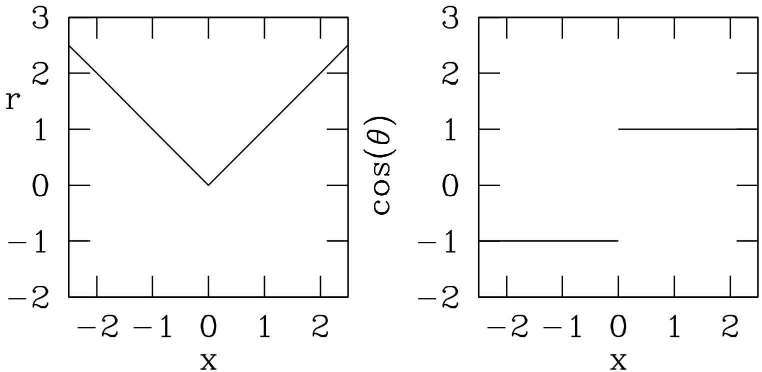
Advantage: no compatibility condition, since Ω_h $(r = 1)$ has no boundary.

Used by Willis & Barenghi (JFM 2002) and by Hollerbach (2007)

Polar coordinates I: regularity at origin

In Cartesian coordinates, monomials $x^k y^n$ are analytic.

In polar coordinates, products $r^j e^{im\theta}$ are generally *not* analytic:



r $\cos(\theta)$

General analytic form:

$$f(r,\theta) = \sum_{m\geq 0} \sum_{\substack{j\geq m \\ j+m \text{ even}}} f_{jm} r^j e^{im\theta} \qquad + \text{c.c.}$$

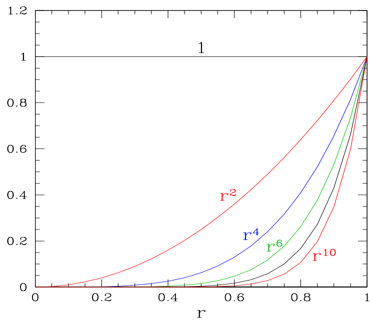
can be derived in many ways, e.g.:

- **converting monomials $x^k y^n = (r\cos\theta)^k (r\sin\theta)^n$**
- **acting repeatedly with Laplacian $(\frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\partial_{\theta\theta})$ on $(r^j e^{im\theta})$ and eliminating singular contributions**
- **examining expansions of Bessel functions**

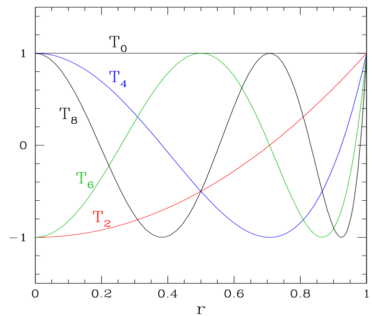
(This is precisely analogous to poles in spherical coordinates and the constraint $\ell \geq |m|$ for spherical harmonics.)

Not imposing $j+m$ even, $j \geq m$ seems not to generate errors (singular functions are just "carried along" without polluting results) but 3/4 of the basis functions are useless.

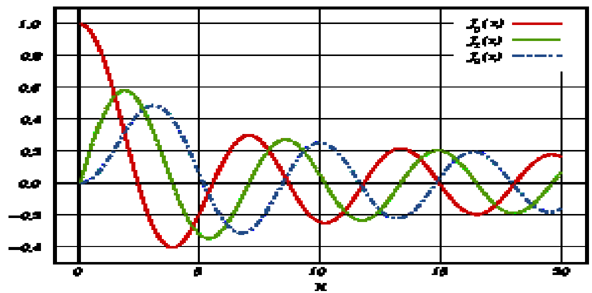
How to impose regularity at $r = 0$?



