Noise Induced Bistability of Parametric Surface Waves

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We report an experimental study on the effect of an external multiplicative noise on a subcritical bifurcation leading to the parametric amplification of surface waves. We show that the probability density function of the wave amplitude in the presence of noise has two maxima that do not correspond to any of the deterministic states. When the deterministic forcing is varied in the presence of noise, these most probable values give two new branches in the bifurcation diagram that involve a much larger difference in oscillation amplitude. The bistable region is also strongly enlarged. This noise induced bistability can be understood in the general framework of noise induced transitions.

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It has been known for a long time that random fluctuations of the control parameters of a dynamical system, or multiplicative noise, may generate surprising effects, such as stabilization by noise [1,2], noise induced transitions [3,4], stochastic resonance [5], etc. Although these effects have been predicted for a great variety of systems, only a few quantitative experiments have been performed on the effect of noise on the threshold of instabilities. Previous studies involve electronic oscillators [6], spin waves in ferrites and antiferromagnets [7,8], and electroconvection in nematic liquid crystals [9,10], but only the effect of noise on supercritical bifurcations has been considered thus far. Although stabilization of a parametric instability by noise has been reported with electronic oscillators [6], it has been also shown that noise may generate large amplitude bursts before the deterministic threshold leading to on-off intermittency dynamics [10].

In this Letter, we study the effect of multiplicative noise on a subcritical bifurcation. We consider the parametric amplification of waves on the surface of a vertically vibrated fluid layer (the Faraday instability [11]). The study of this system has been motivated by the following features: First, it is known that, with an appropriate choice of the vibration frequency, this system undergoes an abrupt and hysteretic transition from a static state to a finite amplitude oscillatory regime when the vibration amplitude is increased [12]; second, contrary to the case of electronic parametric oscillators, many damped modes are present at instability onset. These modes, which can be adiabatically eliminated for deterministic systems, are continuously excited by noise and may influence the bifurcation diagram [1].

A schematical diagram of the experimental setup is shown in Fig. 1. The fluid container is an aluminum vessel of dimensions $15 \times 2 \times 1$ cm^3 filled with distilled water. The meniscus is avoided by pinning the fluid surface at the edge of the container [13]. To prevent liquid evaporation and contamination of the surface, the fluid container is closed with a Plexiglas plate and its temperature is controlled by circulating water at 12.5 ± 0.1 °C. An electromagnetic vibration exciter, driven by a frequency synthesizer, provides a clean vertical acceleration (horizontal acceleration less than 1% of the vertical one). The vertical acceleration is measured by a piezoelectric accelerometer and a charge amplifier.

Even in the presence of noise, the pattern grows or vanishes coherently in space and consists of a single mode, at half the excitation frequency. We thus use a local optical detection technique [14] to measure the wave amplitude. An He-Ne laser beam is focused on the fluid surface, and its deflection by the surface oscillation is recorded with a position sensitive detector. The reflected light spot oscillates between two extrema whose distance is proportional to the amplitude of the surface wave. The photodetector provides a voltage which is proportional to the position of the incoming beam, thus to the height of the surface.

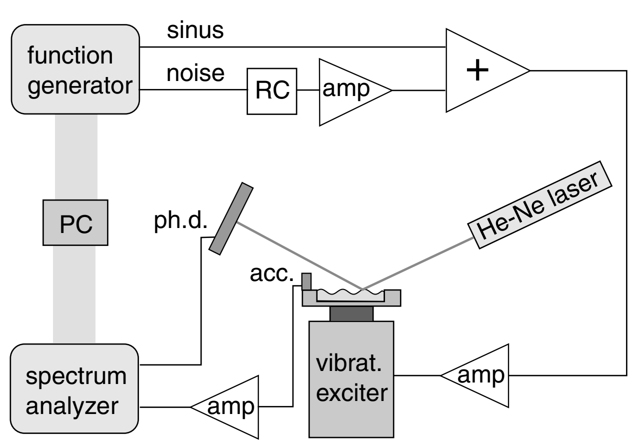

FIG. 1. Schematical diagram of the experimental setup (ph.d.: photodetector; acc.: accelerometer).

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A two channel function generator provides the sinusoidal and the noise signal. The noise is filtered, amplified, and then summed to the sinusoidal excitation. Using a spectrum analyzer, we check that the filtered noise is large band with a cutoff frequency around 1 kHz, so that it can be considered as a white noise around the typical frequencies of the sinusoidal signal (~60 Hz). The two channel spectrum analyzer records the signals of the photodetector and of the accelerometer, so that the amplitude of the instability and the corresponding acceleration are monitored at the same time. When noise is added to the sinusoidal forcing, the acceleration in the reference frame of the fluid container is $g_{\mathrm{eff}} = g + a\cos(\Omega t) + \xi(t)$, where g is the acceleration of gravity. Thus, both the sinusoidal forcing and the noise $\xi(t)$ parametrically force gravity-capillary waves. The amplitude of each mode obeys a Mathieu equation in the linear approximation for a fluid of small viscosity [15].

