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# Codimension-two points

# The 1:2 mode interaction

System with $O(2)$ symmetry with competing wavenumbers $m = 1$ and $m = 2$
Solutions approximated as:

$$u(\theta, t) = z_1(t)e^{i\theta} + z_2(t)e^{2i\theta} + \bar{z}_1(t)e^{-i\theta} + \bar{z}_2(t)e^{-2i\theta}$$

with $z_1(t) = x_1(t) + iy_1(t) = r_1(t)e^{i\phi_1(t)}$, $z_2(t) = x_2(t) + iy_2(t) = r_2(t)e^{i\phi_2(t)}$

$O(2)$ generated by rotation by $\theta_0$ and reflection about $\theta = 0$:

$$\begin{aligned} S_{\theta_0}(z_1, z_2) &= (e^{i\theta_0}z_1, e^{2i\theta_0}z_2) \\ \kappa(z_1, z_2) &= (\bar{z}_1, \bar{z}_2) \end{aligned}$$

Define

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{F}{\to} \begin{pmatrix} \bar{z}_1 z_2 \\ -z_1^2 \end{pmatrix}$$

Show that $F$ is equivariant with respect to $O(2)$:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{F}{\to} \begin{pmatrix} \bar{z}_1 z_2 \\ -z_1^2 \end{pmatrix} \overset{\kappa}{\to} \begin{pmatrix} z_1 \bar{z}_2 \\ -\bar{z}_1{}^2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{\kappa}{\to} \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \overset{F}{\to} \begin{pmatrix} z_1 \bar{z}_2 \\ -\bar{z}_1{}^2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{F}{\to} \begin{pmatrix} \bar{z}_1 z_2 \\ -z_1^2 \end{pmatrix} \overset{S_{\theta_0}}{\to} \begin{pmatrix} e^{i\theta_0} \bar{z}_1 z_2 \\ -e^{2i\theta_0} z_1^2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{S_{\theta_0}}{\to} \begin{pmatrix} e^{i\theta_0} z_1 \\ e^{2i\theta_0} z_2 \end{pmatrix} \overset{F}{\to} \begin{pmatrix} e^{-i\theta_0} \bar{z}_1 e^{2i\theta_0} z_2 \\ -e^{2i\theta_0} z_1^2 \end{pmatrix}$$

Essentially $1+1=2$ and $2-1=1$

Dynamical system for evolution of $z_1$, $z_2$ is:

$$\begin{aligned} \dot{z}_1 &= \bar{z}_1 z_2 + z_1 \left(\mu_1 - \alpha_1 |z_1|^2 - \beta_1 |z_2|^2\right) \\ \dot{z}_2 &= -z_1^2 + z_2 \left(\mu_2 - \beta_2 |z_1|^2 - \alpha_2 |z_2|^2\right) \end{aligned}$$

# Steady states

(Phase is arbitrary: $z \to x$)

$$\begin{aligned} 0 &= x_1 \left(x_2 + \left(\mu_1 - \alpha_1 x_1^2 - \beta_1 x_2^2\right)\right) \\ 0 &= -x_1^2 + x_2 \left(\mu_2 - \beta_2 x_1^2 - \alpha_2 x_2^2\right) \end{aligned}$$

Trivial state: $x_1 = x_2 = 0$

Mode-two ("pure mode") state: $x_1 = 0$, $x_2 \neq 0$:

$$x_2^2 = \mu_2 / \alpha_2$$

If $x_1 \neq 0$ then $x_2 \neq 0$! Instead, have "mixed-mode state":

$$\begin{aligned} 0 &= x_2 + \left(\mu_1 - \alpha_1 x_1^2 - \beta_1 x_2^2\right) \\ 0 &= -x_1^2 + x_2 \left(\mu_2 - \beta_2 x_1^2 - \alpha_2 x_2^2\right) \end{aligned}$$

(intersection of two conic sections)

# Stability

Jacobian in Cartesian coordinates (even if $y = 0$, Jacobian must include $y$)

$$\begin{pmatrix} x_2 + \mu_1 - \alpha_1(r_1^2 + 2x_1^2) - \beta_1 r_2^2 & y_2 - \alpha_1 2x_1 y_1 & x_1 - \beta_1 2x_1 x_2 & y_1 - \beta_1 2x_1 y_2 \\ y_2 + \mu_1 - \alpha_1 2x_1 y_1 & -x_2 + \mu_1 - \alpha_1(r_1^2 + 2y_1^2) - \beta_1 r_2^2 & -y_1 - \beta_1 2y_1 x_2 & x_1 - \beta_1 2x_2 y_1 \\ \pm 2x_1 - \beta_2 2x_2 x_1 & \mp 2y_1 + -\beta_2 2x_2 y_1 & \mu_2 - \beta_2 r_1^2 - \alpha_2(r_2^2 + 2x_2^2) & -\alpha_2 2x_2 y_y \\ \pm 2y_1 + \mu_2 - \beta_2 2x_1 y_2 & \pm 2y_1 y_2 & \alpha_2 2x_2 y_2 & \mu_2 - \beta_2 r_1^2 - \alpha_2(r_2^2 + 2y_2^2) \end{pmatrix}$$

Trivial state:

$$\mathcal{J} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}$$

Two 2D eigenspaces. Circle pitchforks at $\mu_1 = 0$ and $\mu_2 = 0$

Mode-two state: $x_1 = y_1 = y_2 = 0$, $x_2 = \pm\sqrt{\mu_2/\alpha_2}$

$$\mathcal{J} = \begin{pmatrix} x_2 + \mu_1 - \beta_1 x_2^2 & 0 & 0 & 0 \\ \mu_1 & -x_2 + \mu_1 - \beta_1 x_2^2 & 0 & 0 \\ 0 & 0 & \mu_2 - 3\alpha_2 x_2^2 & 0 \\ 0 & 0 & 0 & \mu_2 - \alpha_2 x_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} \pm\sqrt{\frac{\mu_2}{\alpha_2}} + \mu_1 - \frac{\beta_1\mu_2}{\alpha_2} & 0 & 0 & \\ 0 & \mp\sqrt{\frac{\mu_2}{\alpha_2}} + \mu_1 - \frac{\beta_1\mu_2}{\alpha_2} & 0 & 0 \\ 0 & 0 & -2\mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues $-2\mu_2$ and 0 are usual results of circle pitchfork.
Other two eigenvalues concern instability to $(x_1, y_1)$.
They are different because mode-two state has a phase (no CP from mode-two).
Mixed-mode branch bifurcates from trivial state at $\mu_1 = 0$
and from mode-two branch at

$$\mu_1 - \beta_1 \frac{\mu_2}{\alpha_2} \pm \sqrt{\frac{\mu_2}{\alpha_2}} = 0$$

# Polar representation

$$z_1(t) = r_1(t)e^{i\phi_1(t)}, z_2(t) = r_2(t)e^{i\phi_2(t)}$$

Evolution equations:

$$\begin{aligned}
(\dot{r}_1 + ir_1\dot{\phi}_1)e^{i\phi_1} &= r_1 e^{-i\phi_1} r_2 e^{i\phi_2} + r_1 e^{i\phi_1}\left(\mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2\right) \\
(\dot{r}_2 + ir_2\dot{\phi}_2)e^{i\phi_2} &= -r_1 e^{i\phi_1} r_1 e^{i\phi_1} + r_2 e^{i\phi_2}\left(\mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2\right)
\end{aligned}$$

Dividing equations by $e^{i\phi_1}$ and by $e^{i\phi_2}$:

$$\begin{aligned}
\dot{r}_1 + ir_1\dot{\phi}_1 &= r_1 r_2 e^{i(\phi_2 - 2\phi_1)} + r_1\left(\mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2\right) \\
\dot{r}_2 + ir_2\dot{\phi}_2 &= -r_1^2 e^{i(2\phi_1 - \phi_2)} + r_2\left(\mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2\right)
\end{aligned}$$

Separating real and imaginary parts and dividing imaginary parts by $r_j \neq 0$:

$$\begin{aligned}
\dot{r}_1 &= r_1 r_2 \cos(\phi_2 - 2\phi_1) + r_1\left(\mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2\right) \\
\dot{\phi}_1 &= r_2 \sin(\phi_2 - 2\phi_1) \\
\dot{r}_2 &= -r_1^2 \cos(2\phi_1 - \phi_2) + r_2\left(\mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2\right) \\
\dot{\phi}_2 &= -(r_1^2/r_2)\sin(2\phi_1 - \phi_2)
\end{aligned}$$

Substitute $\Phi \equiv 2\phi_1 - \phi_2$:

$$\begin{aligned}
\dot{r}_1 &= r_1 \left( r_2 \cos\Phi + \mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2 \right) \\
\dot{r}_2 &= -r_1^2 \cos\Phi + r_2 \left( \mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2 \right)
\end{aligned}$$

$$\left.\begin{array}{lcl} \dot{\phi}_1 & & -r_2 \sin\Phi \\ & = & \\ \dot{\phi}_2 & & -(r_1^2/r_2)\sin\Phi \end{array}\right\} \Longrightarrow \dot{\Phi} = -(2r_2 - r_1^2/r_2)\sin\Phi$$

Suppose $\dot{r}_1 = \dot{r}_2 = \dot{\Phi} = 0$, but $r_1, r_2 \neq 0$

$$\begin{aligned}
0 &= r_2 \cos\Phi + \mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2 \\
0 &= -r_1^2 \cos\Phi + r_2 \left( \mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2 \right) \\
0 &= (2r_2^2 - r_1^2)\sin\Phi
\end{aligned}$$