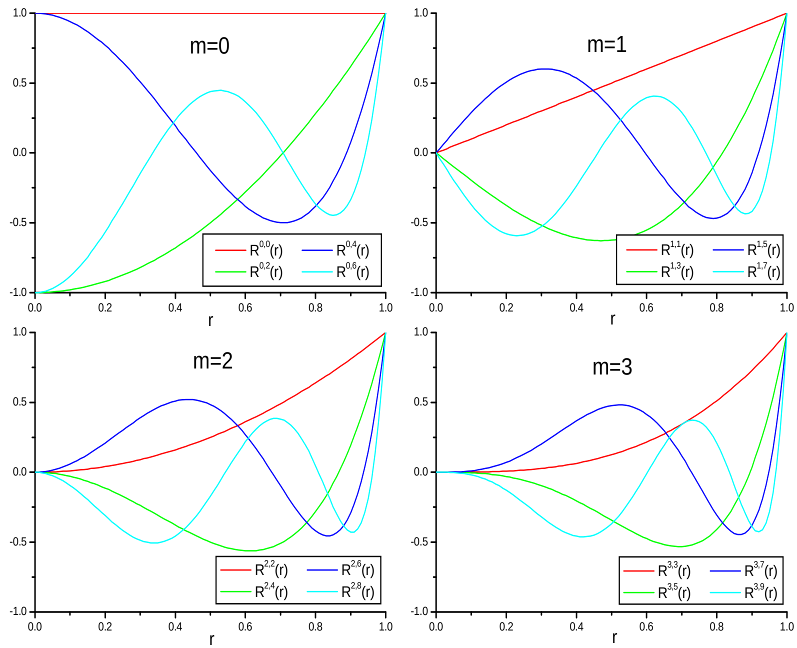
Monomials r^j are ill conditioned.
(Matrix r_i^j for transforming $f(r_i)$ to monomial basis is badly conditioned.)



Chebyshev polynomials $\mathcal{T}_j(r)$ contain all monomials $r^{j'}$, $j' \leq j$, $j' + j$ even



Expansion in Bessel functions $J_m(r)$ displays Gibbs phenomenon at $r = 1$ (no spectral convergence).



Matsushima-Marcus radial functions $\mathcal{Q}_n^m(r)$ (JCP 1994):

- **are *regular* at axis, i.e. $\mathcal{Q}_j^m(r) \sim r^m$ as $r \longrightarrow 0$**
- **matrix $\mathcal{Q}_j^m(r_i)$ is well-conditioned**
- **display spectral convergence**

Polar coordinates II: recursion relations

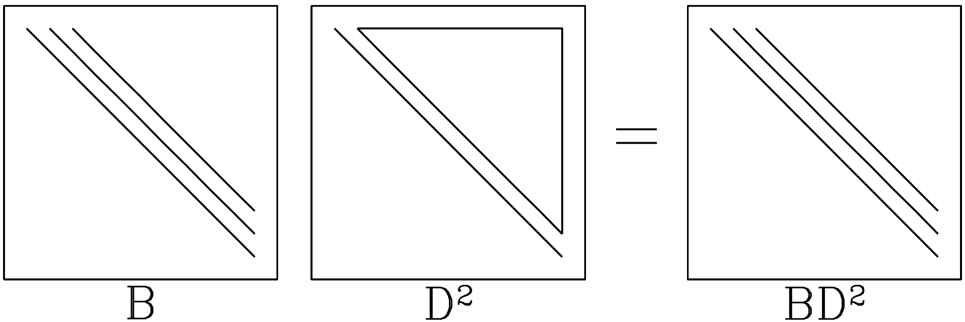
$$\Delta f = \left(\frac{1}{r}\partial_r r \partial_r - \frac{1}{r^2}\partial_\theta^2\right) f = g$$

Fourier mode m:

$$r\partial_r r \partial_r f - m^2 f = r^2 g$$

with f represented in a polynomial radial basis, can be solved in time

$$O(JN_r),\ J = 3\text{, not } O(N_r)^2.$$



Differential operators in curvilinear coordinates in spectral bases are like constant-coefficient differential operators, not general variable-coefficient differential operators; Tuckerman (JCP 1989).

Ideas from linear algebra I: Schur decomposition

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\left(\begin{array}{c} x \\ \hline y \end{array}\right) = \left(\begin{array}{c} z \\ \hline w \end{array}\right)$$

Eliminate y, especially when $w = 0$ or is constant:

$$\begin{array}{rcl} Cx + Dy = w \Longrightarrow & y = D^{-1}(w - Cx) & \Longrightarrow y = -D^{-1}Cx \\ Ax + By = z \Longrightarrow & (A - BD^{-1}C)x = z - BD^{-1}w & \Longrightarrow (A - BD^{-1}C)x = z \end{array}$$

But matrix A may have a nice structure (e.g. banded, block diagonal), which is destroyed by $(A - BD^{-1}C)$.