We performed the fast Fourier transform of both the acceleration and the wave amplitude signals on a frequency span of 400 Hz with a frequency resolution of 1 Hz, thus over a time interval of 1 s. This time scale corresponds to about 30 periods of the wave amplitude but is smaller than the characteristic duration of the wave bursts observed in the vicinity of the bifurcation in the presence of noise. Thus, the time dependence of the amplitude of the waves is well enough resolved in order to compute its probability density function (PDF) and its mean value.

The spectral power of the acceleration signal at Ω gives the rms value of the sinusoidal component a_{rms}. Note that, because of the presence of the noisy component, the measured value of a_{rms} is Gaussian distributed. The average of the spectral power over a frequency band of 30 Hz just above Ω gives a good estimation of the power spectral density of noise:

$$\kappa(\Omega) = 2\int_0^\infty \langle \xi(t)\xi(t+\tau)\rangle \cos(\Omega\tau)\, d\tau\,, \qquad (1)$$

since the spectrum is flat around Ω. Finally, the spectral power of the photodetector signal at $\Omega/2$ gives the rms value A_{rms} of the wave amplitude.

Time recordings of the amplitude of the waves in the presence of noise are displayed in Fig. 2 as the sinusoidal forcing is increased. The frequency of the parametric excitation is $\Omega/2\pi = 60.4$ Hz, corresponding to a negative detuning (roughly a tenth of Hz). Consequently, the f bifurcation without noise is subcritical and the wave amplitude abruptly jumps to a finite value for a rms acceleration $a_c = 5.8 \pm 0.1$ m/s^2 (see below). The noise intensity is $\kappa(\Omega) = 0.10$ mV^2/Hz, where 1 mV^2 corresponds to the square of one acceleration unit (in m/s^2). We can see (Fig. 2a) that noise triggers the instability onset before the deterministic threshold. The temporal signal is characterized by long laminar periods interrupted by intermittent bursts of strong oscillation amplitude. The presence

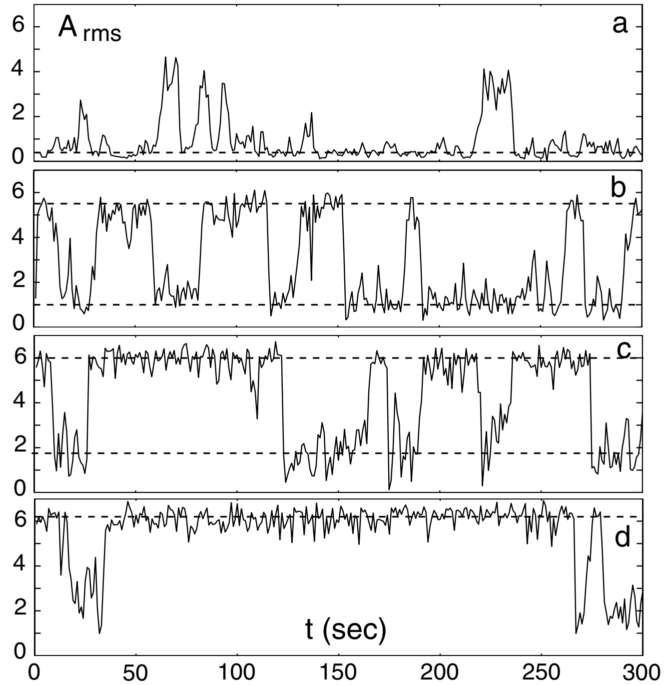

FIG. 2. Temporal evolution of the wave amplitude for a noise intensity $\kappa(\Omega) = 0.10$ mV^2/Hz. The sinusoidal forcing corresponds to an average acceleration $\langle a_{\mathrm{rms}}\rangle = 5.7$, 5.9, 6.2, and 6.4 m/s^2 in (a), (b), (c), and (d), respectively.

of this rare but large event makes the average value of the wave amplitude different from zero before the deterministic threshold. Note, however, that the most probable value is zero. When the average acceleration is increased, the amplitude begins to switch between two values indicated by the two dotted lines in Figs. 2b and 2c. Both values are increasing when the amplitude of the sinusoidal forcing is increased, and the system spends more and more time in the vicinity of the largest one. The PDFs corresponding to the time recordings of Fig. 2 are displayed in Fig. 3. At first sight, one may think that the effect of noise consists of triggering transitions between the two metastable states that exist in the vicinity of a subcritical bifurcation. We emphasize below that this is not the case.