# Mixed modes

$\Phi = 0, \pi \Longrightarrow \sin\Phi = 0 \Longrightarrow \dot{\phi}_1 = \dot{\phi}_2 = 0 \Longrightarrow$ steady states:

$$\begin{aligned}
0 &= \pm r_2 + \mu_1 - \alpha_1 r_1^2 - \beta_1 r_2^2 \\
0 &= \mp r_1^2 + r_2 \left( \mu_2 - \beta_2 r_1^2 - \alpha_2 r_2^2 \right)
\end{aligned}$$

# Traveling Waves

$$\begin{aligned} 0 &= \dot{\Phi} = -(2r_2 - r_1^2/r_2)\sin\Phi \\ \sin\Phi \neq 0 \Longrightarrow 0 &= 2r_2^2 - r_1^2 \Longrightarrow r_1^2 = 2r_2^2 \\ 0 &= \dot{\Phi} \equiv 2\dot{\phi}_1 - \dot{\phi}_2 \end{aligned}$$

Definition:

$$u(\theta, t) = u(\theta - ct, 0)$$

$$\begin{aligned} u(\theta, t) &= r_1(t)e^{i(\phi_1(t)+\theta)} + r_2(t)e^{i(\phi_2(t)+2\theta)} + \text{ complex conjugate} \\ u(\theta - ct, 0) &= r_1(0)e^{i(\phi_1(0)+\theta-ct)} + r_2(0)e^{i(\phi_2(0)+2(\theta-ct))} + \text{ complex conjugate} \end{aligned}$$

$$\Longrightarrow \begin{cases} r_1(t) = r_1(0) \quad \text{and} \quad \phi_1(t) = \phi_1(0) - ct \\ r_2(t) = r_2(0) \quad \text{and} \quad \phi_2(t) = \phi_2(0) - 2ct \end{cases}$$

$$\Longrightarrow 2\phi_1(t) - \phi_2(t) = 2\phi_1(0) - \phi_2(0) \Longrightarrow \Phi(t) = \Phi(0)$$

$$
\begin{aligned}
0 &= 2r_2^2 - r_1^2 & \Longrightarrow r_1^2 = 2r_2^2 \\
0 &= \dot{r}_1 = r_2 \cos\Phi + \mu_1 - \alpha_1 2r_2^2 - \beta_1 r_2^2 \\
0 &= \dot{r}_2 = -2r_2^2 \cos\Phi + r_2\left(\mu_2 - \beta_2 2r_2^2 - \alpha_2 r_2^2\right)
\end{aligned}
$$

Add $2\times$ blue equation to $(1/r_2)\times$ green equation

$$
\begin{aligned}
0 &= 2\mu_1 + \mu_2 - (4\alpha_1 + 2\beta_1 + 2\beta_2 + \alpha_2) r_2^2 \\
r_2^2 &= \frac{2\mu_1 + \mu_2}{4\alpha_1 + 2\beta_1 + 2\beta_2 + \alpha_2}
\end{aligned}
$$

Can also obtain:

$$
\cos\Phi = \frac{\mu_1(2\alpha_2 + \beta_2) - \mu_2(2\alpha_1 + \beta_1)}{\left[(2\mu_1 + \mu_2)(4\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2)\right]^{1/2}}
$$

Traveling waves bifurcate from mixed mode branch when $|\cos\Phi| = 1 \Longleftrightarrow$

$$
\left(\mu_1(2\alpha_2 + \beta_2) - \mu_2(2\alpha_1 + \beta_1)\right)^2 = (2\mu_1 + \mu_2)(4\alpha_1 + 2\beta_1 + 2\alpha_2 + \beta_2)
$$

# Time-dependent states

- Traveling waves via Hopf bif from mixed-mode branch

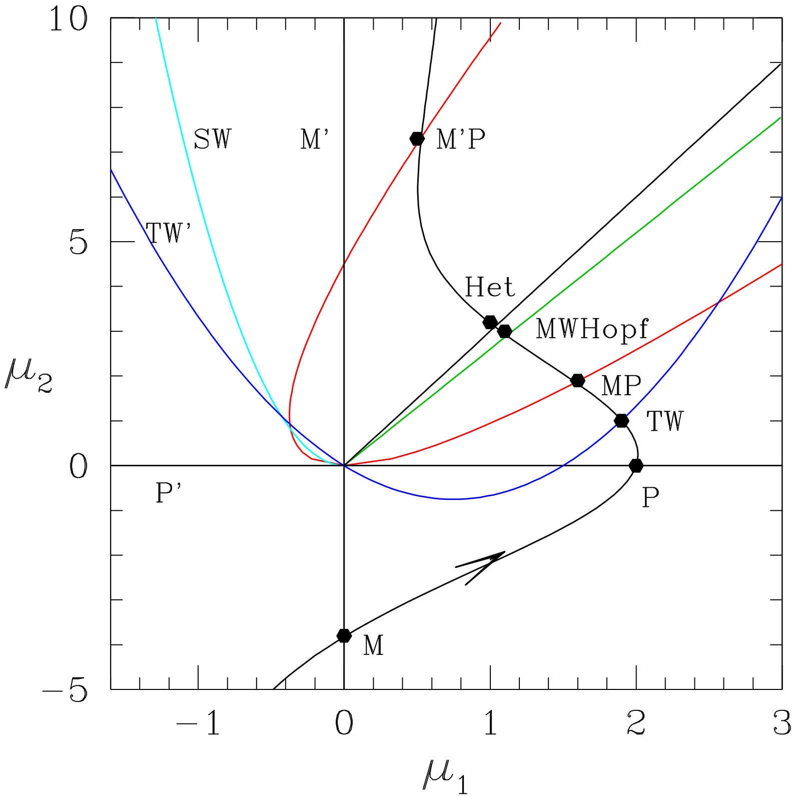
- Modulated waves via secondary Hopf bif from traveling waves

- Heteroclinic orbit connects two opposite-phase mode-two saddles with eigenvalues $-\lambda_- < 0 < \lambda_+$

Can prove orbit is stable if $\lambda_- > \lambda_+$, i.e. if contraction more important than expansion

$$-\left(\mu_1 - \beta_1 \frac{\mu_2}{\alpha_2} - \sqrt{\frac{\mu_2}{\alpha_2}}\right) > \mu_1 - \beta_1 \frac{\mu_2}{\alpha_2} + \sqrt{\frac{\mu_2}{\alpha_2}}$$

$$\Longleftrightarrow \beta_1 \frac{\mu_2}{\alpha_2} > \mu_1$$



10
5
0
−5
−1
0
1
2
3
$\mu_2$
$\mu_1$
SW
M'
M'P
TW'
Het
MWHopf
MP
TW
P'
P
M

# Takens-Bogdanov normal form

Meeting of Hopf and steady bifurcations

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -\mu_1 x + \mu_2 y - x^3 - x^2 y
\end{aligned}
$$

Steady states:

$$
\begin{aligned}
0 &= y \\
0 &= -\mu_1 x - x^3 \Longrightarrow x = \pm\sqrt{-\mu_1}
\end{aligned}
$$

Jacobian:

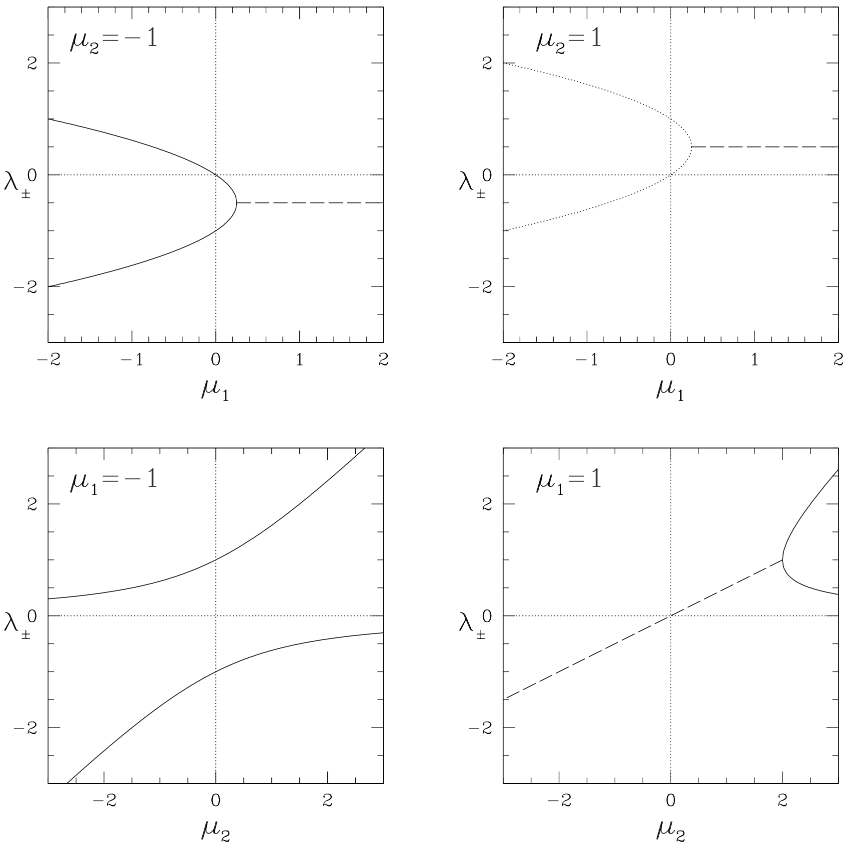
$$
\begin{aligned}
\mathcal{J} &= \begin{pmatrix} 0 & 1 \\ \mu_1 - 3x^2 - 2xy & \mu_2 - x^2 \end{pmatrix} \\
\mathcal{J}(0,0) &= \begin{pmatrix} 0 & 1 \\ \mu_1 & \mu_2 \end{pmatrix} \Longrightarrow \lambda = \frac{\mu_2}{2} \pm \sqrt{\left(\frac{\mu_2}{2}\right)^2 - \mu_1}
\end{aligned}
$$