Ideas from linear algebra II: Sherman-Morrison-Woodbury

$$\begin{pmatrix}\text{Influence}\\ \text{matrix}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{Green's}\\ \text{functions}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{Sherman-Morrison-}\\ \text{Woodbury formula}\end{pmatrix}$$

Don't be afraid of a little coupling!

$$\begin{aligned} (H + VW^T)^{-1} &= H^{-1} - H^{-1} V C^{-1} W^T H^{-1} \\ C &\equiv I + W^T H^{-1} V \end{aligned}$$

$$\begin{pmatrix} & & \\ & H & \\ & & \end{pmatrix} + \begin{pmatrix} V \\ \\ \end{pmatrix} \begin{pmatrix} & W^T & \end{pmatrix}$$

More easily inverted (e.g. decoupled) — J columns — J rows

a rank-J change

Matrix C is $J \times J$ and is calculated in pre-processing step.
Rank-J matrix VW^T can change some rows or columns.

Magnetic field

$$
\begin{aligned}
0 &= \hat{\mathrm{e}} \cdot \mathrm{g}_B = (\partial_t - \frac{1}{Re_m}\Delta)\Delta_h \phi_B - \hat{\mathrm{e}} \cdot \mathrm{s}_B \\
0 &= \hat{\mathrm{e}} \cdot \nabla \times \mathrm{g}_B = (\partial_t - \frac{1}{Re_m}\Delta)\Delta_h \psi_B - \hat{\mathrm{e}} \cdot \nabla \times \mathrm{s}_B
\end{aligned}
$$

Cylinder surrounded by external vacuum:

$$
\begin{aligned}
\mathrm{B}^{vac} &= \nabla \phi^{vac} \\
\Delta \phi^{vac} &= 0 \\
\mathrm{B}^{vac}|_{(r,z)\to\infty} &= 0
\end{aligned}
$$

Matching to unknown $\nabla \phi^{vac}$ at boundary (increases BCs needed by 1):

$$
[\mathrm{B}]_{\partial\Omega} \equiv \mathrm{B}|_{\partial\Omega} - \mathrm{B}^{vac}|_{\partial\Omega} = 0
$$

Vertical directions: order 2+2=4 $\Longrightarrow$ Need 4/2+1=3 BCs at $z = \pm h/2$

Horizontal directions: order 4+4=8 $\Longrightarrow$ Need 8/2+1=5 BCs at $r = 1$

Compatibility at $r = 1$:

$$0 = \hat{e}_r \cdot g_B = \left(\partial_t - \frac{1}{Rm}\left(\Delta - \frac{1}{r^2}\right)\right) B_r + \frac{1}{Rm}\frac{2}{r^2}\partial_\theta B_\theta$$

Gauge at $r = 1$:

$$0 = (\partial_z \phi - \phi^{vac})$$

value of ϕ^{vac}

Matching conditions at $r = 1$:

$$0 = B_r - B_r^{vac} = \frac{1}{r}\partial_\theta \psi + \partial_r (\partial_z \phi - \phi^{vac})$$

normal deriv of ϕ^{vac}

$$0 = B_\theta - B_\theta^{vac} = -\partial_r \psi + \frac{1}{r}\partial_\theta (\partial_z \phi - \phi^{vac}) = -\partial_r \psi$$

$$0 = B_z - B_z^{vac} = -\Delta \phi + \partial_z (\partial_z \phi - \phi^{vac}) = -\Delta \phi$$

Matching conditions at $z = \pm h/2$:

$$\psi = \text{const}$$

$$0 = \partial_z \phi - \phi^{vac}$$

value of ϕ^{vac}

$$0 = -\Delta_h \phi - \partial_z \phi^{vac}$$

normal deriv of ϕ^{vac}

Eliminating External Problem

Dudley & James: Poloidal-toroidal formulation, spherical harmonics

Isakov & Dormy: 3D formulation, finite elements, Greens functions

General framework: Dirichlet-to-Neumann mapping

Given a domain Ω with boundary $\partial\Omega$ and Dirichlet data $\phi|_{\partial\Omega}$, can solve:

$$\begin{aligned} \Delta\phi &= 0 && \text{in } \Omega \\ \phi &= \phi|_{\partial\Omega} && \text{on } \partial\Omega \end{aligned}$$

then evaluate normal derivative on boundary:

$$(\hat{\mathrm{n}} \cdot \nabla\phi)|_{\partial\Omega}$$

This is a linear mapping:

$$\mathcal{F} : \phi|_{\partial\Omega} \longrightarrow (\hat{\mathrm{n}} \cdot \nabla\phi)|_{\partial\Omega}$$

In our case, solve:

$$\Delta\phi^{vac} = 0 \qquad \text{outside of cylinder}$$
$$\nabla\phi^{vac} = 0 \qquad \text{at } \infty$$
$$\phi^{vac} = [\partial_z \phi]_{\partial\Omega} \qquad \text{on } \partial\Omega \text{ (surface of cylinder)}$$