The amplitude of the waves measured in the absence of noise is shown with open circles in the bifurcation diagram displayed in Fig. 4. As said above, there is an abrupt jump from zero to finite amplitude for $a_c = 5.8 \pm 0.1$ m/s^2. The subcritical nature of the bifurcation is due to negative detuning, i.e., $\nu = \Omega/2 - \omega(k_c) < 0$, where $\omega(k_c)$ is the frequency of the wave with the critical wave number k_c. When $\nu < 0$, a finite amplitude oscillation can be parametrically tuned because its frequency is a decreasing function of its amplitude [12]. The bistable region in the vicinity of this subcritical bifurcation is rather small but is strongly enlarged in the presence of noise. The most probable values of the wave amplitude in the presence of noise [$\kappa(\Omega) = 0.10$ mV^2/Hz as in Fig. 2] are plotted in Fig. 4. We observe a lower branch that continuously

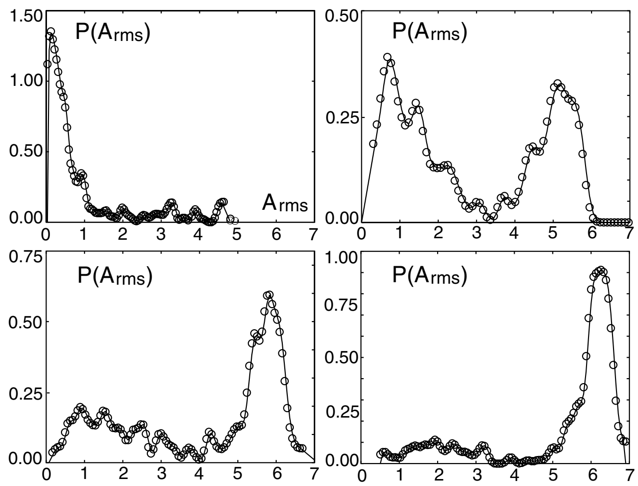

FIG. 3. PDFs of the wave amplitude for a noise intensity $\kappa(\Omega) = 0.10\ \mathrm{mV^2/Hz}$ and for different values of the deterministic forcing: $\langle a_{\mathrm{rms}} \rangle = 5.7$, 5.9, 6.2, and 6.4 $\mathrm{m/s^2}$ in (a), (b), (c), and (d), respectively.

increases from zero and an upper branch which exists for a rms acceleration above $a_1 = 5.9 \pm 0.1\ \mathrm{m/s^2}$. Thus, two branches of solutions, displaying a bistable region, are created by noise. The two dotted lines mark the beginning of the bistable region and the point $a_2 = 6.2 \pm 0.1\ \mathrm{m/s^2}$ above which the probability of visiting the lower branch has decreased to 10% of the probability of visiting the upper one.

The average wave amplitude $\langle A_{\mathrm{rms}} \rangle$ is shown with crosses in Fig. 4. We first observe that $\langle A_{\mathrm{rms}} \rangle$ increases continuously from zero when the average acceleration, $\langle a_{\mathrm{rms}} \rangle$, is increased. However, $\langle A_{\mathrm{rms}} \rangle$ does not interpolate between the two branches of the deterministic bifurcation diagram as it would do if the only effect of noise were to generate random transitions between these two states. It of course interpolates between the two branches for the most probable value of the wave amplitude. The same bifurcation diagram is displayed in Fig. 5 for a larger noise intensity [$\kappa(\Omega) = 0.18\ \mathrm{mV^2/Hz}$]. The behavior is qualitatively the same but, rather surprisingly, the noise induced bistable region is shifted to higher values of the average acceleration. The value of the slope of $\langle A_{\mathrm{rms}} \rangle$ versus the average acceleration decreases.

The effect of multiplicative noise on parametric instabilities has been considered for a long time. Both an analytical linear stability analysis of the zero solution of the Mathieu equation with a forcing involving harmonic and noisy components [16], and an experiment with an electronic parametric oscillator [6], have been performed. Our experimental results differ in two important aspects from the previous studies: First, we consider the effect of noise on a subcritical bifurcation; second, our system is spatially extended and thus involves many modes that may be excited by noise contrary to a parametric oscillator described by a Mathieu-type equation. We emphasize that both ingredients are necessary for the observation of noise induced bistability.

We have studied the effect of noise on a supercritical Faraday instability by choosing a positive detuning, $\nu = \Omega/2 - \omega(k_c) > 0$. In that case, the PDFs of the wave amplitude never display two local maxima.