$\mathcal{J}$ is Jordan block at codimension-two point $\mu_1 = \mu_2 = 0$

Hopf bifurcation at $\mu_2 = 0$ for $\mu_1 > 0$

Pitchfork bifurcation at $\mu_1 = 0$

Real eigenvalues coalesce to form complex conjugate pair
At collision, imaginary part is zero
At a nearby Hopf bifurcation, frequency is near zero
$\Longrightarrow$ period is near infinity

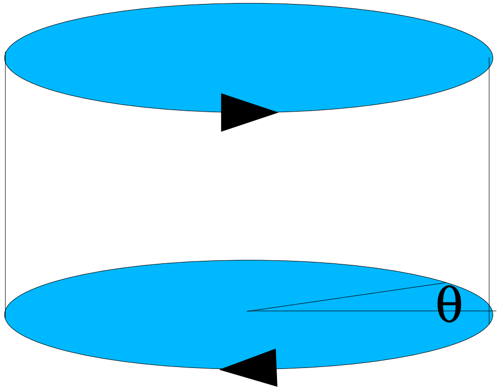


$\mu_2 = -1$
$\mu_2 = 1$
$\mu_1 = -1$
$\mu_1 = 1$
$\lambda_\pm$
$\mu_1$
$\mu_2$
2
0
−2
−1
1

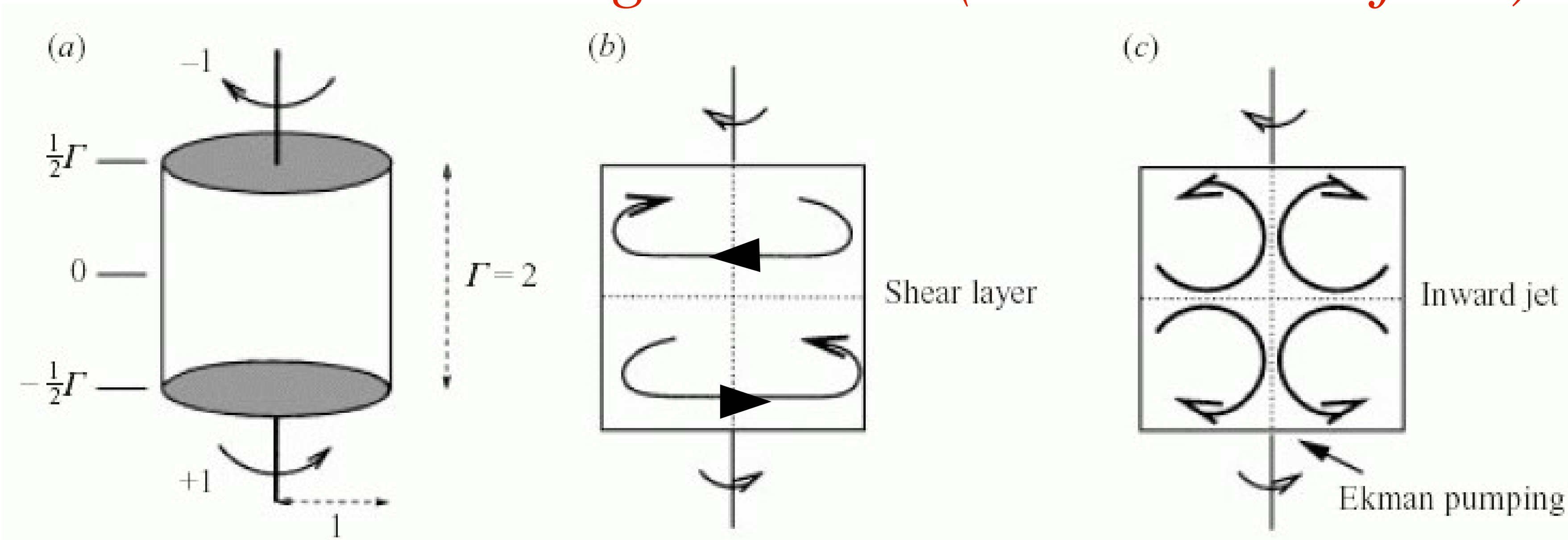
# Heteroclinic cycles in the French washing machine

*Caroline Nore*
*Laurette Tuckerman*
*Olivier Daube*
*Shihe Xin*

LIMSI-CNRS, France



# *The French Washing Machine (von Karman flow)*




Symmetry Group:
Rotations in θ and
Combined reflection in z and θ
Rot/Ref don't commute ⇒ O(2)

*Douady, Brachet, Couder, Fauve et al*
*Le Gal et al, Rabaud et al, Daviaud et al.*
*Gelfgat et al, Lopez & Marques et al*

# *Numerical Methods*

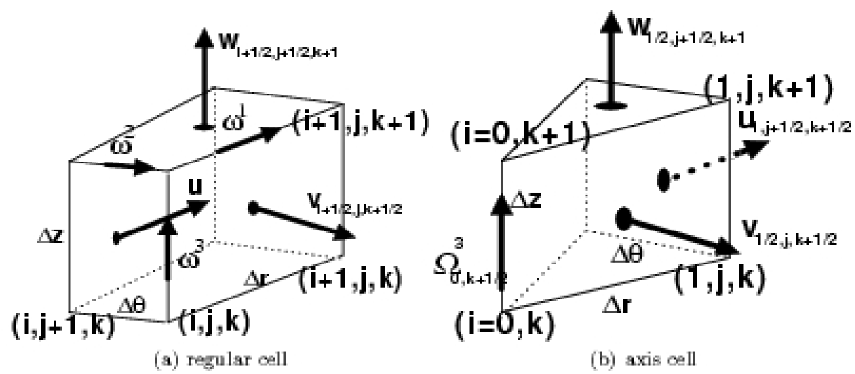


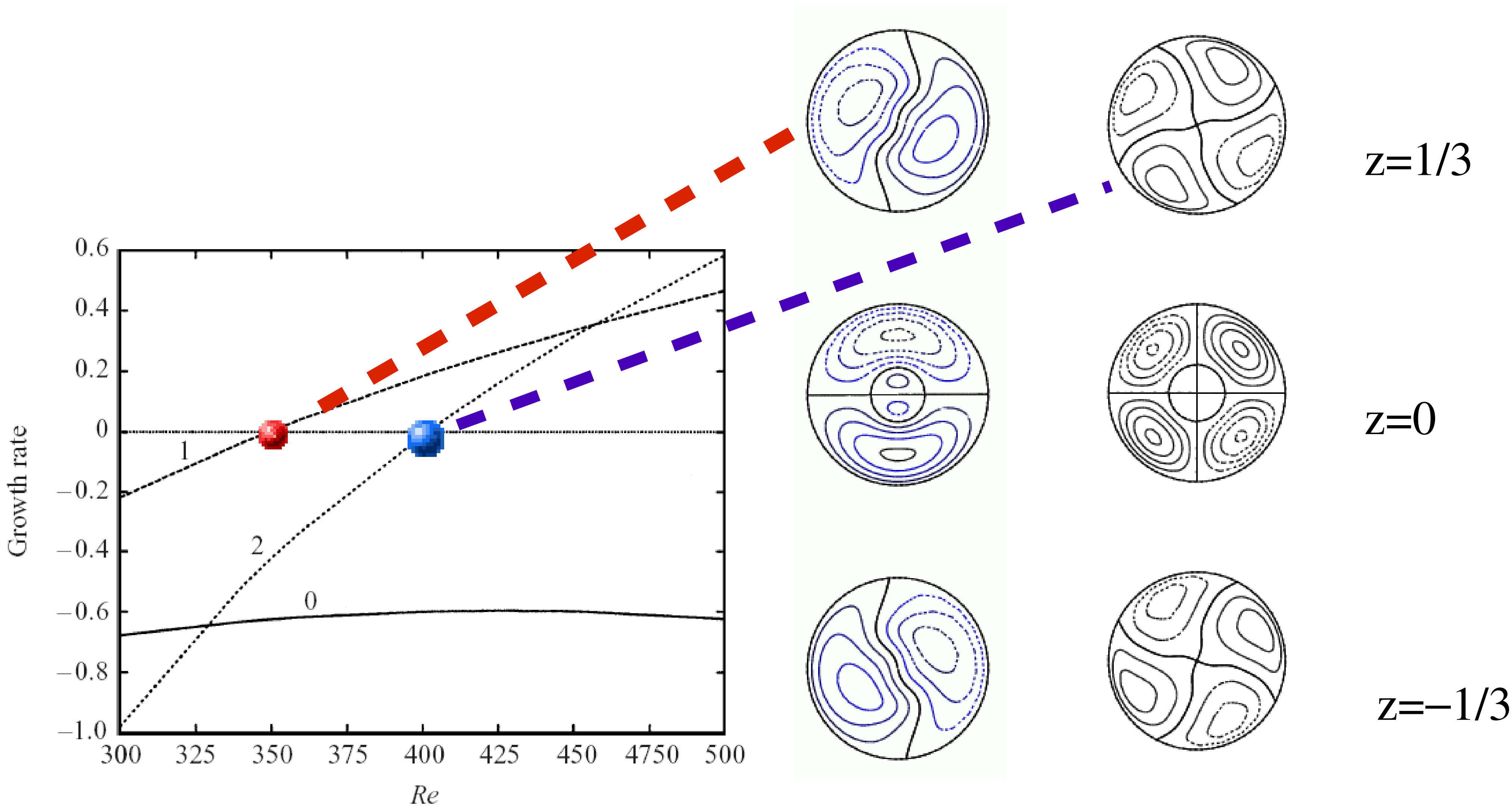

Figure 1: The MAC grid

Time-integration code for Navier-Stokes eqns by Daube

Spatial: finite differences in (r,z), Fourier in θ
Temporal: 2$^{nd}$ order backward difference formula