Set:

$$\mathcal{F}_r\left(\{\partial_z\phi|_{\partial\Omega}\}\right) \equiv \partial_r\phi^{vac}|_{r=1}$$
$$\mathcal{F}_z^{\pm}\left(\{\partial_z\phi|_{\partial\Omega}\}\right) \equiv \partial_z\phi^{vac}|_{z=\pm h/2}$$

Replace:

$$\left\{\begin{array}{l} 0 = [\partial_z\phi - \phi^{vac}]_{r=1} \\ 0 = \left[\frac{1}{r}\partial_\theta\psi + \partial^2_{rz}\phi - \partial_r\phi^{vac}\right]_{r=1} \\ 0 = [\partial_z\phi - \phi^{vac}]_{z=\pm h/2} \\ 0 = [\Delta_h\phi + \partial_z\phi^{vac}]_{z=\pm h/2} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} 0 = \left[\frac{1}{r}\partial_\theta\psi + \partial^2_{rz}\phi\right]_{r=1} \\ \qquad -\mathcal{F}_r(\{\partial_z\phi|_{\partial\Omega}\}) \\ 0 = [\Delta_h\phi]_{z=\pm h/2} \\ \qquad +\mathcal{F}_z^{\pm}(\{\partial_z\phi|_{\partial\Omega}\}) \end{array}\right\}$$

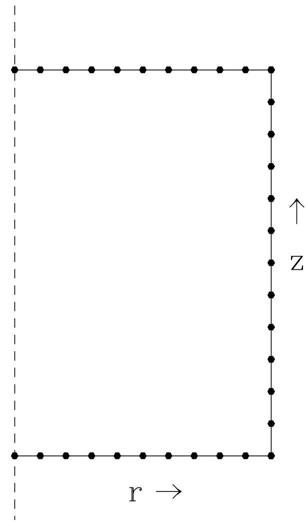
ϕ^{vac} no longer appears in boundary conditions, but need $\mathcal{F}$.

Must now represent Dirichlet-to-Neumann mapping $\mathcal{F}$!

Dudley & James:

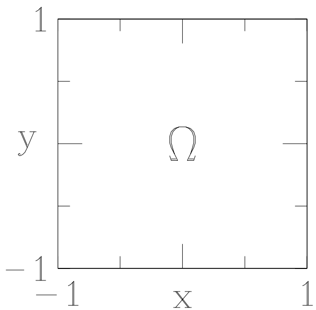
For exterior of a sphere, $\mathcal{F}$ is diagonal in basis of spherical harmonics. Also holds for other doubly-periodic surfaces.

In geometry with one periodic direction (e.g. finite cylinder) $\mathcal{F}$ is block diagonal in Fourier basis, with one block per azimuthal wavenumber m.



Can be a difficult problem.

Analogous 2D Scalar Problem (Boronski, JCP 2007)



$$\Delta\Phi = \rho \quad \textbf{in } \Omega \qquad \Delta\phi = 0 \quad \textbf{outside } \Omega$$

Boundary conditions:

$$\begin{aligned} \Phi(\mathbf{x}) - \phi(\mathbf{x}) &= 0 \qquad \mathbf{x} \in \partial\Omega \\ \partial_n\Phi(\mathbf{x}) - \partial_n\phi(\mathbf{x}) &= 0 \qquad \mathbf{x} \in \partial\Omega \\ \nabla\phi(\mathbf{x}) &\rightarrow 0 \qquad |\mathbf{x}| \rightarrow \infty \end{aligned}$$

Matching internal and external solutions

Particular solution:

$$\Delta\Phi^p = \rho \text{ in } \Omega$$

$$\Phi^p|_{\partial\Omega} = 0$$

Homogeneous solution:

(b_j scans over all boundary points or basis functions)

$$\Delta\Phi^h_j = 0 \text{ in } \Omega$$

$$\Phi^h_j|_{\partial\Omega} = b_j$$

External solution:

$$\Delta\phi_j = 0 \text{ outside } \Omega$$

$$\nabla\phi_j|_\infty = 0$$

$$\phi_j|_{\partial\Omega} = b_j$$

Matching conditions between $\Phi = \Phi^p + \Phi^h$ and ϕ:

$$(\Phi - \phi)|_{\partial\Omega} = \Phi^p|_{\partial\Omega} + \sum_j c_j \left(\Phi^h_j - \phi_j\right)\Big|_{\partial\Omega} = 0$$

already satisfied by construction

$$\left.\frac{\partial(\Phi - \phi)}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial\Phi^p}{\partial n}\right|_{\partial\Omega} + \sum_j c_j \left.\frac{\partial(\Phi^h_j - \phi_j)}{\partial n}\right|_{\partial\Omega} = 0$$

constitutes system of equations for c_j

Preprocessing: construct ϕ_j (don't save), Φ^h_j, $\left.\frac{\partial(\Phi^h_j - \phi_j)}{\partial n}\right|_{\partial\Omega}$