We have also studied an electronic parametric oscillator governed by a Mathieu equation with a cubic nonlinearity [17]. In the case of negative detuning, i.e., for

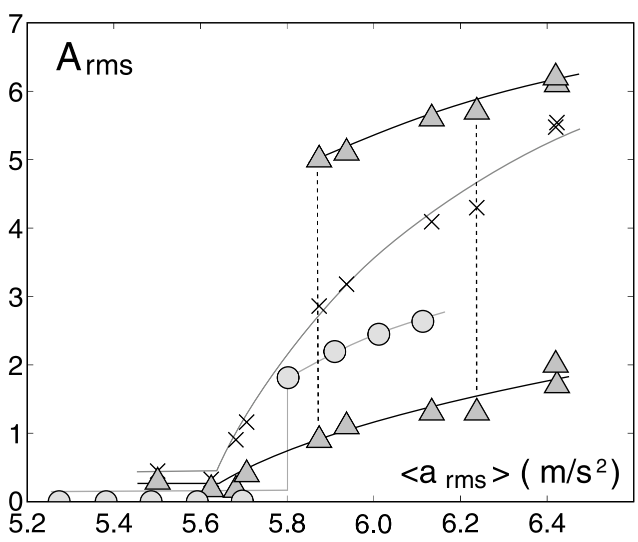

FIG. 4. Bifurcation diagram for the amplitude of the surface waves. The wave amplitude A_{rms} is measured in arbitrary units. Circles are measured in the absence of noise. Crosses correspond to the average value $\langle A_{\mathrm{rms}} \rangle$ measured for a noise intensity $\kappa(\Omega) = 0.10\ \mathrm{mV^2/Hz}$. For the same set of measurements, triangles represent the most probable values of the corresponding PDF. Lines are guides for the eyes.

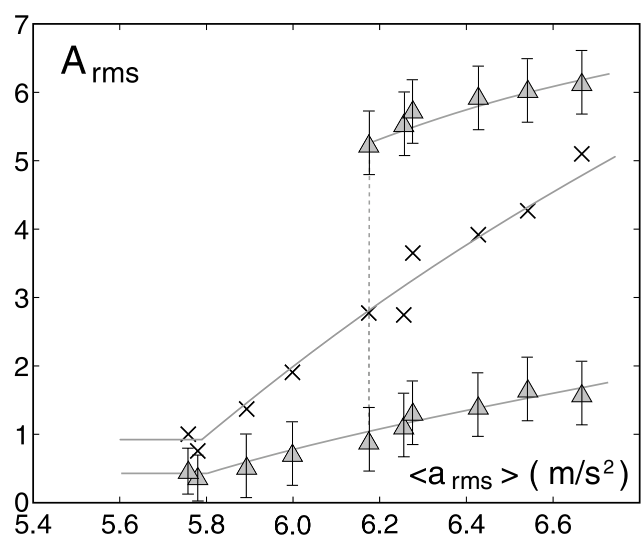

FIG. 5. Bifurcation diagram (crosses) of the average wave amplitude for $\kappa(\Omega) = 0.18\ \mathrm{mV^2/Hz}$. Triangles represent the most probable values of the corresponding PDF. Lines are guides for the eyes.

a subcritical bifurcation, two maxima of the PDF of the oscillation amplitude may be observed in the presence of multiplicative noise. However, these maxima correspond to the two metastable states of the deterministic system. For this low-dimensional dynamical system, the effect of noise is just to generate random transitions between the two metastable states of the deterministic system. If the mean value of the oscillation amplitude is taken as an order parameter, the bifurcation remains first order for a low external noise level, but becomes continuous at high enough noise. This is analogous in the time domain, to the phenomenon observed in phase transitions, where it has been shown that microscopic random impurities or other types of spatial disorder may produce rounding of a first-order phase transition [18–21].

Besides rounding the transition, the effect of noise on the mean wave amplitude is twofold. When the system is below the deterministic threshold, noise triggers random bursts of large amplitude oscillations. On the other hand, when waves are developed above the deterministic threshold, noise decreases their mean amplitude. This is due to the decorrelation of the phase that randomly detunes the waves from parametric resonance. This effect increases at high noise intensity and the system visits more and more the lower branch of the bistable region. This explains why the slope of the average wave amplitude decreases when the noise intensity is increased. This twofold effect of noise may also explain the enlargement of the bistable region.

In conclusion, we have shown that the bistable region that exists in the vicinity of a subcritical bifurcation to parametrically amplified waves may be strongly enlarged by multiplicative noise. This effect does not exist for a low-dimensional parametric oscillator and thus requires several coupled modes. We note that "noise-enhanced multistability" has been reported in a simple model of coupled oscillators but also requires additive noise [22]. The structure of this model being of a very different nature than the one involved in parametric amplification, we expect that noise induced bistability can be observed with other subcritical pattern-forming instabilities.

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