Adaptations:
-Steady state solving via Newton for axisymmetric flows
-Linear stability about axisymmetric and 3D flows

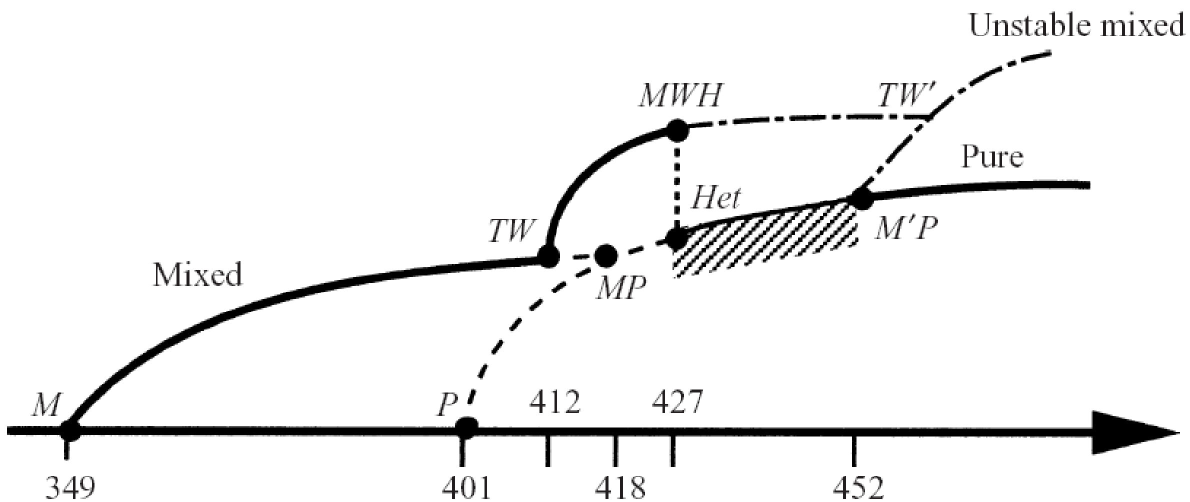
# *Linear Stability of Basic Axisymmetric Flow*




m=1
mixed mode
Re=355

m=2
pure mode
Re=410

# *Bifurcation Diagram for 1:2 mode interaction*




## *Normal Form*

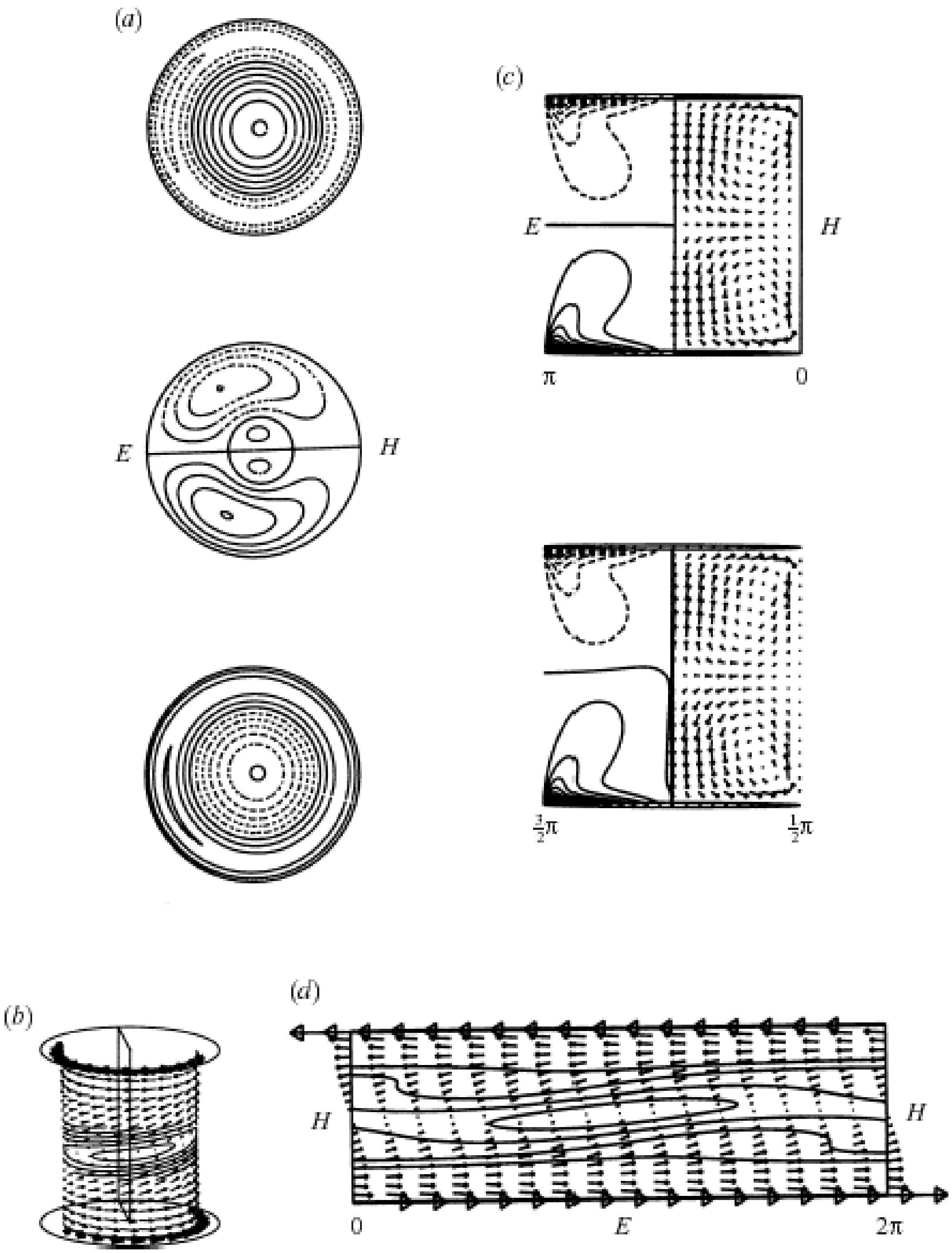
$$\dot{z}_1 = \overline{z_1} z_2 + z_1(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2)$$

$$\dot{z}_2 = \pm z_1^2 + z_2(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2)$$