Each timestep: solve for Φ^p, $\{c_j\}$

Validation: electrostatic example

$$\rho_m(r,\theta) = r^m e^{-r^2/\delta^2}\cos(m\theta) \qquad r \equiv |\mathrm{x}| = \sqrt{x^2+y^2},\ \ \theta \equiv \arg(x+iy)$$

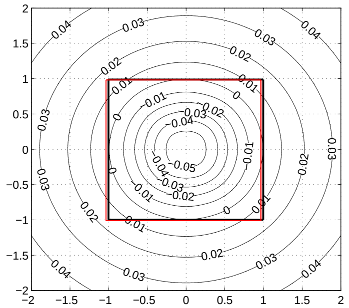
Exact solutions in unbounded domain **(ρ_m very small outside of square):**

$$\begin{aligned}
\Phi_{m=0}(r,\theta) &= \frac{\delta^2}{4}\left[Ei\left(1,\frac{r^2}{\delta^2}\right) + 2\log(r)\right] \\
\Phi_{m=1}(r,\theta) &= \frac{\delta^4}{4r}\left[e^{-\frac{r^2}{\delta^2}} - 1\right]\cos\theta \\
\Phi_{m=2}(r,\theta) &= \frac{\delta^4}{4r^2}\left[\left(\delta^2 + r^2\right)e^{-\frac{r^2}{\delta^2}} - \delta^2\right]\cos 2\theta
\end{aligned}$$

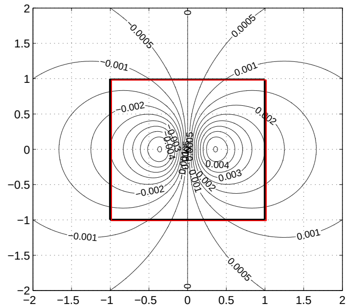
where $Ei(a,z) \equiv \int_1^\infty e^{-tz} t^{-a}\, dt$

Error:

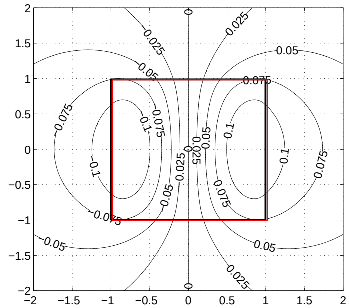
$$E_m(N) = \sup_{r,\theta} \frac{|\Phi_m(r,\theta) - \Phi_m^N(r,\theta)|}{|\Phi_m(r,\theta)|}$$



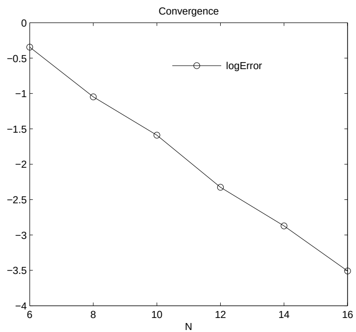
$\Phi^{N=8}_{m=0}$ **for** $\delta^2 = 0.15$



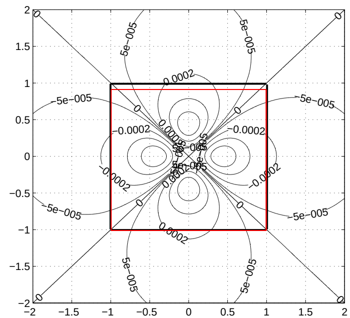
$\Phi^{N=16}_{m=1}$ **for** $\delta^2 = 0.1$



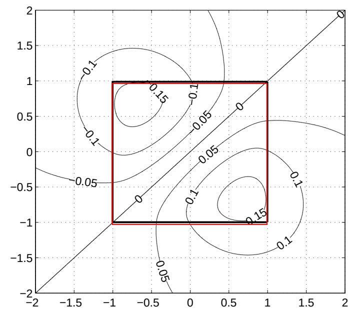
$\Phi^{N=16}_{m=1}$ **for** $\delta^2 = 2$



$\log E_{m=0}(N)$ **for** $\delta^2 = 0.1$

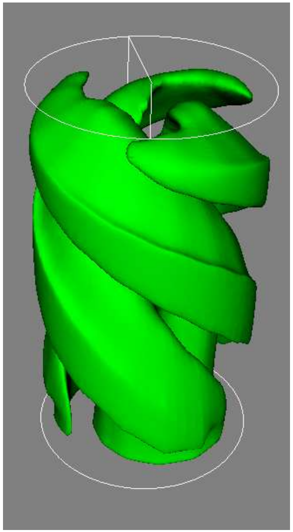


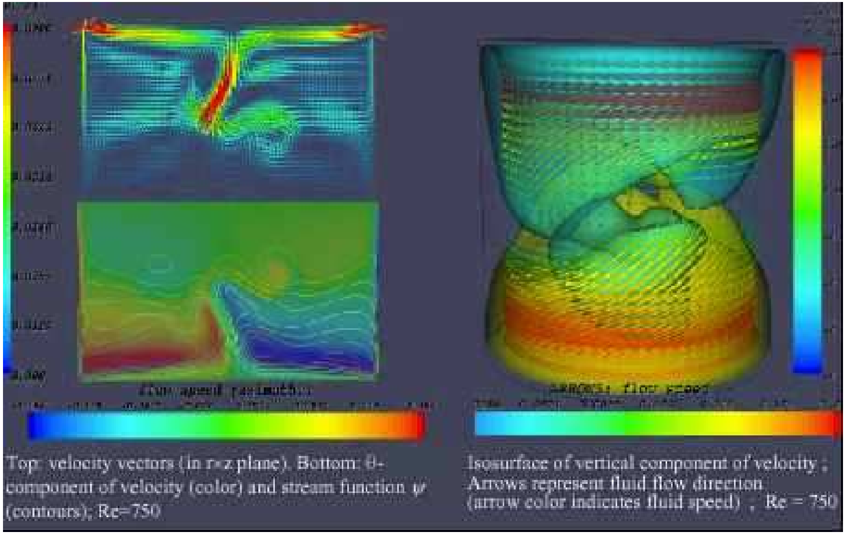
$\Phi^{N=16}_{m=2}$ **for** $\delta^2 = 0.1$



Rotation by 45°

Visualizations of Hydrodynamic Simulations





Rotor-stator

$Re_c \approx 2150$

$h = 3.5$

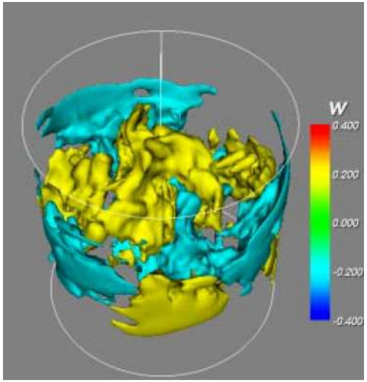
Exactly counter-rotating disks

2D simulation

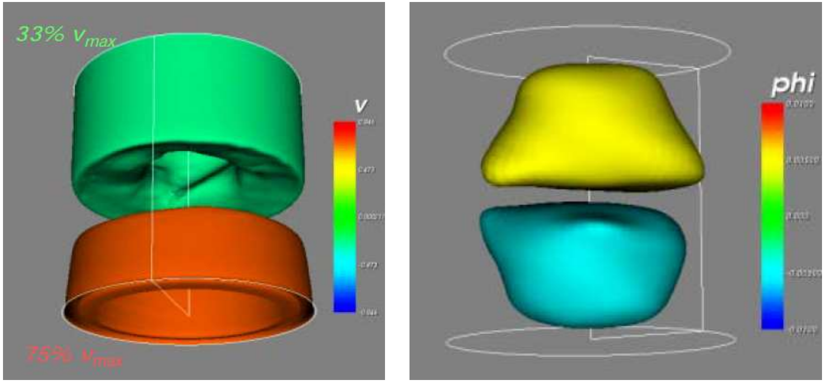
3D simulation

$Re = 750, h = 2$

Turbulent flow at Re=5000 with exact counter-rotation



Instantaneous flow



Temporal average contains two cells

Influence Matrix for Nested Elliptic Problems

$$\left(\partial_t - \frac{1}{Re}\Delta\right)\Delta_h\psi = \hat{e}_z \cdot \nabla \times s \equiv s_\psi$$

$$\left(\partial_t - \frac{1}{Re}\Delta\right)\Delta\Delta_h\phi = -\hat{e}_z \cdot \nabla \times \nabla \times s \equiv s_\phi$$