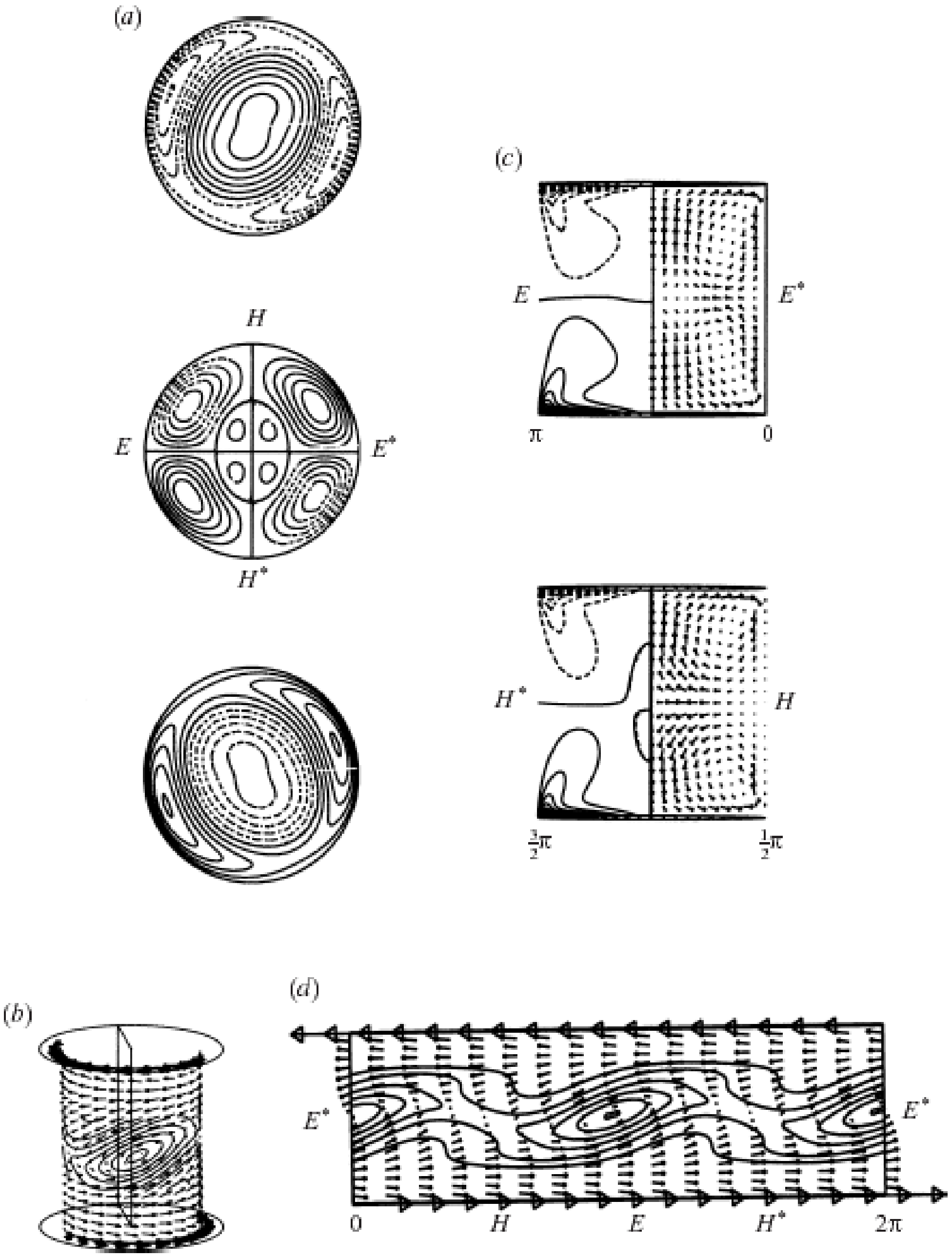
quadratic terms

*Armbruster, Guckenheimer & Holmes; Proctor & Jones (1988)*

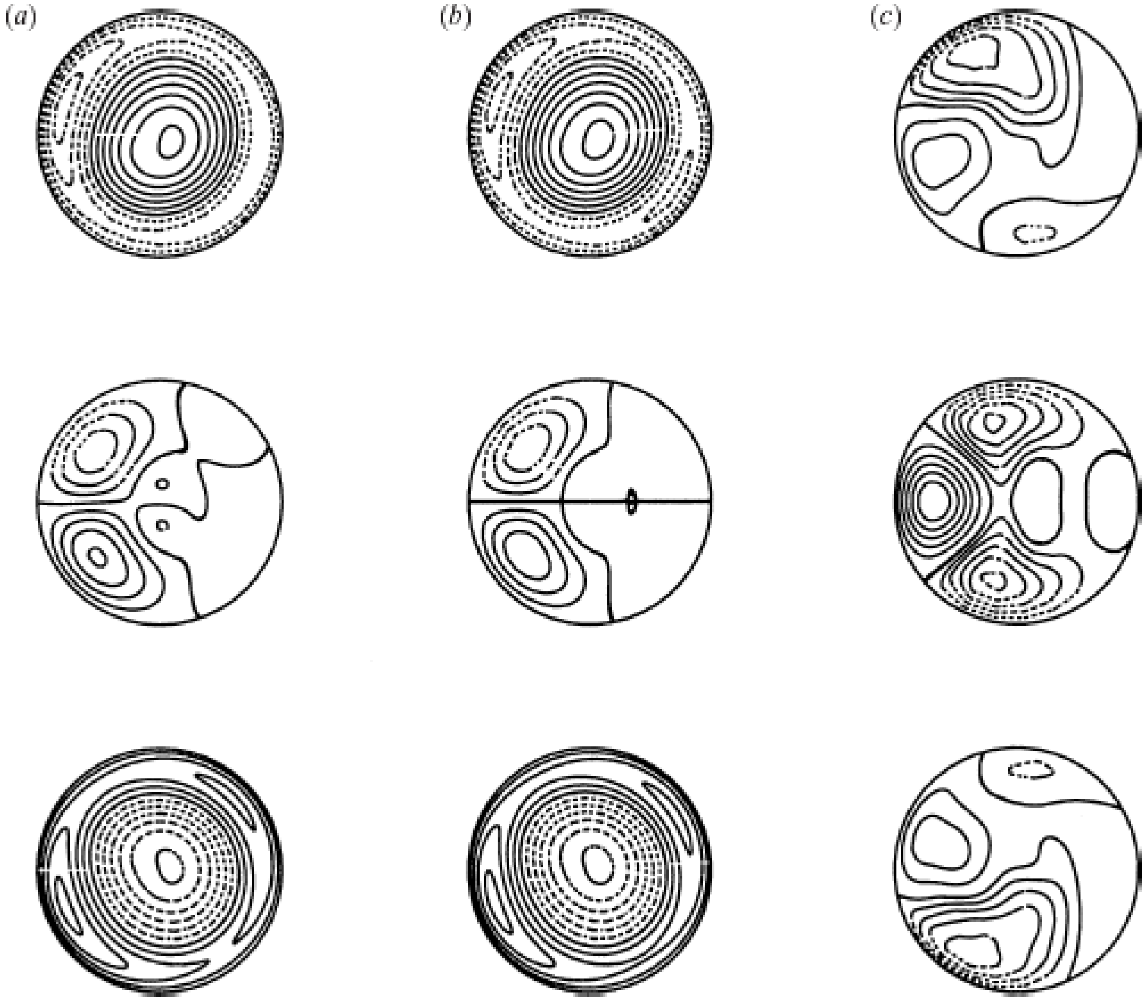
# *Mixed Mode (from m=1 eigenvector)*



# *Pure Mode (from m=2 eigenvector)*



# *Travelling Waves (Re=415)*



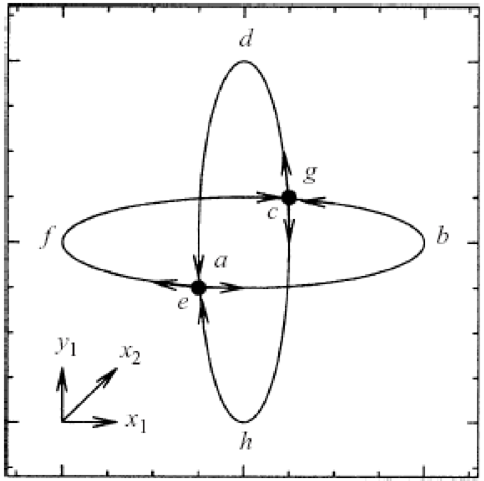

TW = Mixed Mode + Eigenvector

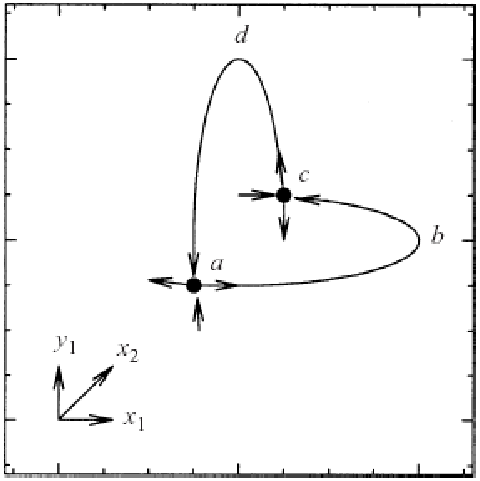
Reflection-Symmetric Antisymmetric

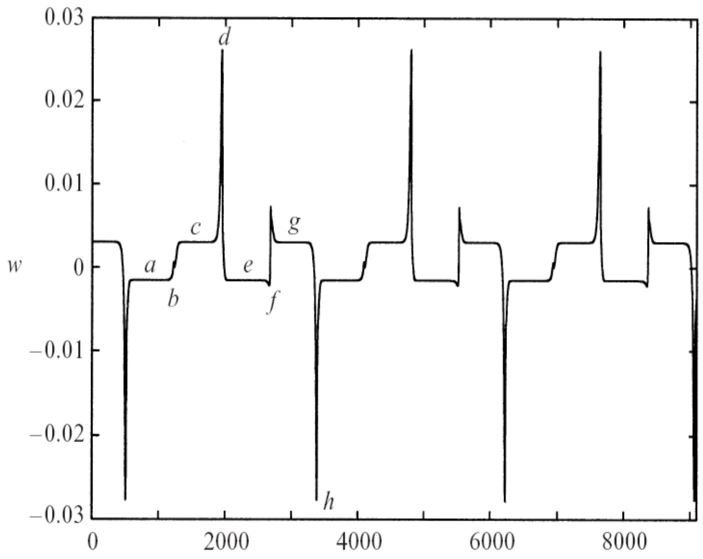
# Two types of heteroclinic cycles

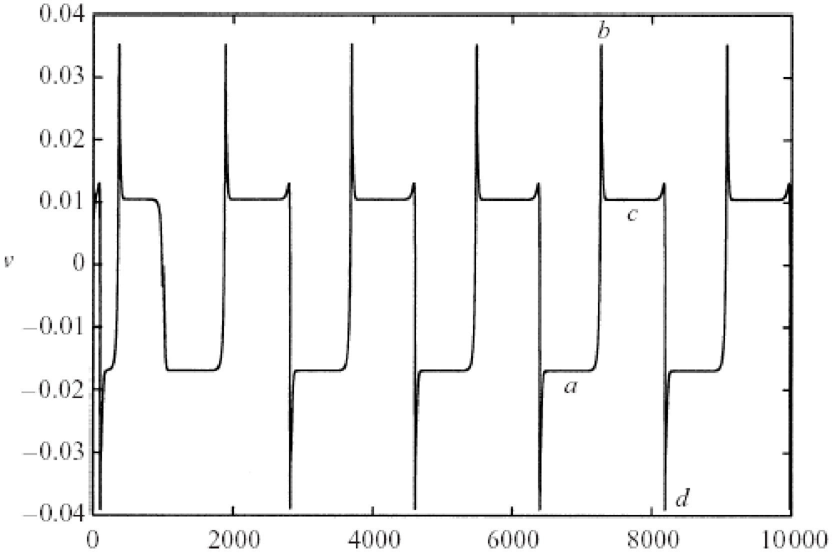
## 4 plateaus

## 2 plateaus

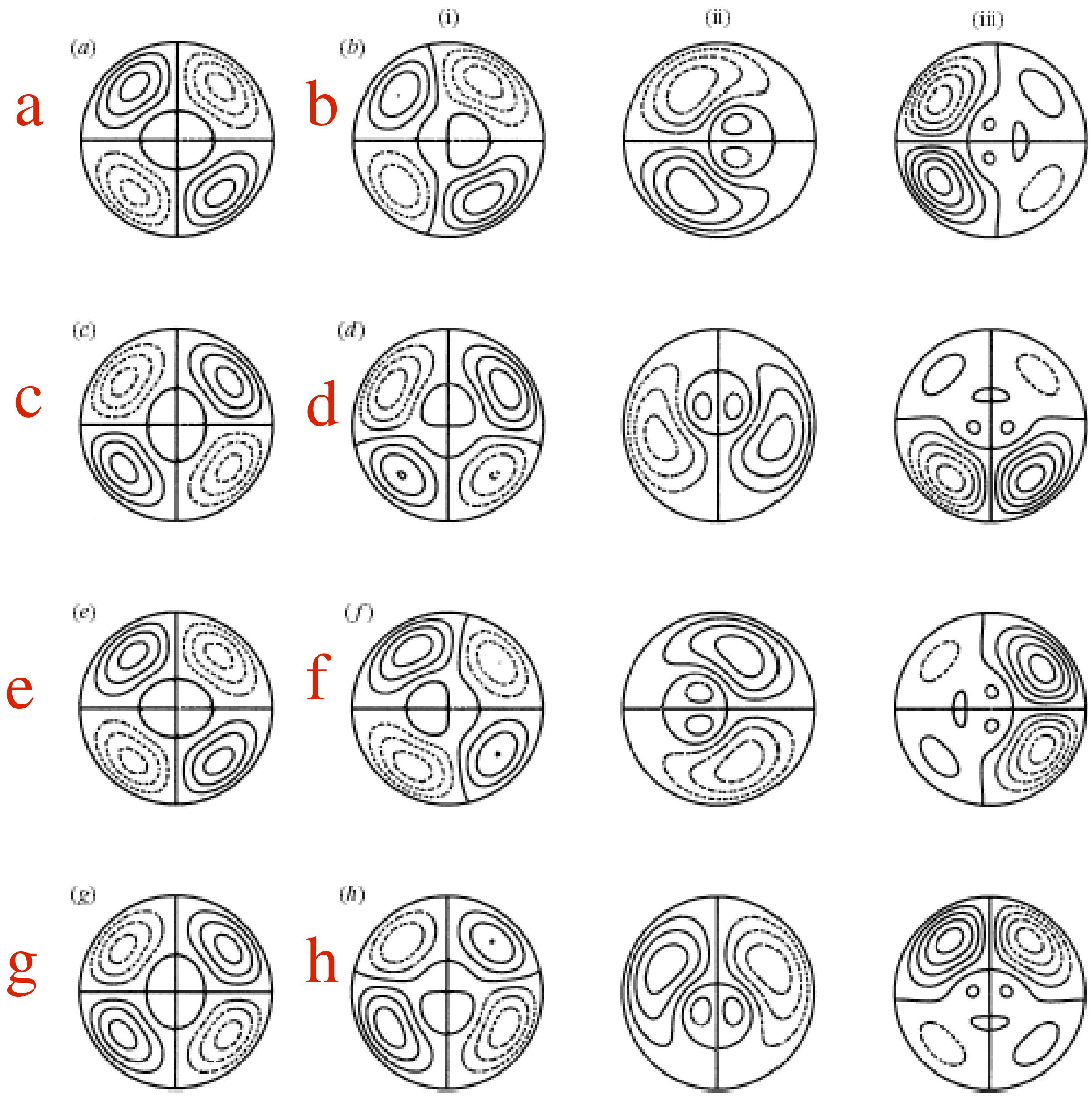


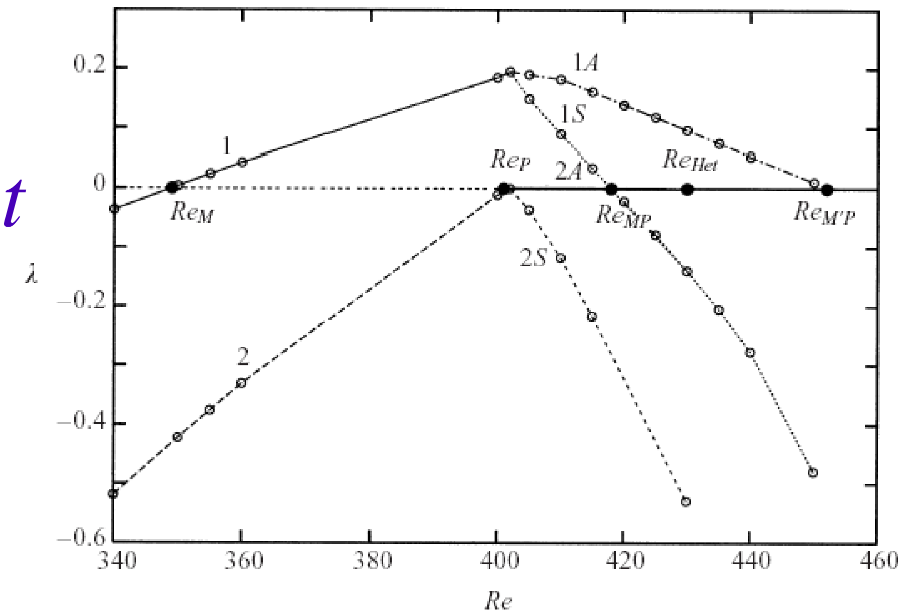



# *Heteroclinic Cycle (Re=430)*

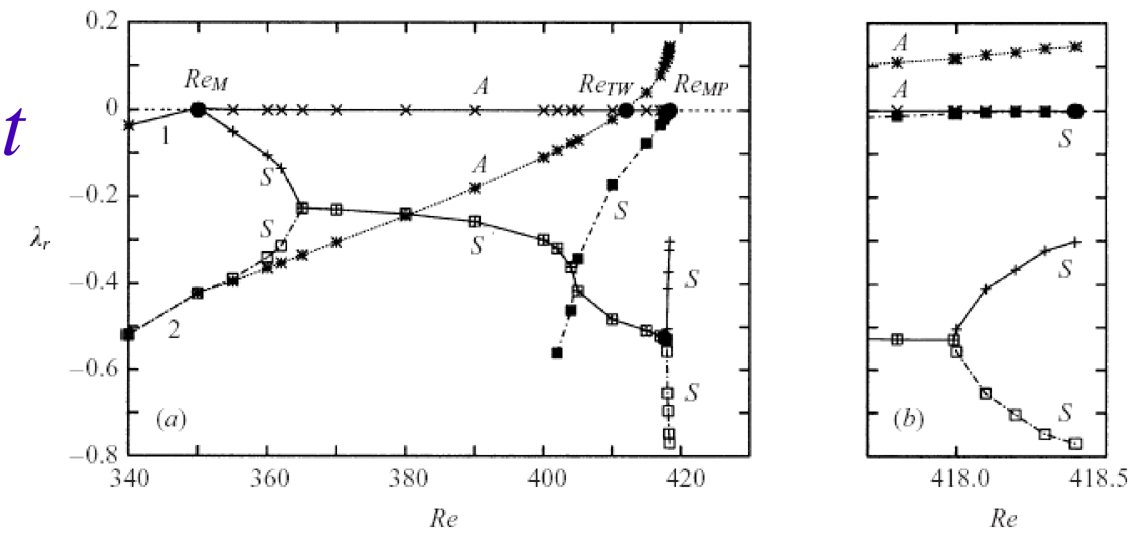


# *Linear stability analysis about nonaxisymmetric flows*

*Eigenvalues about pure mode*




*Eigenvalues about mixed mode*



# *Conclusion*

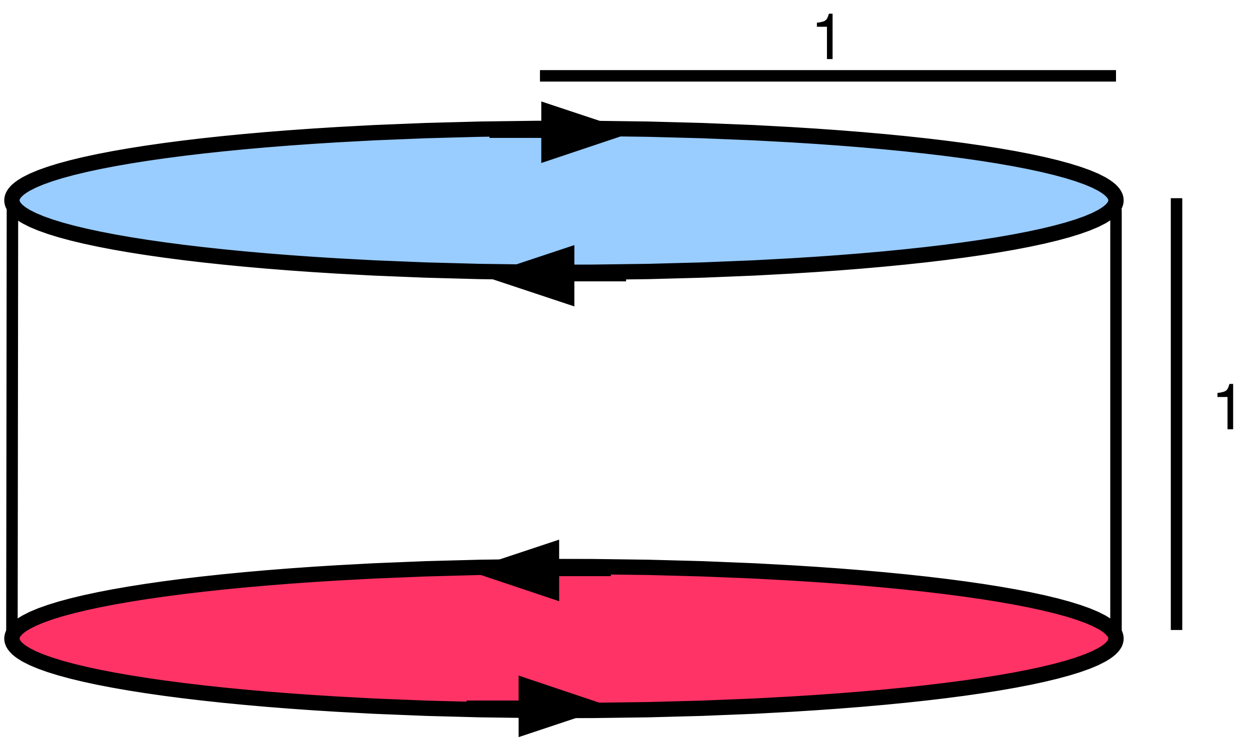
*counter-rotating von Karman flow with diameter=height is almost perfect realisation of 1:2 mode interaction*

- *steady states (mixed and pure modes)*
- *travelling waves*
- *robust heteroclinic cycles of two kinds*
- *possible Kelvin-Helmholtz instability mechanism*