At $r = 1$:

$$\frac{1}{r}\partial_\theta\psi + \partial_z\partial_r\phi = 0 \qquad (u_r)$$

$$\partial_r\psi = 0 \qquad (u_\theta)$$

$$\Delta_h\phi = 0 \qquad (u_z)$$

$$\phi = 0 \qquad \text{(gauge)}$$

$$\partial^2_{rz}\Delta_h\psi - \frac{1}{r}\partial_\theta\Delta\Delta_h\phi = 0 \qquad \text{(compatibility)}$$

At $z = \pm h/2$:

$$\Delta_h\psi = -\frac{1}{r}\partial_r(r^2\Omega_\pm) \qquad (\hat{e}_z \cdot \nabla \times u_h)$$

$$\partial_z\Delta_h\phi = 0 \qquad (\nabla_h \cdot u_h)$$

$$\Delta_h\phi = 0 \qquad (\hat{e}_z \cdot u)$$

$$b_{f_\psi}(z) \equiv r^{-1}\partial_\theta\psi + \partial_z\partial_r\phi = 0 \Longrightarrow \begin{array}{ll} (\partial_t - \frac{1}{Re}\Delta) f_\psi = s_\psi & \\ f_\psi = -\frac{1}{r}\partial_r(r^2\Omega_\pm) & \text{at } z = \pm h/2 \\ f_\psi = \sigma_{f_\psi}(z) & \text{at } r = 1 \end{array}$$

$$\begin{array}{ll} \Delta_h\psi = f_\psi & \\ \partial_r\psi = 0 & \text{at } r = 1 \end{array}$$

$$(\partial_t - \frac{1}{Re}\Delta) g_\phi = s_\phi$$

$$b_{g_\phi}(z) \equiv \partial^2_{rz} f_\psi - r^{-1}\partial_\theta g_\phi = 0 \Longrightarrow \quad g_\phi = \sigma_{g_\phi}(z) \quad \text{at } r = 1$$

$$b_{g_\phi}(r) \equiv \partial_z f_\phi = 0 \Longrightarrow \quad g_\phi = \sigma_{g_\phi}(r) \quad \text{at } z = \pm h/2$$

$$\begin{array}{ll} \Delta f_\phi = g_\phi & \\ f_\phi = 0 & \text{at } r = 1 \\ f_\phi = 0 & \text{at } z = \pm h/2 \end{array}$$

$$\begin{array}{ll} \Delta_h\phi = f_\phi & \\ \phi = 0 & \text{at } r = 1 \end{array}$$

Preprocessing step:

Calculate solutions to homogeneous problem ($\mathbf{s} = \mathbf{0}$) with complete set of Dirichlet boundary conditions (all possible values of $\boldsymbol{\sigma}_{f_\psi}, \boldsymbol{\sigma}_{g_\phi}$)

For each homogeneous solution, calculate values of unsatisfied conditions $\boldsymbol{b}_{f_\psi}$, $\boldsymbol{b}_{g_\phi}$ on boundary.

Collect and invert to form influence matrix of size $\mathbf{2}(\boldsymbol{N_z} + \boldsymbol{N_r}) \times \mathbf{2}(\boldsymbol{N_z} + \boldsymbol{N_r})$

Each timestep:

Calculate solutions to inhomogeneous problem ($\mathbf{s} \neq \mathbf{0}$) with homogeneous Dirichlet boundary conditions $\boldsymbol{\sigma}_{f_\psi} = \boldsymbol{\sigma}_{g_\phi} = \mathbf{0}$

Calculate values of unsatisfied conditions $\boldsymbol{b}_{f_\psi}$, $\boldsymbol{b}_{g_\phi}$ on boundary.

Multiply by influence matrix to obtain appropriate values of $\boldsymbol{\sigma}_{f_\psi}, \boldsymbol{\sigma}_{g_\phi}$

Solve inhomogeneous problem with corrected Dirichlet boundary values.