# Convection + counter-rotation
# (Rayleigh-Bénard + von Kármán)

Tuckerman with Bordja, Cruz Navarro, Martin Witkowski, Barkley

Pr=1    $\Gamma$=1    axisymmetric




Symmetry: $R_\pi$

$z \to z_{mid} - z$

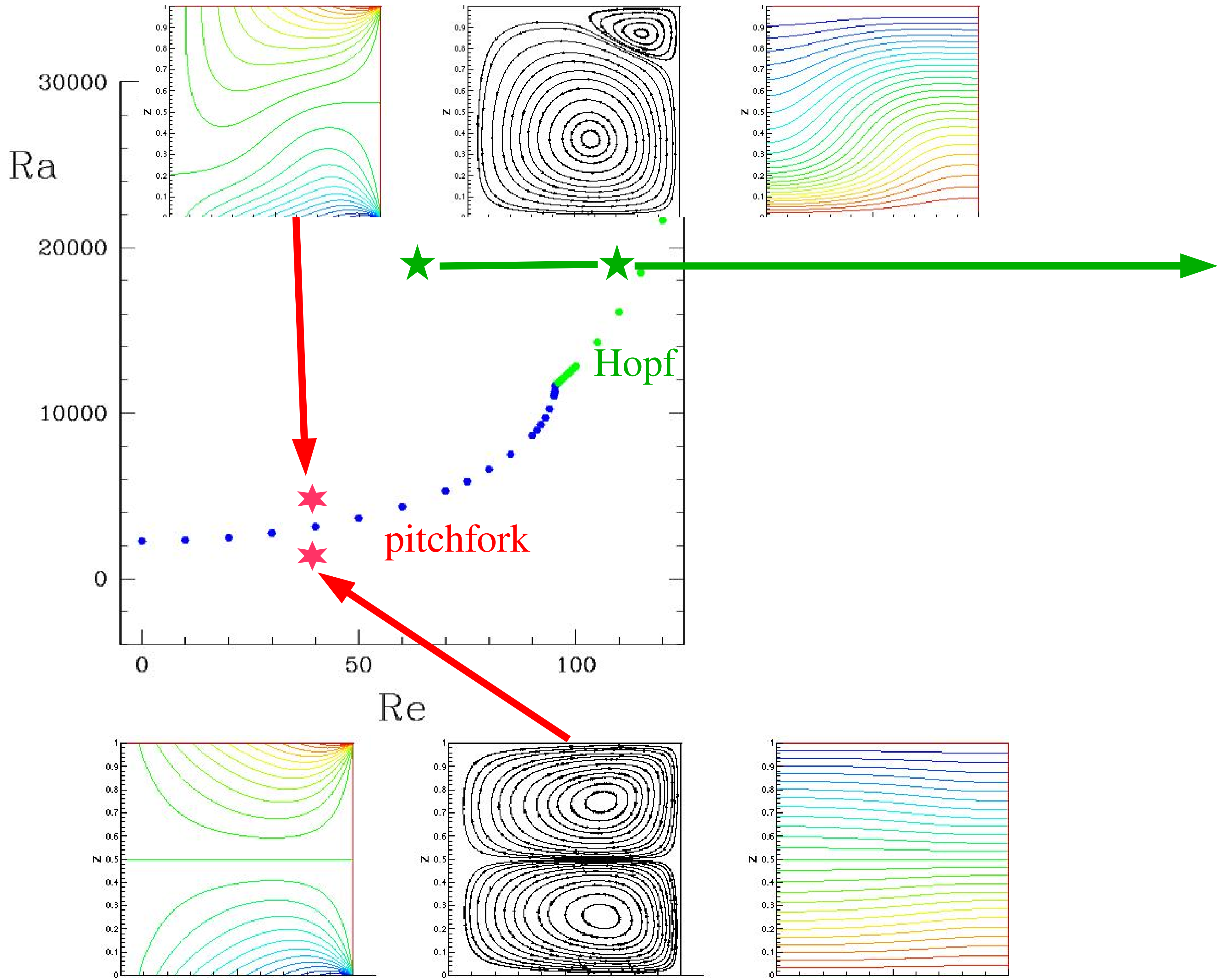
$T \to T_{mid} - T$

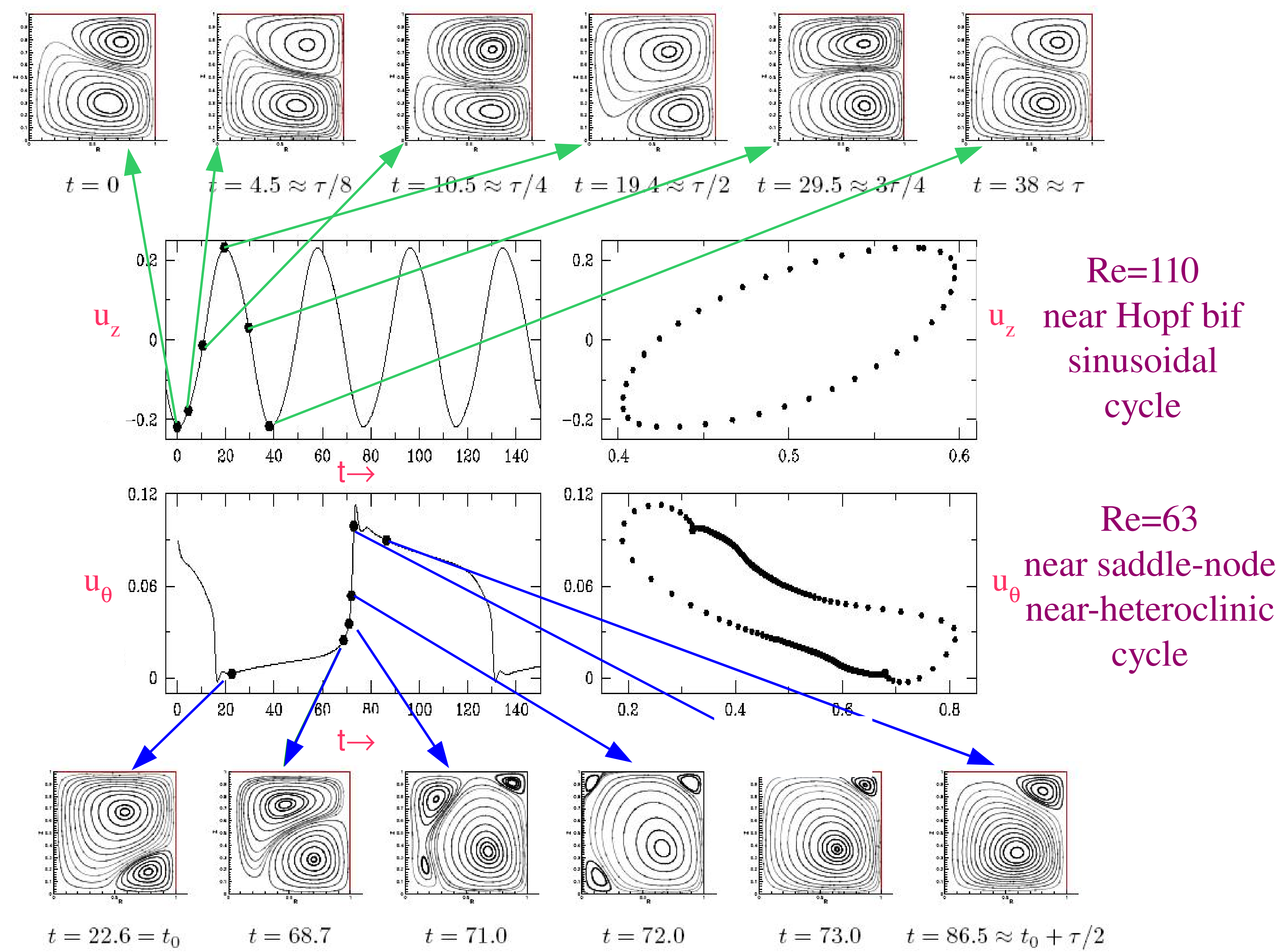
$\theta \to \theta_{mid} - \theta$

$u_z \to -u_z$

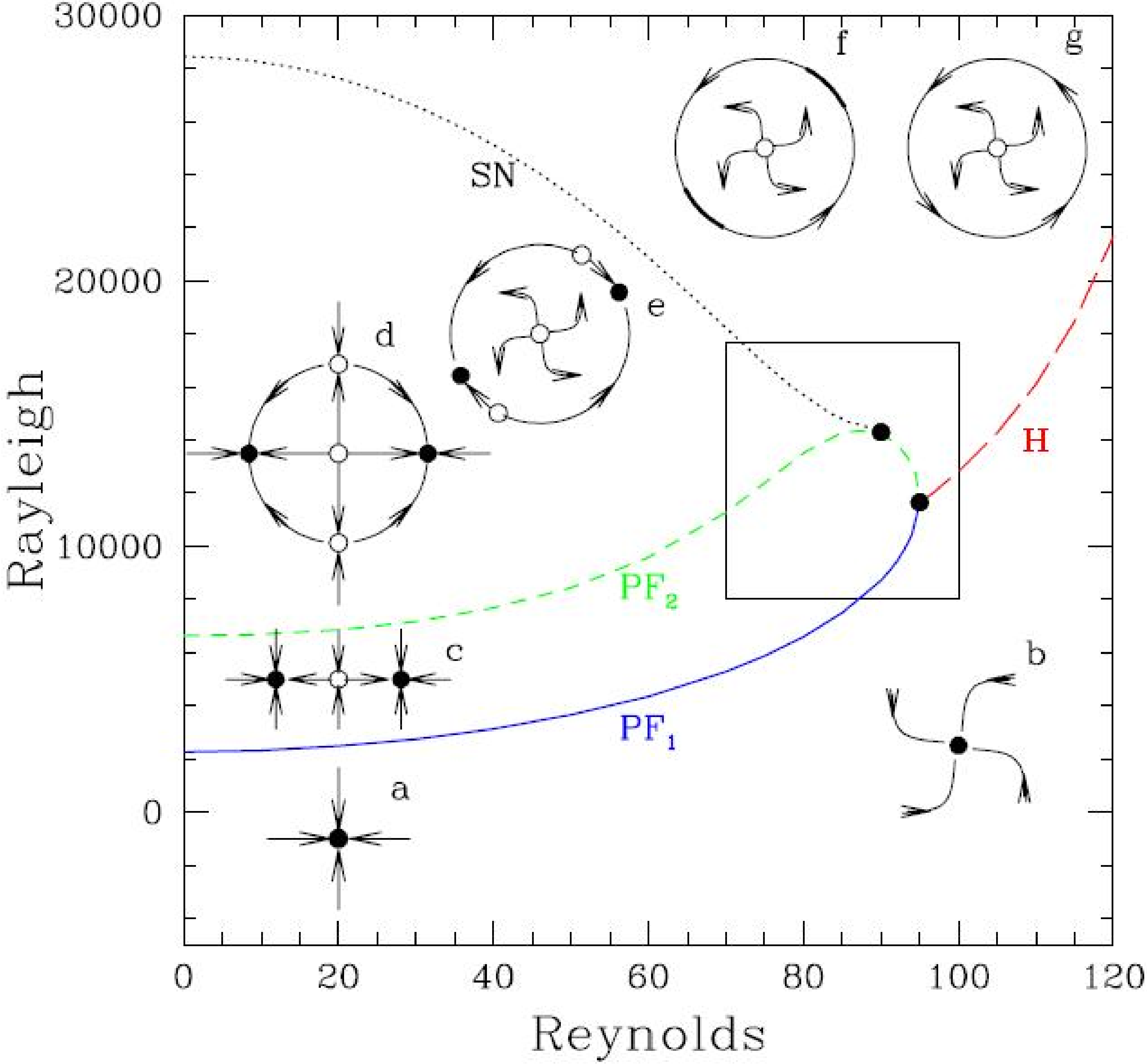
$u_\theta \to -u_\theta$

(Nore et al. 2003)

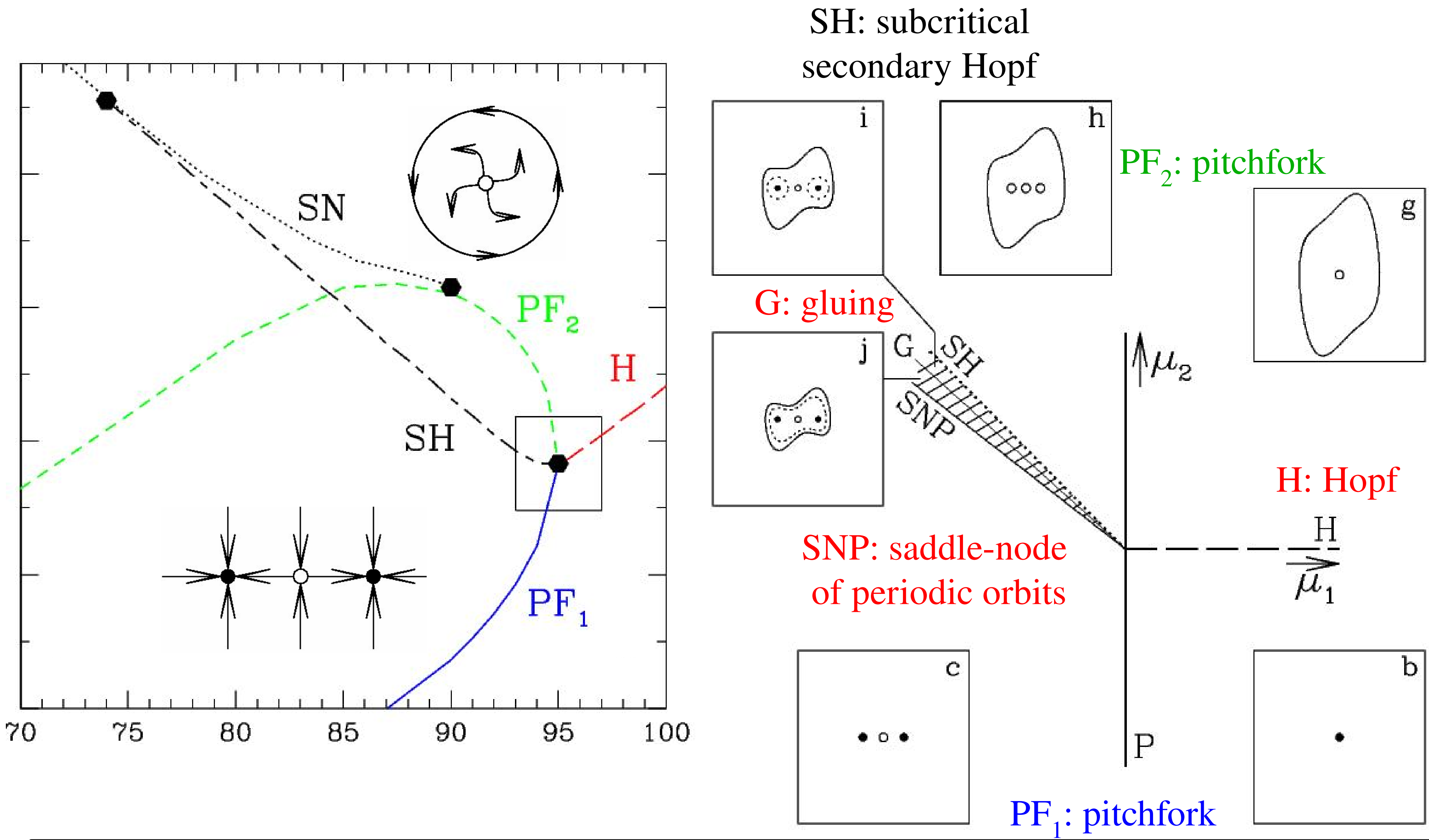

Ra
30000
20000
10000
0
Re
0
50
100
Hopf
pitchfork



$t = 0$
$t = 4.5 \approx \tau/8$
$t = 10.5 \approx \tau/4$
$t = 19.4 \approx \tau/2$
$t = 29.5 \approx 3\tau/4$
$t = 38 \approx \tau$
$u_z$
$t \rightarrow$
Re=110
near Hopf bif
sinusoidal
cycle
$u_\theta$
Re=63
near saddle-node
near-heteroclinic
cycle
$t = 22.6 = t_0$
$t = 68.7$
$t = 71.0$
$t = 72.0$
$t = 73.0$
$t = 86.5 \approx t_0 + \tau/2$



30000
20000
10000
0
0
20
40
60
80
100
120
Rayleigh
Reynolds
SN
PF2
PF1
H
a
b
c
d
e
f
g




Takens-Bogdanov normal form:

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = \mu_1 x - \mu_2 y - \delta x^3 - x^2 y$$

# Eigenvalues: from real to complex