External harmonic functions

Poisson eqn over all space $[-\infty, \infty]^d$ with source on $\partial\Omega$:

$$\Delta\phi(\mathrm{x}) = f(\mathrm{x}) \qquad f(\mathrm{x}) = 0 \quad \text{for } \mathrm{x} \notin \partial\Omega$$
$$\phi(\mathrm{x}) = 0 \qquad \text{as } |\mathrm{x}| \to \infty$$

Solve using Green's functions:

$$\Delta G(\mathrm{x};\mathrm{x}') = \delta|\mathrm{x}-\mathrm{x}'|$$
$$\text{2D:} \quad G(\mathrm{x};\mathrm{x}') = -\frac{1}{2\pi}\ln|\mathrm{x}-\mathrm{x}'|$$
$$\text{3D:} \quad G(\mathrm{x};\mathrm{x}') = -\frac{1}{4\pi}\frac{1}{|\mathrm{x}-\mathrm{x}'|}$$
$$\phi(\mathrm{x}) = \int \rho(\mathrm{x}')G(\mathrm{x};\mathrm{x}')\,dx'$$

Repeat for complete basis b_j on boundaries and set

$$\phi(\mathrm{x}) = \sum_j c_j \phi_j(x) \qquad \text{with } c_j \text{ to be determined}$$

Construction of External Harmonic Functions

$$\int_a^b G(x;x')\,\sigma(x')\,dx' = f(x)$$ Fredholm integral eqn of 1st kind for surface source σ

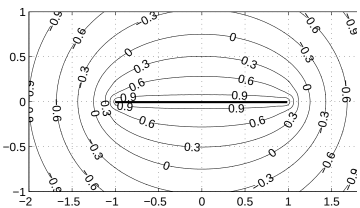
$G(x;x') = \ln|x - x'| \Longrightarrow$ Carleman's equation. Has solution (!) :

$$\sigma(x) = \frac{1}{\pi^2\sqrt{(x-a)(b-x)}}\left[\int_a^b \frac{\sqrt{(t-a)(b-t)}\,f'(t)\,dt}{t-x} + \frac{1}{\ln(1/2)}\int_a^b \frac{f(t)\,dt}{\sqrt{(t-a)(b-t)}}\right]$$

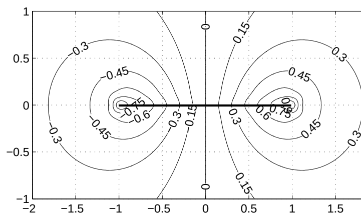
For $[a,b] = [-1,1]$, $b_j(x) = \mathcal{T}_j(2x)$, solution becomes (!) :

$$\sigma_j(x) = A_j \frac{\mathcal{T}_j(x)}{\pi\sqrt{1-x^2}} \quad ; \quad A_j = \begin{cases} -j & j > 0 \\ [\ln(1/2)]^{-1} & j = 0 \end{cases}$$

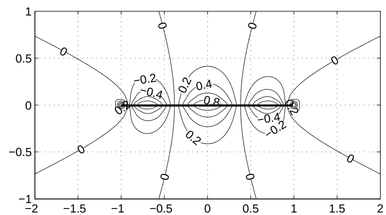
$$\phi_j^x(x,y) = \int_{-1}^{1} \ln\,|(x-x')\hat{e}_x + y\hat{e}_y|\,\sigma_j(x')\,dx'$$



$\phi_0^x(x,y)$



$\phi_1^x(x,y)$



$\phi_4^x(x,y)$

External harmonic solution is sum of $\boldsymbol{2(J+K)}$ basis functions:

$$\boldsymbol{\phi^{J,K}(\mathrm{x}) = \sum_{j=0}^{J-1} \left[c_j^{x,-} \phi_j^x \left(\mathrm{x} + \hat{\mathrm{e}}_x\right) + c_j^{x,+} \phi_j^x \left(\mathrm{x} - \hat{\mathrm{e}}_x\right) \right]}$$
$$\boldsymbol{+ \sum_{k=0}^{K-1} \left[c_k^{y,-} \phi_k^y \left(\mathrm{x} + \hat{\mathrm{e}}_y\right) + c_k^{y,+} \phi_k^y \left(\mathrm{x} - \hat{\mathrm{e}}_y\right) \right]}$$

where

$$\boldsymbol{\phi_k^y(\mathrm{x}) = \int_{-1}^{1} \ln \left|\mathrm{x} - y' \hat{\mathrm{e}}_y\right| \sigma_k(y')\, dy'}$$

Evaluate normal derivatives via singular integrals:

$$\boldsymbol{\frac{\partial \phi_j^x}{\partial n}(x,y) = \frac{\partial}{\partial y} \int_{-1}^{1} \frac{1}{2} \ln \left[(x - x')^2 + y^2\right] \sigma_j(x')\, dx'}$$
$$\boldsymbol{= \int_{-1}^{1} \frac{y}{(x - x')^2 + y^2} \sigma_j(x')\, dx'}$$

Internal harmonic solution is:

$$\boldsymbol{\Phi^{J,K}(\mathrm{x}) = \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \Phi_{jk} T_j(x) T_k(y)}$$